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ORIGINAL ARTICLE

Bilinear control of high frequencies for a 1D Schrödinger equation

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Abstract In this article, we consider a 1D linear Schrödinger equation with potential V and bilinear control. Under appropriate assumptions on V, we prove the exact controllability of high frequencies, in H^3, locally around any H^3-trajectory of the free system. In particular, any initial state in H^3 can be steered to a regular state, for instance a finite sum of eigenfunctions of $(-\Delta + V)$. This fact, coupled with a previous result due to Nersesyan, proves the global exact controllability of the system in H^3, under appropriate assumptions.

Keywords Schrödinger equation · Quantum control · Control of PDE · Bilinear control

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1 Introduction

1.1 Main result

In this article, we consider the 1D Schrödinger equation

$$\begin{cases} i\partial_t\psi(t,x) = \Big(-\partial_x^2 + V(x) - u(t)\mu(x)\Big)\psi(t,x)\,, & (t,x)\in(0,T)\times(0,1)\,,\\ \psi(t,0)=\psi(t,1)=0\,, & t\in(0,T)\,,\\ \psi(0,x)=\psi_0(x)\,, & x\in(0,1)\,, \end{cases} \tag{1}$$

where $V,\mu\in L^\infty((0,1),\mathbb{R})$ and $u:(0,T)\to\mathbb{R}$. It is a bilinear control system in which the state is ψ and the control is u.

To state our results, we first need to introduce few notations and recall well-posedness results. We denote by $\langle .,.\rangle$ the $L^2((0,1),\mathbb{C})$-scalar product,

$$\langle f,g\rangle := \Re\left(\int_0^1 f(x)\overline{g(x)}dx\right),$$

by A_V the operator

$$D(A_V) := H^2\cap H_0^1((0,1),\mathbb{C})\,,\qquad A_V := -\partial_x^2 + V\,,$$

(which is assumed to be positive: replacing $V(x)$ by $V(x)+C$ which only affects the global phase of ψ) by $(\lambda_{k,V})_{k\in\mathbb{N}^*}$ the nondecreasing sequence of its eigenvalues, by $(\varphi_{k,V})_{k\in\mathbb{N}^*}$ associated eigenfunctions,

$$\begin{cases} -\varphi_{k,V}''(x) + V(x)\varphi_{k,V}(x) = \lambda_{k,V}\varphi_{k,V}(x)\,,\ x\in(0,1)\,,\\ \varphi_{k,V}(0)=\varphi_{k,V}(1)=0\,,\\ \|\varphi_{k,V}\|_{L^2(0,1)}=1\,, \end{cases} \tag{2}$$

by $\mathbb{P}_{K,V}$, for $K\in\mathbb{N}^*$, the projection

$$\left|\begin{array}{l} \mathbb{P}_{K,V} : L^2((0,1),\mathbb{C}) \to \overline{\mathrm{Span}_{\mathbb{C}}(\varphi_{k,V};k\geqslant K)}\,,\\ \qquad\qquad\quad \xi \qquad\quad\ \mapsto \xi - \sum_{k=1}^{K-1}\langle\xi,\varphi_{k,V}\rangle\varphi_{k,V} \end{array}\right.$$

by $H^s_{(V)}(0,1)$, for $s>0$, the Sobolev spaces

$$H^s_{(V)}(0,1) := D(A_V^{s/2})\,,\qquad \|\xi\|_{H^s_{(V)}} := \left(\sum_{k=1}^\infty \lambda_{k,V}^s|\langle\xi,\varphi_{k,V}\rangle|^2\right)^{1/2}, \tag{3}$$

which satisfy, in particular,

$$H^3_{(V)}(0,1) = H^3_{(0)}(0,1) = \{\xi\in H^3((0,1),\mathbb{C});\ \xi=\xi''=0 \text{ at } x=0\,,1\}\,,$$

and by $\mathcal{S}$ the unitary $L^2((0,1),\mathbb{C})$-sphere. The following well-posedness result is a consequence of [3, Lemma 1] and the usual fixed point strategy (see [3, Proposition 3] for a the proof with $V = 0$).

Proposition 1 *Let $T > 0$, $V, \mu \in H^3((0,1),\mathbb{R})$, $\psi_0 \in H^3_{(0)}((0,1),\mathbb{C})$ and $u \in L^2((0,T),\mathbb{R})$. There exists a unique solution $\psi \in C^0([0,T], H^3_{(0)}(0,1))$ of (1). Moreover, $\|\psi(t)\|_{L^2(0,1)} = \|\psi_0\|_{L^2}$ for every $t \in [0,T]$.*

The goal of this article is the proof of the following result.

Theorem 1 *Let $T > 0$, $\psi_0 \in H^3_{(0)}((0,1),\mathbb{C}) \cap \mathcal{S}$, $\psi_{ref}(t) := e^{-iA_V t}\psi_0$, and $V, \mu \in H^3((0,1),\mathbb{R})$ be such that*

$$A_V \text{ has a simple spectrum} \tag{4}$$

$$\begin{aligned} &\mu'(1)\varphi'_{k,V}(1) + \mu'(0)\varphi'_{k,V}(0) \neq 0 \text{ and} \\ &\mu'(1)\varphi'_{k,V}(1) - \mu'(0)\varphi'_{k,V}(0) \neq 0\,, \text{ for every } k \in \mathbb{N}^*\,. \end{aligned} \tag{5}$$

1. *There exist $K \in \mathbb{N}^*$, $\delta > 0$ and a C^1-map*

   $$\Gamma : \mathcal{V} \to L^2((0,T),\mathbb{R})$$

   where

   $$\mathcal{V} := \{\psi_f \in \mathbb{P}_{K,V}[H^3_{(0)}(0,1)];\ \|\psi_f - \mathbb{P}_{K,V}[\psi_{ref}(T)]\|_{H^3_{(0)}} < \delta\}$$

   such that
   - $\Gamma\left(\mathbb{P}_{K,V}[\psi_{ref}(T)]\right) = 0$,
   - *for every $\psi_f \in \mathcal{V}$ the solution of (1) with control $u = \Gamma[\psi_f]$ satisfies $\mathbb{P}_{K,V}[\psi(T)] = \psi_f$.*
2. *As a consequence, there exist $K' \geqslant K$ and $u \in L^2((0,T),\mathbb{R})$ such that the solution of (1) satisfies $\mathcal{P}_{K',V}[\psi(T)] = 0$; in particular, $\psi(T,.) \in H^4_{(V)}((0,1),\mathbb{C})$.*

This result allows to prove the global exact controllability of (1) in $H^3_{(V)}((0,1),\mathbb{C})$, instead of $H^4_{(V)}((0,1),\mathbb{C})$ in [14] (or $H^{3+}_{(V)}((0,1),\mathbb{C})$ as can be proved by following the original proof [16]).

Corollary 1 *Let $V, \mu \in H^4((0,1),\mathbb{R})$ that satisfy (5) and*

$$\exists C > 0 \text{ such that } |\langle \mu\varphi_{1,V}, \varphi_{k,V}\rangle| \geqslant \frac{C}{k^3}\,, \forall k \in \mathbb{N}^*\,, \tag{6}$$

$$\lambda_{k,V} - \lambda_{1,V} \neq \lambda_{p,V} - \lambda_{q,V}\,, \forall k,p,q \in \mathbb{N}^* \text{ such that } \{1,k\} \neq \{p,q\}\,. \tag{7}$$

For every $\psi_0, \psi_f \in H^3_{(V)}((0,1),\mathbb{C}) \cap \mathcal{S}$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that the solution of (1) satisfies $\psi(T) = \psi_f$.

Let us now comment about the conclusion and hypothesis of Theorem 1. First, why to control only the high frequencies? The main interest of this strategy is that we can perform this control close to any given free trajectory. It could then happen that the linearization close to this trajectory contains a lot of directions lost for the control. Yet, these lost directions are necessary in finite number and we can control all the state, except a finite-dimensional subspace.

Concerning the hypothesis, we first notice they are likely to be generic in (V, μ) for an appropriate topology; see [3, Appendix A] for such a proof for a similar problem. Also, even in the case where the assumptions are not fulfilled, there are many "tricks" already available in the litterature to face this problem, see [16] or [2]. For instance, replacing $u(t)$ by $u(t) - \delta$ and V by $V + \delta\mu$ may allow very often to fulfill our assumptions, up to changing the eigenfunctions by $\varphi_{k,V+\delta\mu}$. Note also that perturbation arguments (see (18) below) allow to get that $\mu'(1)\varphi'_{k,V}(1) \pm \mu'(0)\varphi'_{k,V}(0) = \sqrt{2}(k\pi)\left[\mu'(1)(-1)^k \pm \mu'(0)\right] + \mathcal{O}_{k\to+\infty}(1)$. In particular, if $\mu'(1) \pm \mu'(0) \neq 0$, assumption (5) is always fulfilled for large k. Yet, the important example $\mu(x) = x$ does not satisfy this assumptions, for reasons of symmetry. But in [2], it was possible to break this symmetry. We believe that it should still be possible in our context with similar arguments.

The main idea of the proof is a decomposition of the control inspired by [18]. We decompose the control in two components: one term which contains the boundary terms and is close to a control from the boundary and another remainder term which contains the interior part. It turns out that the effective part of the control actually comes from the boundary term, the internal part being compact from the point of view of control. It satisfies, for instance, the condition of noncontrolability of [1]. The "boundary" control is then sufficient to control the large frequencies.

1.2 Bibliographical comments

The Schrödinger equation with bilinear control has been widely studied in the litterature. The multi-d model writes

$$\begin{cases} (i\partial_t + \Delta - V)\psi(t,x) = u(t)\mu(x)\psi(t,x)\,, & (t,x) \in (0,T)\times\Omega\,, \\ \psi(t,x) = 0\,, & (t,x) \in (0,T)\times\partial\Omega\,, \end{cases} \tag{8}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \in \mathbb{N}^*$, $V, \mu : \Omega \to \mathbb{R}$ are given functions, the state ψ lives in the $L^2(\Omega, \mathbb{C})$-sphere, denoted $\mathcal{S}$ and the control is the real-valued function $u : (0, T) \to \mathbb{R}$.

1.2.1 A negative result

A negative control result was proved by Turinici in [19], as a consequence of a general result by Ball, Marsden and Slemrod in [1]. It states that, for $V = 0$, for a given function $\mu \in C^2(\Omega, \mathbb{R})$, for a given initial condition $\psi_0 \in (H^2 \cap H^1_0)(\Omega, \mathbb{C}) \cap \mathcal{S}$, and by using controls $u \in L^r_{loc}((0,\infty), \mathbb{R})$ with $r > 1$, one may only reach a subset of $(H^2 \cap H^1_0)(\Omega) \cap \mathcal{S}$ that has an empty interior in $(H^2 \cap H^1_0)(\Omega, \mathbb{C}) \cap \mathcal{S}$.

Recently, Boussaid, Caponigro and Chambrion extended this negative result to the case of controls in $L^1_{loc}((0,\infty),\mathbb{R})$, see [7]. However, this negative result is actually due to a bad choice of functional setting, as emphasized in the next paragraph.

It is interesting to compare these negative results to our result. Indeed, they prove that our result is optimal concerning the attainable set of (8) for integer Sobolev regularity.

Concerning intermediate regularity, $s \in [2, 3]$, it seems that for $\mu \in C^3(\Omega, \mathbb{R})$ the counterexamples in [19] could be extended to $H^s_{(0)}$ for $s < 5/2$ which is the last space for which the multiplication by μ is continuous. For the specific example of $\mu(x) = x$ (which is not directly covered by our paper but gives some interesting insights), the results of [7] proves that the attainable set cannot contain $H^{5/2}_{(0)}$ even for control Radon measure. This leaves the possibility of controllability for $5/2 < s < 3$. This might be possible for some controls in L^1. Yet, we believe that for a control in L^2, the regularity $H^3_{(0)}$ is optimal since it is the case for the linearized equation.

1.2.2 Local exact results in 1-d

Beauchard proved in [2] the exact controllability of Eq. (8), locally around the ground state in H^7, with controls $u \in H^1((0, T), \mathbb{R})$ in large time T, in the case $N = 1$, $\Omega = (-1/2, 1/2)$, $\mu(x) = x$ and $V = 0$. The proof of [2] relies on Coron's return method and Nash–Moser theorem.

Reference [3] improves this result and establishes the exact controllability of Eq. (8), locally around the ground state in H^3, with controls $u \in L^2((0, T), \mathbb{R})$, in arbitrary time $T > 0$, and with generic functions μ when $N = 1$, $\Omega = (0, 1)$, $V = 0$. This result can be extended to an arbitrary potential V, as explained in [14]. The proof relies on a smoothing effect, that allows to conclude with the inverse mapping theorem (instead of Nash–Moser's one).

Then, Morancey and Nersesyan developed this strategy to control a Schrödinger equation with a polarizability term [13] and a finite number of Schrödinger equations with one control [12,14].

1.2.3 Global approximate results in N-d

Three strategies have been developed to study approximate controllability for Eq. (8).

The first strategy is a variational argument introduced by Nersesyan in [15]. It proves, under appropriate assumptions on (V, μ), that any initial condition in $H^{3+}_{(0)}(\Omega, \mathbb{C}) \cap \mathcal{S}$ can be steered to the ground state, approximately in H^3, with smooth controls $u \in C^\infty_c((0, T), \mathbb{R})$, in large time T, in arbitrary dimension N.

Note that, in 1D, this result can be coupled with the previous local exact controllability results. Then, under appropriate assumptions on (V, μ), we get that any initial condition in $H^{3+}_{(0)}((0, 1), \mathbb{C}) \cap \mathcal{S}$ can be steered to the ground state, exactly, in large time $T > 0$, with controls $u \in L^2((0, T), \mathbb{R})$. See [16] for one Eq. (8), [13] for a Schrödinger equation with a polarizability term, [14] for a finite number of Schrödinger equations with the same control.

A second strategy consists in deducing approximate controllability in regular spaces (containing H^3) from exact controllability results in infinite time by Nersesyan and Nersisyan [10]

A third strategy, due to Chambrion, Mason, Sigalotti, and Boscain [9], relies on geometric techniques for the controllability of the Galerkin approximations. It proves (under appropriate assumptions on V and μ) the approximate controllability of (8) in L^2, with piecewise constant controls. The hypotheses of this result were refined by Boscain, Caponigro, Chambrion and Sigalotti in [4]. The approximate controllability is proved in higher Sobolev norms in [7] for one equation, and in [6] for a finite number of equations with one control. For more details and more references about the geometric techniques, we refer the reader to the recent survey [5].

1.3 Structure of this article

In Sect. 2, we give the main steps of the proof of Theorem 1. Two intermediary results are stated and used in this proof, but proved later, in Sects. 3 and 4. Finally, in Sect. 5, we prove Corollary 1.

2 Proof of the main result

In this section, $V, \mu, T, \psi_0, \psi_{ref}$ are fixed and satisfy the assumptions of Theorem 1. The first statement of this theorem comes by applying the inverse mapping theorem to the map

$$\left|\begin{array}{rl}\Theta_K : L^2((0,T),\mathbb{R}) & \to \mathbb{P}_{K,V}[H^3_{(0)}(0,1)] \\ u & \mapsto \mathbb{P}_{K,V}[\psi(T)]\end{array}\right.$$

where ψ solves (1). Adapting the proof of [3, Proposition 3] to the case $V \neq 0$, we see that Θ_K is a C^1-map and

$$\left|\begin{array}{rl}d\Theta_K(0) : L^2((0,T),\mathbb{R}) & \to \mathbb{P}_{K,V}[H^3_{(0)}(0,1)] \\ v & \mapsto \mathbb{P}_{K,V}[\Psi(T)]\end{array}\right.$$

where Ψ solves the linearized system

$$\begin{cases} i\partial_t\Psi(t,x) = \Big(-\partial_x^2 + V(x)\Big)\Psi(t,x) - v(t)\mu(x)\psi_{ref}(t,x)\,, & (t,x)\in(0,T)\times(0,1)\,, \\ \Psi(t,0) = \Psi(t,1)\,, & t\in(0,T)\,, \\ \Psi(0,x) = 0\,, & x\in(0,1)\,. \end{cases} \tag{9}$$

Thus, to prove Theorem 1.1, it suffices to prove that, for K large enough, $d\Theta_K(0)$ has a continuous right inverse between the following spaces

$$d\Theta_K(0)^{-1} : \mathbb{P}_{K,V}[H^3_{(0)}(0,1)] \to L^2((0,T),\mathbb{R})\,.$$

To this aim, we introduce the decomposition $\mu(x)\psi_{ref}(t,x) = (\mu_1+\mu_2)(t,x)$ where $\mu_2 \in C^0([0,T], H^3_{(0)}(0,1))$ solves

$$\begin{cases} (-\partial_x^2 + V)^2\mu_2 = (-\partial_x^2 + V)^2[\mu\psi_{ref}]\,, & (t,x) \in (0,T)\times(0,1)\,, \\ \mu_2(t,\sigma) = \partial_x^2\mu_2(t,\sigma) = 0\,, & (t,\sigma) \in (0,T)\times\{0,1\}\,, \end{cases} \tag{10}$$

and

$$\mu_1(t,x) := \mu(x)\psi_{ref}(t,x) - \mu_2(t,x)\,, \quad \forall (t,x) \in (0,T)\times(0,1)\,, \tag{11}$$

i.e.,

$$\begin{cases} (-\partial_x^2 + V)^2\mu_1 = 0\,, & (t,x) \in (0,T)\times(0,1)\,, \\ \mu_1(t,\sigma) = 0\,, & (t,\sigma) \in (0,T)\times\{0,1\}\,, \\ \partial_x^2\mu_1(t,\sigma) = 2\mu'(\sigma)\partial_x\psi_{ref}(t,\sigma) & (t,\sigma) \in (0,T)\times\{0,1\}\,. \end{cases} \tag{12}$$

This decomposition is inspired by [18]. Then,

$$d\Theta_K(0).v = \Big(\mathcal{L}_K + \mathcal{K}_K\Big)(v)$$

where

$$\left|\begin{aligned} \mathcal{L}_K\,:\, L^2((0,T),\mathbb{R}) &\to \mathbb{P}_{K,V}[H^3_{(0)}(0,1)] \\ v &\mapsto \mathbb{P}_{K,V}[\Psi_1(T)] \end{aligned}\right.$$

$$\left|\begin{aligned} \mathcal{K}_K\,:\, L^2((0,T),\mathbb{R}) &\to \mathbb{P}_{K,V}[H^3_{(0)}(0,1)] \\ v &\mapsto \mathbb{P}_{K,V}[\Psi_2(T)] \end{aligned}\right.$$

and, for $j = 1, 2$,

$$\begin{cases} i\partial_t\Psi_j(t,x) = \Big(-\partial_x^2 + V(x)\Big)\Psi_j(t,x) - v(t)\mu_j(t,x)\,, & (t,x) \in (0,T)\times(0,1)\,, \\ \Psi_j(t,0) = \Psi_j(t,1)\,, & t \in (0,T)\,, \\ \Psi_j(0,x) = 0\,, & x \in (0,1)\,. \end{cases} \tag{13}$$

By [3, Lemma 1], for every $v \in L^2((0,T),\mathbb{R})$, $\Psi_j \in C^0([0,T], H^3_{(0)}(0,1))$, and thus, $\mathcal{L}_K, \mathcal{K}_K$ are continuous operators. The following 2 results will be proved in Sects. 3 and 4.

Proposition 2 *There exist $K_* \in \mathbb{N}^*$, $\mathcal{C} > 0$ and a decreasing sequence $(\mathcal{H}_K)_{K \geqslant K_*}$ of closed vector subspaces of $L^2((0,T),\mathbb{R})$ satisfying*

$$\bigcap_{K \geqslant K^*} \mathcal{H}_K = \{0\}\,, \tag{14}$$

such that for every $K \geqslant K_*$, the operator $\mathcal{L}_K : \mathcal{H}_K \to \mathbb{P}_K[H^3_{(0)}(0,1)]$ is an isomorphism and

$$\|\mathcal{L}_K^{-1}\|_{\mathbb{P}_{K,V}[H^3_{(0)}]\to L^2} \leqslant \mathcal{C}\,. \tag{15}$$

Proposition 3 *For every $K \in \mathbb{N}^*$, the operator $\mathcal{K}_K : L^2((0,T),\mathbb{R}) \to \mathbb{P}_K[H^3_{(0)}(0,1)]$ is compact.*

To end the proof of Theorem 1.1, it suffices to prove the existence of $K \geqslant K_*$ such that $d\Theta_K(0) = (\mathcal{L}_K + \mathcal{K}_K)$ is an isomorphism from $\mathcal{H}_K$ to $\mathbb{P}_{K,V}[H^3_{(0)}(0,1)]$.

Working by contradiction, we assume that, for every $K \geqslant K_*$, $(\mathcal{L}_K + \mathcal{K}_K) : \mathcal{H}_K \to \mathbb{P}_K[H^3_{(0)}(0,1)]$ is not an isomorphism. By Fredholm alternative (see Theorem VI of [8]) on $I + \mathcal{L}_K^{-1}\mathcal{K}_K$, there exists a sequence $(v_K)_{K \geqslant K_*}$ such that

$$v_K \in \mathcal{H}_K\,, \quad \|v_K\|_{L^2} = 1\,, \quad (\mathcal{L}_K + \mathcal{K}_K)(v_K) = 0\,, \quad \forall K \geqslant K_*\,.$$

Then, by (14)

$$v_K \rightharpoonup 0 \text{ in } L^2((0,T),\mathbb{R}) \tag{16}$$

and

$$\begin{aligned} 1 = \|v_K\|_{L^2(0,T)} &= \|\mathcal{L}_K^{-1} \circ \mathcal{L}_K(v_K)\|_{L^2(0,T)} \text{ because } v_K \in \mathcal{H}_K \\ &\leqslant \mathcal{C}\|\mathcal{L}_K(v_K)\|_{H^3_{(0)}(0,1)} \text{ by (15)} \\ &\leqslant \mathcal{C}\|\mathcal{K}_K(v_K)\|_{H^3_{(0)}(0,1)} \\ &\leqslant \mathcal{C}\|\mathcal{K}_1(v_K)\|_{H^3_{(0)}(0,1)} \underset{K\to\infty}{\longrightarrow} 0 \end{aligned}$$

because $\mathcal{K}_1$ is compact. This is a contradiction.

To prove the second statement of Theorem 1, one considers $K' \geqslant K$ such that $\|\mathbb{P}_{K',V}(\psi_{ref}(T))\|_{H^3_{(0)}} < \delta$ and applies statement 1 to $\psi_f := (\mathbb{P}_{K,V} - \mathbb{P}_{K',V})[\psi_{ref}(T)]$.

3 Ingham inequality

The goal of this section is to prove Proposition 2, by reducing the problem to an Ingham inequality. First, we recall useful estimates (see [17, Theorem 4 Chap. 2]).

$$\lambda_{k,V} = (k\pi)^2 + \int_0^1 V(x)dx + r_k \quad \text{where } \sum_{k=1}^{\infty} r_k^2 < \infty\,, \tag{17}$$

$$\exists C = C(V) > 0 \text{ such that } \|\varphi'_{k,V} - \varphi'_{k,0}\|_{L^\infty(0,1)} \leqslant C\,, \quad \forall k \in \mathbb{N}^*\,. \tag{18}$$

By the Duhamel formula, we have

$$\Psi_j(T) = i\sum_{k=1}^{\infty}\int_0^T v(t)\langle \mu_j(t), \varphi_{k,V}\rangle e^{-i\lambda_{k,V}(T-t)}dt\,\varphi_{k,V}\,. \tag{19}$$

For every $t \in (0,T)$, the function $x \mapsto \mu_1(t,x)$ solves a ordinary differential equation of order 4 with continuous coefficients, because $V \in H^3((0,1),\mathbb{R})$ (see (12)); thus, $\mu_1(t,.) \in C^4([0,1],\mathbb{C})$ and the following integrations by parts are legitimate

$$\begin{aligned}\langle \mu_1(t), \varphi_{k,V}\rangle &= \tfrac{1}{\lambda_{k,V}^2}\int_0^1 \mu_1(t,x)\Big(-\partial_x^2 + V(x)\Big)^2 \varphi_{k,V}(x)dx \\ &= \tfrac{2}{\lambda_{k,V}^2}\Big(\mu'(1)\partial_x\psi_{ref}(t,1)\varphi_{k,V}'(1) - \mu'(0)\partial_x\psi_{ref}(t,0)\varphi_{k,V}'(0)\Big).\end{aligned}$$

Thus, for a given target $\Psi_f \in \mathbb{P}_K[H^3_{(0)}(0,1)]$ and a function $v \in L^2((0,T),\mathbb{R})$, the equality $\mathcal{L}_K(v) = \Psi_f$ is equivalent to the moment problem

$$\int_0^T v(t) f_k(t)dt = \frac{\lambda_{k,V}^2}{2k\pi}\langle \Psi_f, \varphi_{k,V}\rangle e^{i\lambda_{k,V}T}, \quad \forall k \geqslant K, \tag{20}$$

where

$$f_k(t) := \Big(\mu'(1)\partial_x\psi_{ref}(t,1)\varphi_{k,V}'(1) - \mu'(0)\partial_x\psi_{ref}(t,0)\varphi_{k,V}'(0)\Big)\frac{e^{i\lambda_{k,V}t}}{k\pi}, \forall k \in \mathbb{N}^*.$$

Note that the right-hand side of (20) belongs to l^2 thanks to (17) and (3). Let

$$\mathcal{H}_K^{\mathbb{C}} := \mathrm{Adh}_{L^2((0,T),\mathbb{C})}\Big(\mathrm{Vect}\{f_k;\, |k| \geqslant K\}\Big) \text{ and } \mathcal{H}_K := \mathcal{H}_K^{\mathbb{C}} \cap L^2((0,T),\mathbb{R}) \tag{21}$$

where $f_k(t) := \overline{f_{-k}(t)}, \forall k \leqslant -1$. Clearly, (14) is satisfied. The following Ingham inequality - that will be proved later on - proves that, for K large enough, $(f_k)_{|k|\geqslant K}$ is a Riesz basis of $\mathcal{H}_K^{\mathbb{C}}$.

Proposition 4 *There exists $K_* \in \mathbb{N}^*$ and $\mathcal{C}_1, \mathcal{C}_2 > 0$ such that*

$$\mathcal{C}_1\|b\|_{l^2} \leqslant \left(\int_0^T \left|\sum_{|k|\geqslant K_*} b_k f_k(t)\right|^2 dt\right)^{1/2} \leqslant \mathcal{C}_2\|b\|_{l^2}, \quad \forall b \in l^2(\mathbb{Z}_{K_*},\mathbb{C}), \tag{22}$$

where $\mathbb{Z}_{K_*} := \{k \in \mathbb{Z};\, |k| \geqslant K_*\}$.

This proposition has 3 consequences: for every $K \geqslant K_*$

- for every $(d_k)_{|k|\geqslant K} \in l^2(\mathbb{Z}_K,\mathbb{C})$, there exists a unique function $v \in \mathcal{H}_K^{\mathbb{C}}$ such that

$$\int_0^T v(t) f_k(t)dt = d_k, \quad \forall |k| \geqslant K, \tag{23}$$

- in particular, if $d_{-k} = \overline{d_k}$ for every k, then v is real-valued (consequence of uniqueness); this proves that $\mathcal{L}_K : \mathcal{H}_K \to \mathbb{P}_{K,V}[H^3_{(0)}(0,1)]$ is bijective,
- moreover, this candidate is the unique solution in $L^2((0,T),\mathbb{R})$ of the moment problem (23) with minimal $L^2(0,T)$-norm; this proves that the sequence $\Big(\|\mathcal{L}_K^{-1}\|_{\mathbb{P}_{K,V}[H^3_{(0)}]\to\mathcal{H}_K}\Big)_{K\geqslant K_*}$ is decreasing, and thus, (15) holds.

which ends the proof of Proposition 2.

Proof of Proposition 4
Step 1: We prove that the 2 functions $g_\pm : (0,T) \to \mathbb{C}$ *defined by*

$$g_\pm(t) := \mu'(1)\partial_x \psi_{ref}(t,1) \pm \mu'(0)\partial_x \psi_{ref}(t,0)$$

are not identically zero on $(0,T)$.

It is a consequence of (4), (5) and the explicit expression

$$g_\pm(t) = \sum_{k=1}^{\infty} \Big(\mu'(1)\varphi'_{k,V}(1) \pm \mu'(0)\varphi'_{k,V}(0)\Big)\langle \psi_0, \varphi_{k,V}\rangle e^{-i\lambda_{k,V}t}\,.$$

More precisely, (4) implies that all the $\lambda_{k,V}$ are distinct, while (17) implies an asymptotic spectral gap. This allows to apply some Ingham type inequality (see, for instance, Haraux [11] or Theorem 6 of [3] that we extend by density). This implies

$$\|g_\pm\|^2_{L^2(0,T)} \geq C \sum_{k=1}^{\infty} \left| \Big(\mu'(1)\varphi'_{k,V}(1) \pm \mu'(0)\varphi'_{k,V}(0)\Big)\langle \psi_0, \varphi_{k,V}\rangle \right|^2. \tag{24}$$

Yet, since $\psi_0 \neq 0$, there exists k so that $\langle \psi_0, \varphi_{k,V}\rangle \neq 0$. Combined with (5), this gives $\|g_\pm\|_{L^2(0,T)} \neq 0$.

Step 2: We prove the existence of $K_0, \mathcal{C}_1^0, \mathcal{C}_2^0 > 0$ *such that*

$$\mathcal{C}_1^0 \|b\|_{l^2} \leqslant \left(\int_0^T \left| \sum_{|k|\geqslant K_0} b_k h_k(t) \right|^2 dt \right)^{1/2} \leqslant \mathcal{C}_2^0 \|b\|_{l^2}\,, \quad \forall b \in l^2(\mathbb{Z}_{K_0}, \mathbb{C})\,, \tag{25}$$

where

$$h_k(t) := \Big((-1)^k \mu'(1)\partial_x \psi_{ref}(t,1) - \mu'(0)\partial_x\psi_{ref}(t,0)\Big)e^{i\lambda_{k,V}t}\,, \quad \forall k \in \mathbb{N}^*\,,$$

Thanks to (4) and (17), for every $0 \leqslant \tau_1 < \tau_2 < \infty$, there exists $\mathcal{C}'_j = \mathcal{C}'_j(\tau_1,\tau_2) > 0$ such that

$$\mathcal{C}'_1 \|b\|_{l^2} \leqslant \left(\int_{\tau_1}^{\tau_2} \left| \sum_{|k|\geqslant 1} b_k e^{\pm i\lambda_{k,V}t} \right|^2 dt \right)^{1/2} \leqslant \mathcal{C}'_2 \|b\|_{l^2}\,, \quad \forall b \in l^2(\mathbb{Z} - \{0\}, \mathbb{C})\,, \tag{26}$$

where $\lambda_{k,V} := -\lambda_{k,V}$, $\forall k \leqslant -1$ (see [11]).

Let $(b_k)_{|k|\geqslant K}$ be a sequence of complex numbers with finite support. We have

$$\begin{aligned}\left\|\sum_{|k|\geqslant K} b_k h_k\right\|^2_{L^2(0,T)} &= \left\|g_+(t)\sum_{|k|\text{odd}\geqslant K} b_k e^{i\lambda_{k,V}t}\right\|^2_{L^2(0,T)} \\ &\quad+\left\|g_-(t)\sum_{|j|\text{even}\geqslant K} b_j e^{i\lambda_{j,V}t}\right\|^2_{L^2(0,T)} \\ &\quad-2\Re(\mathcal{T}_K)\end{aligned} \tag{27}$$

where

$$\mathcal{T}_K := \sum_{|k|\text{odd}\geqslant K}\ \sum_{|j|\text{even}\geqslant K} b_k\overline{b_j}\int_0^T g_+(t)g_-(t)e^{i(\lambda_{k,V}-\lambda_{j,V})t}dt\,.$$

For any $x\in[0,1]$, the map $t\mapsto \partial_x\psi_{ref}(t,x)$ belongs to $H^1(0,T)$; indeed,

$$\partial_t\partial_x\psi_{ref}(t,x) = -i\sum_{k=1}^{\infty}\lambda_{k,V}\langle\psi_0,\varphi_{k,V}\rangle e^{-i\lambda_{k,V}t}\varphi'_{k,V}(x);$$

thus, by (18) and (26)

$$\int_0^T |\partial_t\partial_x\psi_{ref}(t,x)|^2dt \leqslant \mathcal{C}'_2(0,T)^2\sum_{k=1}^{\infty}|\lambda_{k,V}(k\pi+C)\langle\psi_0,\varphi_{k,V}\rangle|^2 \leqslant C'\|\psi_0\|^2_{H^3_{(0)}}\,.$$

Therefore, the maps $g_\pm$ belong to $H^1((0,T),\mathbb{C})$, which is an algebra; thus, there exists $C>0$ such that (integration by part)

$$\left|\int_0^T g_+(t)g_-(t)e^{i\omega t}dt\right| \leqslant \frac{C}{|\omega|},\quad \forall|\omega|\geqslant 1\,.$$

Then, by Cauchy–Schwarz inequality,

$$|\mathcal{T}_K| \leqslant C\left(\sum_{|k|\text{odd}\geqslant K}|b_k|^2\right)^{1/2}\left(\sum_{|j|\text{even}\geqslant K}|b_j|^2\right)^{1/2}\sqrt{\epsilon_K}$$

where

$$\epsilon_K := \sum_{|k|\text{odd}\geqslant K}\ \sum_{|j|\text{even}\geqslant K}\frac{1}{(\lambda_{k,V}-\lambda_{j,V})^2}\,.$$

is finite and converges to zero when $K\to\infty$. Indeed, by (17), there exists C , $K'>0$ such that $|\lambda_{k,V}-\lambda_{j,V}|\geqslant C|k^2-j^2|$ for every odd integer $k\geqslant K'$ and even integer $j\geqslant K'$. Moreover, using the decomposition

$$\frac{1}{(k^2-j^2)^2} = \frac{1}{4k^2}\left(\frac{1}{(j-k)^2}+\frac{1}{(j+k)^2}\right) - \frac{1}{4k^3}\left(\frac{1}{j-k}-\frac{1}{j+k}\right)$$

we get

$$\sum_{k \text{ odd } \geqslant K} \sum_{j \text{ even} \geqslant K} \frac{1}{(k^2-j^2)^2} \leqslant C' \sum_{k \text{ odd} \geqslant K} \frac{1}{k^2} \leqslant \frac{C''}{K}.$$

By Step 1, there exists $0 \leqslant \tau_1^{\pm} < \tau_2^{\pm} \leqslant T$ and $m > 0$ such that $|g_{\pm}(t)| \geqslant m$ for every $t \in (\tau_1^{\pm}, \tau_2^{\pm})$. We deduce from (27) and (26) that

$$\left\| \sum_{|k| \geqslant K} b_k h_k \right\|^2_{L^2(0,T)} \geqslant \left(A - C\sqrt{\epsilon_K}\right) \sum_{|k|\geqslant K} |b_k|^2$$

where $A := m^2 \min\{\mathcal{C}_1'(\tau_1^+, \tau_2^+)^2; \mathcal{C}_1'(\tau_1^-, \tau_2^-)^2\}$. This gives the lower bound of (25) with K_0 large enough so that $\mathcal{C}_1^0 := \sqrt{A - C\sqrt{\epsilon_{K_0}}} > 0$. Let $M > 0$ be such that $g_{\pm}(t) \leqslant M$ for every $t \in (0, T)$. We deduce from (27) and (26) that the upper bound of (25) holds with $\mathcal{C}_2^0 := \sqrt{M\mathcal{C}_2'(0,T) + C\sqrt{\epsilon_{K_0}}}$.

Step 3: Conclusion. By (18), there exists $C > 0$ such that

$$\left\| \sum_{|k|\geqslant K} b_k(f_k - h_k) \right\|_{L^2(0,T)} \leqslant C \sum_{|k|\geqslant K} \frac{|b_k|}{k} \leqslant C \left(\sum_{|k|\geqslant K} |b_k|^2 \right) \left(\sum_{|k|\geqslant K} \frac{1}{k^2} \right).$$

We deduce from (25) that, for every $K \geqslant K_0$ and $b \in l^2(\mathbb{Z}_K, \mathbb{C})$, we have

$$\left(\mathcal{C}_1^0 - \frac{2}{K-1}\right)\|b\|_{l^2} \leqslant \left(\int_0^T \left| \sum_{|k|\geqslant K} b_k f_k \right|^2 dt \right)^{1/2} \leqslant \left(\mathcal{C}_2^0 + \frac{2}{K-1}\right)\|b\|_{l^2}$$

which gives the conclusion with any $K_* \geqslant K_0$ large enough so that $\mathcal{C}_1 := \mathcal{C}_1^0 - \frac{2}{K_*-1} > 0$.

4 Compactness property

The goal of this section is to prove Proposition 3. Let $K \in \mathbb{N}^*$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence in $L^2((0,T),\mathbb{R})$ that weakly converges to 0, and is bounded by 1. Then,

$$\|\mathcal{K}_K(v_n)\|^2_{H^3_{(0)}} = \sum_{k\geqslant K} \left| (k\pi)^3 \int_0^T v_n(t) \langle \mu_2(t), \varphi_{k,V} \rangle e^{i\lambda_{k,V} t} dt \right|^2.$$

Each term of this sum converges to zero when $[n \to \infty]$. Moreover, using the explicit expression $\varphi_{k,0}(x) = \sqrt{2}\sin(k\pi x)$, integrations by part (note that $\mu_2 \in C^0([0,T], H^3_{(0)}(0,1))$), Cauchy–Schwarz inequality and (18), we get

$$\begin{aligned}
&\left|(k\pi)^3 \int_0^T v_n(t)\langle \mu_2(t), \varphi_{k,V}\rangle e^{i\lambda_{k,V}t}dt\right| \\
&\leqslant C\left|k^3 \int_0^T v_n(t)\langle \mu_2(t), \varphi_{k,0}\rangle e^{i\lambda_{k,V}t}dt\right| + C\left|k^3 \int_0^T v_n(t)\langle \mu_2(t), \varphi_{k,V}-\varphi_{k,0}\rangle e^{i\lambda_{k,V}t}dt\right| \\
&\leqslant C\left|\int_0^T v_n(t)\langle \partial_x^3\mu_2(t), \sqrt{2}\cos(k\pi x)\rangle e^{i\lambda_{k,V}t}dt\right| + \frac{C}{k}\int_0^T |v_n(t)|\|\mu_2(t)\|_{H^3_{(0)}}dt \\
&\leqslant C\left(\int_0^T |\langle \partial_x^3\mu_2(t), \sqrt{2}\cos(k\pi x)\rangle|^2 dt\right)^{1/2} + \frac{C}{k}\left(\int_0^T \|\mu_2(t)\|^2_{H^3_{(0)}}dt\right)^{1/2}.
\end{aligned}$$

This right-hand side belongs to $l^2(\mathbb{Z}_K)$ and does not depend on n; thus, by the dominated convergence theorem $\mathcal{K}_K(v_n) \underset{n\to\infty}{\longrightarrow} 0$ in $H^3_{(0)}(0,1)$.

5 Global exact controllability in $H^3_{(0)}(0, 1)$

The following result is proved in [14, Theorem 5.1], by following the proof developed in the original article [16].

Proposition 5 *Let $V, \mu \in H^4((0,1),\mathbb{R})$ that satisfy (6) and (7). Then for every $\psi_0, \psi_f \in H^4_{(V)}((0,1),\mathbb{C}) \cap \mathcal{S}$, there exists $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that the solution of (1) satisfies $\psi(T) = \psi_f$.*

Proof of Corollary 1 Starting from an initial condition $\psi_0 \in H^3_{(0)}$, we first use a control $u \in L^2((0,T_1),\mathbb{R})$ to reach a function $\psi(T_1) \in H^4_{(V)}(0,1)$, thanks to the second statement of Theorem 1. Then, by the previous proposition, there exists a control $u \in L^2((T_1,T_2),\mathbb{R})$ that steers the solution from $\psi(T_1)$ to $\psi(T_2) = \varphi_{1,V}$.

Given a target $\psi_f \in H^3_{(0)}$, thanks to the previous result and the time reversibility of the Schrödinger equation (i.e., (ψ, u) is a trajectory $\Rightarrow$ $(\overline{\psi}(T-t), u(T-t))$ is a trajectory) there exists $u \in L^2((T_2,T_3),\mathbb{R})$ that steers the solution from $\psi(T_2) = \varphi_{1,V}$ to $\psi(T_3) = \psi_f$. $\square$

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