EXACT CONTROLLABILITY OF ANISOTROPIC 1D PARTIAL DIFFERENTIAL EQUATIONS IN SPACES OF ANALYTIC FUNCTIONS

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Abstract. In this article, we prove a local controllability result for a general class of 1D partial differential equations on the interval $(0,1)$. The PDEs we consider take the form $\partial_t^N y = \zeta_M \partial_x^M y + f(x, y, \partial_x y, ..., \partial_x^{M-1} y)$ where $1 \leq N < M$, $\zeta_M \in \mathbb{C}^*$, and f is some linear or nonlinear term of lower order. In this context, we prove a local controllability result between states that are analytic functions. If some boundary conditions are prescribed, a similar local controllability result holds between analytic functions satisfying some compatibility conditions that are natural for the existence of smooth solutions of the considered PDE. The proof is performed by studying a nonlinear Cauchy problem in the spatial variable with data in some spaces of Gevrey functions and by investigating the relationship between the jet of space derivatives and the jet of time derivatives. We give various examples of applications, including the (good and bad) Boussinesq equation, the Ginzburg-Landau equation, the Kuramoto-Sivashinsky equation and the Korteweg-de Vries equation.

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Contents

1. Introduction

For M, $N \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ fixed with $M > N$ and y a function defined on $[0,1] \times [0,T]$, with value in $\mathbb{R}$, we consider the abstract dynamical system

$$\partial_t^N y = P\,y + f(x, y, \partial_x y, ..., \partial_x^{M-1} y), \qquad x \in [0,1],\ t \in [0,T], \tag{1.1}$$

$$BY^x(0,t) = 0, \qquad t \in [0,T], \tag{1.2}$$

$$Y^t(x,0) = Y_0(x), \qquad x \in [0,1], \tag{1.3}$$

with

$$Y^x(x,t) := (y(x,t), \partial_x y(x,t), ..., \partial_x^{M-1} y(x,t)), \tag{1.4}$$

$$Y^t(x,t) := (y(x,t), \partial_t y(x,t), ..., \partial_t^{N-1} y(x,t)), \tag{1.5}$$

$$P := \sum_{j=0}^{M} \zeta_j \partial_x^j, \tag{1.6}$$

where $\zeta_j \in \mathbb{R}$ for $0 \le j \le M$ and $\zeta_M \neq 0$, $Y_0 \in C^\infty([0,1])^N$, $B \in \mathbb{R}^{v \times M}$ is a fixed real matrix of size $v \times M$, and $v \in \mathbb{N}$ is the number of boundary conditions that we require to be equal to zero. (If $v = 0$, it indicates that there is no boundary condition at $x = 0$.) Finally, we assume $f \in C^\infty(\mathbb{R}^{M+1}; \mathbb{R})$ and f is analytic with respect to all its arguments in a *neighborhood* of $\vec{0}_{\mathbb{R}^{M+1}}$. More precisely, we assume that

$$f(x, 0, \dots, 0) = 0, \quad \forall x \in (-4, 4), \tag{1.7}$$

and

$$f(x, \vec{y}) := \sum_{(\vec{p},r) \in \mathbb{N}^{M+1}} a_{\vec{p},r} \vec{y}^{\vec{p}} x^r = \sum_{(\vec{p},r) \in \mathbb{N}^{M+1}} a_{\vec{p},r} y_0^{p_0} y_1^{p_1} \cdots y_{M-1}^{p_{M-1}} x^r \tag{1.8}$$

with $\vec{y} = (y_0, y_1, ..., y_{M-1})$, $(x, \vec{y}) \in (-4,4)^{M+1}$, and $\vec{p} = (p_0, \dots, p_{M-1}) \in \mathbb{N}^M$ where the coefficients $a_{\vec{p},r}$ are such that

$$|a_{\vec{p},r}| \le \frac{C_a}{b^{|\vec{p}|} b_2^r}, \qquad \forall r \in \mathbb{N},\ \forall \vec{p} \in \mathbb{N}^M, \tag{1.9}$$

for some constants

$$C_a > 0, \quad b > 4, \quad \text{and } b_2 > 4. \tag{1.10}$$

Note that $a_{\vec{0},r} = 0$ for all $r \in \mathbb{N}$ by (1.7). For $\vec{p} \in \mathbb{N}^M$, we define

$$A_{\vec{p}}(x) := \sum_{r \in \mathbb{N}} a_{\vec{p},r} x^r, \quad |x| < b_2.$$

We infer from (1.8) and (1.9) that

$$f(x,\vec{y}) \;=\; \sum_{\substack{\vec{p}\,\in\,\mathbb{N}^M\\ |\vec{p}|\,>\,0}} A_{\vec{p}}(x)\vec{y}^{\vec{p}} = \sum_{\substack{\vec{p}\,\in\,\mathbb{N}^M\\ |\vec{p}|\,>\,0}} A_{\vec{p}}(x)y_0^{p_0}y_1^{p_1}\dots y_{M-1}^{p_{M-1}}, \tag{1.11}$$

$$|A_{\vec{p}}(x)| \le \frac{C_a}{b^{|\vec{p}|}}\frac{1}{1-\frac{|x|}{b_2}}, \quad |x|<b_2. \tag{1.12}$$

Among the many physically relevant instances of (1.1) satisfying (1.7)-(1.10), we can mention

(1) the *Korteweg-de Vries (KdV) equation*

$$\partial_t y = \partial_x^3 y + \partial_x y + y\partial_x y;$$

(2) the *"good"* $(-)$ *or "bad"* $(+)$ *Boussinesq equation*

$$\partial_t^2 y = \pm\partial_x^4 y + \partial_x^2 y - \partial_x^2(y^2);$$

(3) the *Kuramoto-Sivashinsky (KS) equation*

$$\partial_t y + \partial_x^4 y + \partial_x^2 y + y\partial_x y = 0.$$

With a few modifications in the framework, we can also treat

(4) the *complex Ginzburg-Landau (GL) equation*

$$\partial_t y = e^{i\theta}\partial_x^2 y + e^{i\varphi}|y|^2 y \quad \text{where} \quad \theta,\varphi\in\mathbb{R}.$$

The exact controllability result has to be stated in a space of analytic functions (see [25] for the linear heat equation). For given $R>1$ and $C>0$, we denote by $\mathcal{N}_{R,C}$ and $\mathcal{R}_{R,C}$ the sets

$$\mathcal{N}_{R,C} := \left\{(\alpha_n)_{n\ge 0}\in\mathbb{C}^{\mathbb{N}};\, |\alpha_n|\le C\frac{n!}{R^n},\ \ \forall n\ge 0\right\}\subset\mathbb{C}^{\mathbb{N}}, \tag{1.13}$$

$$\mathcal{R}_{R,C} := \left\{z:[-1,1]\to\mathbb{C}:\ \exists(\alpha_n)_{n\ge 0}\in\mathcal{N}_{R,C} \text{ with } z(x)=\sum_{n=0}^{\infty}\alpha_n\frac{x^n}{n!},\ \ \forall x\in[-1,1]\right\}. \tag{1.14}$$

Let us denote by $H(\Omega)$ the space of holomorphic functions in Ω, and let us introduce the Hardy space $H_R^\infty := H(B(0,R))\cap L^\infty(B(0,R))$, which is a Banach space for the norm $\|\cdot\|_{L^\infty(B(0,R))}$ (see [33]). Let

$$\mathcal{B}_{R,C} := \{z:[-1,1]\to\mathbb{C};\ \exists f\in H_R^\infty,\ \|f\|_{L^\infty(B(0,R))}\le C,\ f_{|\,[-1,1]}=z\}.$$

Observe that

$$\mathcal{B}_{R,C}\subset\mathcal{R}_{R,C}\subset\mathcal{B}_{r,C(1-\frac{r}{R})^{-1}} \text{ for } 1<r<R \text{ and } C>0.$$

For the proof, see below Lemma 6.1.

We say that a function $h\in C^\infty([t_1,t_2])$ is *Gevrey of order* $s\ge 0$ *on* $[t_1,t_2]$, and we write $h\in G^s([t_1,t_2])$, if there exist some positive constants C,R such that

$$|\partial_t^p h(t)| \le C\frac{(p!)^s}{R^p}, \quad \forall t\in[t_1,t_2],\ \forall p\in\mathbb{N}.$$

Similarly, we say that a function $y \in C^\infty([x_1,x_2]\times[t_1,t_2])$ is *Gevrey of order s_1 in x and s_2 in t*, with $s_1, s_2 \geq 0$, and we write $y \in G^{s_1,s_2}([x_1,x_2]\times[t_1,t_2])$, if there exist some positive constants C, R_1, R_2 such that

$$|\partial_x^{p_1}\partial_t^{p_2} y(x,t)| \leq C\frac{(p_1!)^{s_1}(p_2!)^{s_2}}{R_1^{p_1}R_2^{p_2}}, \quad \forall (x,t)\in[x_1,x_2]\times[t_1,t_2],\ \forall (p_1,p_2)\in\mathbb{N}^2. \tag{1.15}$$

The suitable time Gevrey regularity in our situation is

$$\lambda := \frac{M}{N} > 1.$$

Before giving our results, we need to define a set of compatibility conditions. The initial data need to belong to a specific set to ensure the existence of smooth solutions issuing from these initial data. Indeed, the equation imposes some relations between the time derivatives of the solutions and the space derivatives of the initial data. Namely, we have the following property whose proof is constructive and mainly consists in taking derivatives in the PDE.

Lemma 1.1. *For any $l \in \mathbb{N}$, there exist a number $m = m(l) \in \mathbb{N}$ and a smooth application $J_l : [-1,1]\times(\mathbb{R}^N)^{m(l)+1} \to \mathbb{R}^M$ such that for any solution $y \in C^\infty([-1,1]\times[t_1,t_2])$ of $\partial_t^N y = Py + f(x, y, \partial_x y, ..., \partial_x^{M-1} y)$, we have*

$$\partial_t^l Y^x = J_l(x, Y^t, \partial_x Y^t, ..., \partial_x^m Y^t)) \quad \text{on } [-1,1]\times[t_1,t_2]. \tag{1.16}$$

Definition 1. Let J_l, $l \in \mathbb{N}$, be the vector functions defined in Lemma 1.1. We define the following **compatibility set**

$$\mathcal{C} := \left\{ Y_0 \in C^\infty([0,1])^N;\quad BJ_l(x, Y_0, \partial_x Y_0, ..., \partial_x^{m(l)} Y_0)\Big|_{x=0} = 0, \quad \forall l \in \mathbb{N} \right\}. \tag{1.17}$$

The compatibility set $\mathcal{C}$ plays an important role in the exact controllability of system (1.1)-(1.3). Since the PDE (1.1) is time-invariant, we can check that the condition (1.17) is the same at any time. In particular

- for any smooth solution y of (1.1)-(1.2), we have that $Y^x(t) \in \mathcal{C}$ for any $t \in [0,T]$. (See below Lemma 4.4.)
- if y is a smooth solution to (1.1) such that $Y^x(t) \in \mathcal{C}$ for any $t \in [0,T]$, then y satisfies the boundary condition (1.2). (See below Lemma 4.5.)

If we want to consider the boundary controllability of the PDE (1.1) subject to the boundary conditions (1.2), it is therefore very natural to consider initial and final data in the space $\mathcal{C}$. We will derive a controllability result by considering small amplitude analytic functions in $\mathcal{C}$.
The main result in this paper is the following *local exact controllability result.*

Theorem 1.2. *Let $f = f(x,\vec{y})$ be as in (1.7)-(1.10) with $b, b_2 > \hat{R} := 4Me^{(\lambda e)^{-1}}$ Let $R > \hat{R}$ and $T > 0$. Then there exists some number $\hat{C} > 0$ such that for all $Y_0, Y_1 \in (\mathcal{R}_{R,\hat{C}})^N \cap \mathcal{C}$, there exists a smooth solution y of (1.1)-(1.3) defined for all $(x,t) \in [0,1]\times[0,T]$ and satisfying $Y^t(x,T) = Y_1(x)$ for all $x \in [0,1]$. Furthermore, we have $y \in G^{1,\lambda}([0,1]\times[0,T])$.*

We stress that Theorem 1.2 can be applied to **any** PDE with less derivatives in time than in space, even if the corresponding initial boundary value problem is **not well-posed**. For instance, the backward heat equation $\partial_t y = -\partial_x^2 y$ and the bad Boussinesq equation $\partial_t^2 y = \partial_x^4 y + \partial_x^2 y - \partial_x^2(y^2)$ are concerned.

It is difficult in general to describe explicitly $\mathcal{C}$ (see Section 2.1 for the KdV equation). However, the set $\mathcal{C}$ can be precisely described in the following cases:

- If $B = 0$ (i.e. no boundary conditions at $x = 0$), then $\mathcal{C} = C^\infty([0,1])^N$ (i.e. all smooth initial data are allowed)
- If $f = 0$ (linear PDE with constant coefficients), then the compatibility set reads

$$\mathcal{C} = \Big\{ Y_0 = (y_0, y_1, \dots, y_l, \dots, y_{N-1}) \in C^\infty([0,1])^N \text{ such that } BP^j Y_0^{x,l}(0) = 0, \\ \forall j \in \mathbb{N}, \forall l = 0, \dots, N-1 \Big\}$$

when we denoted $Y_0^{x,l}(x) := (y_l(x), \dots, \partial_x^{M-1} y_l(x))$ as in (1.4). We refer to Proposition 2.8 for a precise statement and for the proof.
- if $M \in 2\mathbb{N}$ and $P = \sum_{j=0}^{M/2} \zeta_{2j} \partial_x^{2j}$ (i.e. P contains only even derivatives), some parity arguments can be used under some symmetry assumptions about the non-linearity, as it is shown in the following proposition.

Proposition 1.3. *Assume that $M \in 2\mathbb{N}$ and $P = \sum_{j=0}^{M/2} \zeta_{2j} \partial_x^{2j}$.*

(1) If the boundary conditions $BY^x(0,t) = 0$ reduce to $\partial_x^{2j} y(0,t) = 0$ for $2j \le M-1$, and if for all $x \in [-1,1]$ and all $(y_0, \dots, y_M) \in (-4,4)^{M+1}$ we have

$$f(-x, -y_0, \dots, (-1)^{i+1} y_i, \dots, y_{M-1}) = -f(x, y_0, \dots, y_{M-1}) \tag{1.18}$$

then

$$\mathcal{C} = \big\{ Y_0 = (y_0, y_1, \dots, y_{N-1}) \in C^\infty([0,1])^N; \quad \partial_x^{2j} y_l(0) = 0 \quad \forall j \in \mathbb{N}, \ \forall l = 0, \dots. N-1 \big\}$$

(2) If the boundary conditions $BY^x(0,t) = 0$ reduce to $\partial_x^{2j+1} y(0,t) = 0$ for $2j+1 \le M-1$, and if for all $x \in [-1,1]$ and all $(y_0, \dots, y_M) \in (-4,4)^{M+1}$ we have

$$f(-x, y_0, \dots, (-1)^i y_i, \dots, -y_{M-1}) = f(x, y_0, \dots, y_{M-1}) \tag{1.19}$$

then

$$\mathcal{C} = \big\{ Y_0 = (y_0, y_1, \dots, y_{N-1}) \in C^\infty([0,1])^N; \quad \partial_x^{2j+1} y_l(0) = 0, \quad \forall j \in \mathbb{N}, \forall l = 0, \dots, N-1 \big\}.$$

Note that in the last two cases, the intersection of $\mathcal{C}$ with the set of analytic functions is a set of functions that admit odd (respectively even) extensions. Note also that the "good" and "bad" Boussinesq equations satisfy only (1.19), while the Ginzburg-Landau equation satisfies both (1.18) and (1.19).

Remark 1. *(1) The constant $\hat{R} := 4Me^{(\lambda e)^{-1}}$ is probably not optimal, but we aimed to provide an explicit (reasonable) constant. For the linear heat equation, it is known that the optimal constant is $\hat{R} := 1$ with a diamond-shaped domain of analyticity (see* [4, 13, 14]*).*

(2) If f is linear in the variables $\vec{y} = (y_0, \dots, y_{M-1})$, then the PDE (1.1) *is linear and the smallness assumption on the amplitude of the initial and final data can be removed, as long as $Y^0, Y^1 \in (\mathcal{R}_{R,\hat{C}})^N \cap \mathcal{C}$ for some $\hat{C} \in (0, +\infty)$. In particular, for $f(x, \vec{y}) = V(x) y_0$, Theorem 1.2 applies for any equation of the form $\partial_t^N y = P y + V(x) y$ where V is analytic*

in a sufficiently large ball. The compatibility set $\mathcal{C}$ may depend on V. Note however that both conditions (1.18) *and* (1.19) *are satisfied if V is even (i.e. $V(x) = V(-x)$ for all $x \in [-1,1]$). Theorem 1.2 applies for instance for the linear heat equation $\partial_t y = \partial_x^2 y + V(x)y$ without any smallness assumption about the potential $V(x)$, giving that the reachable space from zero contains functions that are analytic in some sufficiently large ball. See* [8] *for a more precise result about the reachable space, but under a* smallness assumption about the potential.

Note also that the most relevant term of P is actually the higher order term $\zeta_M \partial_x^M$, since linear lower order terms can be put either in P or in $f(x, y, \partial_x y, ..., \partial_x^{M-1} y)$. Yet, we have chosen to keep this form because $\mathcal{C}$ is easily determined for a linear PDE with constant coefficients.

(3) *The definition of $\mathcal{C}$ seems to depend on some choice of the functions J_l^k. However, the proof of Lemma 1.1 is constructive and therefore it provides an algorithm to define these functions. Moreover, it is possible (see Lemma 4.17 below) to prove that if two functions J_l^k satisfy the property* (1.16) *for all solution y of* (1.1)*, then they coincide in the product of* $[-1,1]$ *and some small ball $B(0,\varepsilon)$ of $(\mathbb{R}^N)^{m(l)+1}$ which is the domain where we are going to use it. In any case, the previous property implies that the functions J_l^k are unique in the class of analytic functions.*

(4) *The paper has been written for a quite general PDE. However, it might certainly be possible to consider more general PDEs, containing for instance time derivatives in the lower order terms, or in the nonlinearity, or some time-dependent coefficients. We did not consider these cases because it would render the proof more technical and more difficult to read.*

The paper is organized as follows. In Section 2, we apply our main results to the Korteweg de-Vries equation, the Boussinesq equation, the Ginzbourg-Landau equation, and the Kuramoto-Sivashinsky equation. Section 3 is concerned with the existence and uniqueness results for the Cauchy problem in the x-variable (Theorem 3.1). The relationship between the jet of space derivatives and the jet of time derivatives at some point (jet analysis) for a solution of (1.1) is studied in Section 4. In particular, we show that the nonlinear equation (1.1) can be (locally) solved forward and backward if the initial data Y_0 can be extended as an analytic function in some ball of $\mathbb{C}$ (Proposition 4.11). Finally, the proofs of Theorem 1.2 and Proposition 1.3 are displayed in Section 5.

2. Examples

In this section, we list a few examples of equations coming from physical models for which our general result applies. The list is of course not exhaustive. Also, we limited ourselves to some models that contain a regularizing effect coming from a parabolic behavior or from smoothing boundary conditions. It is not that Theorem 1.2 is limited to this kind of problems, but for conservative equations (like nonlinear Schrödinger equations, KdV with some specific boundary conditions as in [31], [3] among other works), it is quite likely (and very often it has already been proved) that the controllability can be obtained in much lower regularity. Notice that even in this context, our result can be interesting if we are looking for a very regular control since the control we build is in some Gevrey class.

2.1. **The Korteweg-de Vries equation.** In this section, we are concerned with the controllability of the Korteweg-de Vries (KdV) equation:

$$\partial_t y = \partial_x^3 y + \partial_x y + y\partial_x y, \qquad x \in [0,1], \quad t \in [0,T], \tag{2.1}$$
$$y(1,t) = h(t), \qquad t \in [0,T], \tag{2.2}$$
$$y(0,t) = 0, \qquad t \in [0,T], \tag{2.3}$$
$$\partial_x y(0,t) = 0, \qquad t \in [0,T], \tag{2.4}$$
$$y(x,0) = y^0(x), \qquad x \in [0,1], \tag{2.5}$$

which adapts to our abstract setting (1.1)-(1.3) with $N = 1$, $M = 3$ (hence $\lambda = 3$), $P = \partial_x^3 + \partial_x$ and $f(x, y, \partial_x y, \partial_x^2 y) = y\partial_x y$. Thus $Y^t = y$ and $Y^x = (y, \partial_x y, \partial_x^2 y)$. Note that the change of variables $x \to 1 - x$ transforms (2.1) into the classical form of the KdV equation $\partial_t y + \partial_x^3 y + \partial_x y + y\partial_x y = 0$, and (2.2)-(2.4) into the boundary conditions $y(0,t) = h(t)$ and $y(1,t) = \partial_x y(1,t) = 0$.
It is well-known [10, 31] that system (2.1)-(2.5) is null controllable, and also controllable to the trajectories. Due to the smoothing effect, an exact controllability cannot hold in $L^2(0,1)$. The reachable space for the linearized KdV equation $\partial_t y = \partial_x^3 y + \partial_x y$ supplemented with the boundary conditions (2.2)-(2.4) was described in [22].
By Theorem 1.2, for any $T > 0$ and any $R > \hat{R} := 12e^{(3e)^{-1}}$, there is some number $\hat{C} > 0$ such that for any $y^0, \widetilde{y}^0 \in \mathcal{R}_{R,\hat{C}} \cap \mathcal{C}$, there exists a solution $y \in G^{1,3}([0,1] \times [0,T])$ of (2.1)-(2.5) satisfying $y(x,T) = \widetilde{y}^0(x)$ for all $x \in [0,1]$. Let us now describe more precisely the set $\mathcal{C}$ defined in (1.17). Denote $J_l = (J_{l,1}, J_{l,2}, J_{l,3})$. Recall that $\mathcal{C}$ is given by the conditions $BJ_l(x, y_0, \partial_x y_0, ..., \partial_x^{m(l)} y_0)\Big|_{x=0} = 0$ for all $l \geq 0$, where $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. The following Lemma provides a more precise version of Lemma 1.1 adapted to KdV.

Lemma 2.1. *For any $l \in \mathbb{N}$, $m(l) = 3l + 2$ and there exists a smooth map $H_l : \mathbb{R}^{3l-1} \to \mathbb{R}$ such that*

$$J_{l,1} = y_{3l} + H_l(y_0, y_1, ..., y_{3l-2}), \tag{2.6}$$
$$J_{l,2} = y_{3l+1} + \sum_{i=0}^{3l-2} \frac{\partial H_l}{\partial y_i}(y_0, y_1, ..., y_{3l-2}) y_{i+1}, \tag{2.7}$$
$$J_{l,3} = y_{3l+2} + \sum_{i=0}^{3l-2} \frac{\partial H_l}{\partial y_i}(y_0, y_1, ..., y_{3l-2}) y_{i+2}, + \sum_{i,j=0}^{3l-2} \frac{\partial^2 H_l}{\partial y_j \partial y_i}(y_0, y_1, ..., y_{3l-2}) y_{j+1} y_{i+1}. \tag{2.8}$$

Proof. Clearly $J_{0,1} = y_0$, $J_{0,2} = y_1$, $J_{0,3} = y_2$, so that $m(0) = 2$ and $H_0 = 0$. From (2.1), we infer that

$$\begin{aligned} \partial_t \partial_x y &= \partial_x^4 y + \partial_x^2 y + y\partial_x^2 y + (\partial_x y)^2, \\ \partial_t \partial_x^2 y &= \partial_x^5 y + \partial_x^3 y + y\partial_x^3 y + 3\partial_x y \partial_x^2 y. \end{aligned}$$

Therefore $m(1) = 5$ with

$$J_{1,1} = y_3 + y_1 + y_0 y_1, \quad J_{1,2} = y_4 + y_2 + y_0 y_2 + y_1^2, \quad J_{1,3} = y_5 + y_3 + y_0 y_3 + 3y_1 y_2.$$

Thus $H_1(y_0, y_1) = y_1 + y_0 y_1$. Assume now that $m(l) = 3l + 2$ and that (2.6)-(2.8) hold. Then

$$\begin{aligned} \partial_t^{l+1} y &= \partial_t J_{l,1}(x, y, \partial_x y, ..., \partial_x^{3l+2} y) \\ &= \partial_t\big(\partial_x^{3l} y + H_l(y, \partial_x y, ..., \partial_x^{3l-2} y)\big) \\ &= \partial_t \partial_x^{3l} y + \sum_{i=0}^{3l-2} \frac{\partial H_l}{\partial y_i}(y, \partial_x y, ..., \partial_x^{3l-2} y) \partial_t \partial_x^i y. \end{aligned}$$

Since

$$\partial_x^i (y \partial_x y) = \partial_x^{i+1}(\frac{y^2}{2}) = \frac{1}{2} \sum_{k=0}^{i+1} \left(\begin{array}{c} i+1 \\ k \end{array} \right) \partial_x^k y \partial_x^{i+1-k} y,$$

we obtain

$$\begin{aligned} \partial_t^{l+1} y &= \partial_x^{3l+3} y + \Bigg(\partial_x^{3l+1} y + \frac{1}{2} \sum_{k=0}^{3l+1} \left(\begin{array}{c} 3l+1 \\ k \end{array} \right) \partial_x^k y \partial_x^{3l+1-k} y \\ &\quad + \sum_{i=0}^{3l-2} \frac{\partial H_l}{\partial y_i}(y, \partial_x y, ..., \partial_x^{3l-2} y)\big(\partial_x^{i+3} y + \partial_x^{i+1} y + \frac{1}{2} \sum_{k=0}^{i+1} \left(\begin{array}{c} i+1 \\ k \end{array} \right) \partial_x^k y \partial_x^{i+1-k} y\big) \Bigg). \end{aligned} \tag{2.9}$$

It follows that $J_{l+1,1} = y_{3l+3} + H_{l+1}(y_0, y_1, ..., y_{3l+1})$ with

$$\begin{aligned} H_{l+1} &:= y_{3l+1} + \frac{1}{2} \sum_{k=0}^{3l+1} \left(\begin{array}{c} 3l+1 \\ k \end{array} \right) y_k y_{3l+1-k} \\ &\quad + \sum_{i=0}^{3l-2} \frac{\partial H_l}{\partial y_i}(y_0, y_1, ..., y_{3l-2})\big(y_{i+3} + y_{i+1} + \frac{1}{2} \sum_{k=0}^{i+1} \left(\begin{array}{c} i+1 \\ k \end{array} \right) y_k y_{i+1-k}\big). \end{aligned}$$

Thus (2.6) holds at the step $l+1$. Taking the derivative in x in (2.9) gives (2.7) and (2.8) at the rank $l+1$. Finally $m(l+1) = 3l + 5$. □

Thus $\mathcal{C}$ is the set of the functions $y_0 \in C^\infty([0,1])$ such that $J_{l,1} = J_{l,2} = 0$ for all $l \geq 0$, i.e.

$$\begin{aligned} &y(0) = \partial_x y(0) = 0, \\ &\partial_x^{3l} y(0) = -H_l(y, \partial_x y, ..., \partial_x^{3l-2} y)\Big|_{x=0}, \quad \forall l \in \mathbb{N}^*, \\ &\partial_x^{3l+1} y(0) = -\left(\sum_{i=0}^{3l-2} \frac{\partial H_l}{\partial y_i}(y, \partial_x y, ..., \partial_x^{3l-2} y) \partial_x^{i+1} y \right) \Big|_{x=0} \quad \forall l \in \mathbb{N}^*. \end{aligned}$$

Writing $y_0(x) = \sum_{n=0}^\infty \alpha_n \frac{x^n}{n!}$, we obtain the following conditions for the coefficients α_n:

$$\alpha_0 = \alpha_1 = 0, \tag{2.10}$$

$$\alpha_{3l} = -H_l(\alpha_0, \alpha_1, ..., \alpha_{3l-2}), \quad \forall l \in \mathbb{N}^*, \tag{2.11}$$

$$\alpha_{3l+1} = -\sum_{i=0}^{3l-2} \frac{\partial H_l}{\partial y_i}(\alpha_0, \alpha_1, ..., \alpha_{3l-2}) \alpha_{i+1}, \quad \forall l \in \mathbb{N}^*. \tag{2.12}$$

We conclude that

$$\mathcal{R}_{R,\hat{C}} \cap \mathcal{C} = \Big\{ z : [-1,1] \to \mathbb{C} : \ \exists (\alpha_n)_{n\geq 0} \in \mathcal{N}_{R,\hat{C}} \text{ such that (2.10)-(2.12) hold and } z(x) = \sum_{n=0}^{\infty} \alpha_n \frac{x^n}{n!}, \ \forall x \in [-1,1] \Big\}.$$

Remark 2. *The condition*

$$|\alpha_n| \leq \hat{C} \frac{n!}{R^n} \tag{2.13}$$

has to be satisfied for all $n \in \mathbb{N}$. It is likely (but still to be proved) that if (2.13) *is satisfied for the subsequence $(\alpha_{3l+2})_{l\geq 0}$, eventually for a small constant $\hat{C}$, it is also satisfied for the whole sequence $(\alpha_n)_{n\geq 0}$ (the two other subsequences $(\alpha_{3l})_{l\geq 0}$ and $(\alpha_{3l+1})_{l\geq 0}$ being defined due to* (2.10)-(2.12)*). If it is indeed the case, then the coefficients α_{3l+2} ($l \in \mathbb{N}$) can be chosen "freely" provided that they satisfy* (2.13)*, and hence the set $\mathcal{R}_{R,\hat{C}} \cap \mathcal{C}$ looks like a nonlinear submanifold.*

Theorem 2.2. *Let $R > \hat{R} := 12e^{(3e)^{-1}}$ and $T > 0$. Then there exists some number $\hat{C} > 0$ such that for all functions y^0, $\widetilde{y}^0 \in \mathcal{R}_{R,\hat{C}} \cap \mathcal{C}$, there exist functions $y \in G^{1,3}([0,1] \times [0,T])$ and $h \in G^3([0,T])$ satisfying* (2.1)-(2.5) *together with $y(x,T) = \widetilde{y}^0(x)$ for all $x \in [0,1]$.*

2.2. **Boussinesq equation.** We consider the issue of the exact controllability of two systems involving the (good or bad) Boussinesq equation.

2.2.1. *Neumann boundary conditions.* We first consider the system

$$\begin{cases} \partial_t^2 y = \pm \partial_x^4 y + \partial_x^2 y - \partial_x^2 (y^2), & x \in [0,1], \quad t \in [0,T], \\ \partial_x y(0,t) = 0, & t \in [0,T], \\ \partial_x y(1,t) = v(t), & t \in [0,T], \\ \partial_x^3 y(0,t) = 0, & t \in [0,T], \\ \partial_x^3 y(1,t) = w(t), & t \in [0,T], \\ y(x,0) = y^0(x), & x \in [0,1], \\ y_t(x,0) = y^1(x), & x \in [0,1]. \end{cases} \tag{2.14}$$

If the sign in $\pm$ is $+$, the first equation in (2.14) is called the *bad Boussinesq equation* which is known to be severely ill-posed, even for the linear part. It would therefore be difficult to obtain any controllability result with the standard methods. We shall obtain the following exact controllability result.

Theorem 2.3. *Let $R > \hat{R} := 16e^{(2e)^{-1}}$ and $T > 0$. Then there exists some number $\hat{C} > 0$ such that for all pairs of functions (y^0, y^1), $(\widetilde{y}^0, \widetilde{y}^1) \in (\mathcal{R}_{R,\hat{C}})^2$ which are even with respect to* 0*, there exist functions $y \in G^{1,2}([0,1] \times [0,T])$ and $v, w \in G^2([0,T])$ satisfying* (2.14) *together with $y(x,T) = \widetilde{y}^0(x)$ and $y_t(x,T) = \widetilde{y}^1(x)$ for all $x \in [0,1]$.*

Proof of Theorem 2.3. We apply Theorem 1.2 together with Proposition 1.3 with $\lambda = 4/2 = 2$. Note that the control inputs v, w are just taken as traces of the constructed solution $y \in G^{1,2}([0,1] \times [0,T])$. The regularity of v, w then follows from (1.15).

We need to check that the non-linearity satisfies the right assumption. Since $\partial_x^2(y^2) = 2\left(y\partial_x^2 y + (\partial_x y)^2\right)$, the non-linearity reads $f(x, y_0, y_1, y_2, y_3) = -2(y_0 y_2 + y_1^2)$. As

$$f(-x, y_0, -y_1, y_2, -y_3) = -2(y_0 y_2 + (-y_1)^2) = f(x, y_0, y_1, y_2, y_3),$$

we see that condition (1.19) in Proposition 1.3 is fulfilled. Finally, we notice that for any function $h \in \mathcal{R}_{R,\hat{C}}$, h is even if and only if $\partial_x^{2j+1} h(0) = 0$ for any $j \in \mathbb{N}$. □

2.2.2. *Dirichlet boundary conditions.* If we keep the non-linearity $f(x, y, \partial_x y, \partial_x^2 y, \partial_x^3 y) = -\partial_x^2(y^2)$, then

$$f(-x, -y_0, y_1, -y_2, y_3) = -2(y_0 y_2 + y_1^2) = f(x, y_0, y_1, y_2, y_3),$$

so that condition (1.18) in Proposition 1.3 is not fulfilled. Theorem 1.2 may be applied, but the determination of the compatibility set $\mathcal{C}$ is not obvious.
We consider instead a different non-linearity, namely $f(x, y, \partial_x y, \partial_x^2 y, \partial_x^3 y) = -\partial_x(y^2)$. More precisely, we consider the system

$$\begin{cases} \partial_t^2 y &= \pm\partial_x^4 y + \partial_x^2 y - \partial_x(y^2), & x \in [0,1], \quad t \in [0,T], \\ y(0,t) &= 0, & t \in [0,T], \\ y(1,t) &= v(t), & t \in [0,T], \\ \partial_x^2 y(0,t) &= 0, & t \in [0,T], \\ \partial_x^2 y(1,t) &= w(t), & t \in [0,T], \\ y(x,0) &= y^0(x), & x \in [0,1], \\ y_t(x,0) &= y^1(x), & x \in [0,1]. \end{cases} \tag{2.15}$$

Theorem 2.4. *Let $R > \hat{R} := 16 e^{(2e)^{-1}}$ and $T > 0$. Then there exists some number $\hat{C} > 0$ such that for all pairs of functions (y^0, y^1), $(\widetilde{y}^0, \widetilde{y}^1) \in (\mathcal{R}_{R,\hat{C}})^2$ which are odd with respect to 0, there exist functions $y \in G^{1,2}([0,1]\times[0,T])$ and $v, w \in G^2([0,T])$ satisfying (2.15) together with $y(x,T) = \widetilde{y}^0(x)$ and $y_t(x,T) = \widetilde{y}^1(x)$ for all $x \in [0,1]$.*

Proof of Theorem 2.4. The proof is the same as for Theorem 2.3. Since $\partial_x(y^2) = 2y\partial_x y$, the non-linearity reads $f(x, y_0, y_1, y_2, y_3) = -2y_0 y_1$. From

$$f(-x, -y_0, y_1, -y_2, y_3) = 2y_0 y_1 = -f(x, y_0, y_1, y_2, y_3),$$

we infer that condition (1.18) in Proposition 1.3 is fulfilled. As a function $h \in \mathcal{R}_{R,\hat{C}}$ is odd if and only if $\partial_x^{2j} h(0) = 0$ for any $j \in \mathbb{N}$, the conclusion follows at once. □

2.3. **The complex Ginzburg-Landau equation.** We are concerned with the controllability of the complex Ginzburg-Landau equation with parameters $\theta, \varphi \in \mathbb{R}$. We begin with the control problem with Dirichlet boundary conditions:

$$\begin{cases} \partial_t y &= e^{i\theta}\partial_x^2 y + e^{i\varphi}|y|^2 y, & x \in [0,1], \quad t \in [0,T], \\ y(0,t) &= 0, & t \in [0,T], \\ y(1,t) &= v(t), & t \in [0,T], \\ y(x,0) &= y^0(x) & x \in [0,1]. \end{cases} \tag{2.16}$$

Theorem 2.5. *Let $R > \hat{R} := 8e^{(2e)^{-1}}$ and $T > 0$. Then there exists some number $\hat{C} > 0$ such that for all functions y^0, $\widetilde{y}^0 \in \mathcal{R}_{R,\hat{C}}$ which are odd with respect to 0, there exist $y \in G^{1,2}([0,1] \times [0,T])$ and $v \in G^2([0,T])$ satisfying (2.16) together with $y(x,T) = \widetilde{y}^0(x)$ for all $x \in [0,1]$.*

The control problem with Neumann boundary conditions reads

$$\begin{cases} \partial_t y &= e^{i\theta}\partial_x^2 y + e^{i\varphi}|y|^2 y, & x \in [0,1], \quad t \in [0,T], \\ \partial_x y(0,t) &= 0, & t \in [0,T], \\ \partial_x y(1,t) &= v(t), & t \in [0,T], \\ y(x,0) &= y^0(x) & x \in [0,1]. \end{cases} \tag{2.17}$$

Theorem 2.6. *Let $R > \hat{R} := 8e^{(2e)^{-1}}$ and $T > 0$. Then there exists some number $\hat{C} > 0$ such that for all functions y^0, $\widetilde{y}^0 \in \mathcal{R}_{R,\hat{C}}$ which are even with respect to 0, there exist $y \in G^{1,2}([0,1] \times [0,T])$ and $v \in G^2([0,T])$ satisfying (2.17) together with $y(x,T) = \widetilde{y}^0(x)$ for all $x \in [0,1]$.*

The proof follows the previous cases closely, except that they are complex-valued functions and the nonlinearity $|y|^2 y = y^2 \overline{y}$ cannot be written as a sum (finite or infinite) of powers of the variable y. We describe in Section 7 the modifications that must be performed to get the expected result.

Remark 3. *It might seem problematic to use the nonlinearity $f(z) = |z|^2 z$ which is not holomorphic. The solution we construct satisfies $y(\cdot,t) \in \mathcal{R}_{R,\hat{C}}$, which means that it is real analytic on $[-1,1]$ for any $t \in [0,T]$, in the sense that it agrees with its Taylor expansion at 0, which is enough for the proof we are doing. Indeed, as noticed in Lemma 6.1, it implies that it has a holomorphic extension $z \mapsto y(z,t)$ for $z \in B_{\mathbb{C}}(0,\widetilde{R})$ for some $\widetilde{R} > 0$. The application $x \in [-1,1] \mapsto |y(x,t)|^2 y(x,t)$ is also real analytic and also has a holomorphic extension. Yet, this extension does not coincide with $|y(z,t)|^2 y(z,t)$. In particular, the solution can be extended to $B_{\mathbb{C}}(0,\widetilde{R}) \times [0,T]$ but it is not clear what equation it satisfies on this set. We only know that the solution satisfies the Ginzburg-Landau equation on $[-1,1] \times [0,T]$.*

2.4. **The Kuramoto-Sivashinsky equation.** We investigate the controllability of the Kuramoto-Sivashinsky (KS) equation with boundary conditions of Dirichlet type:

$$\begin{cases} \partial_t y &= -\partial_x^4 y - \partial_x^2 y - y\partial_x y, & x \in [0,1], \quad t \in [0,T], \\ y(0,t) &= 0, & t \in [0,T], \\ y(1,t) &= v(t), & t \in [0,T], \\ \partial_x^2 y(0,t) &= 0, & t \in [0,T], \\ \partial_x^2 y(1,t) &= w(t), & t \in [0,T], \\ y(x,0) &= y^0(x), & x \in [0,1]. \end{cases} \tag{2.18}$$

Theorem 2.7. *Let $R > \hat{R} = 16e^{(4e)^{-1}}$ and $T > 0$. Then there exists some number $\hat{C} > 0$ such that for all functions y^0, $\widetilde{y}^0 \in \mathcal{R}_{R,\hat{C}}$ which are odd with respect to 0, there exist functions $y \in G^{1,4}([0,1] \times [0,T])$ and $v, w \in G^4([0,T])$ satisfying (2.18) together with $y(x,T) = \widetilde{y}^0(x)$ for all $x \in [0,1]$.*

Proof of Theorem 2.7. For $\lambda = 4/1 = 4$, and the non-linearity reads as $f(x, y_0, y_1, y_2, y_3) = -y_0 y_1$. It satisfies

$$f(-x, -y_0, y_1, -y_2, y_3) = y_0 y_1 = -f(x, y_0, y_1, y_2, y_3),$$

which is condition (1.19) in Proposition 1.3. □

The null controllability for the Kuramoto-Sivashinsky equation has been already studied in [1, 2, 12, 18], for different combinations of boundary data, and in the cases where boundary setting agrees with the setting of (1.2), our results are consistent with the known results. However, the critical set of parameters of diffusion appears only in cases when only one control is considered, which is not the case in this paper.

2.5. **The case of a linear PDE with constant coefficients.**

Proposition 2.8. *Assume $f = 0$ (linear PDE with constant coefficients). Then*

$$\mathcal{C} = \Big\{ Y_0 = (y_0, y_1, ..., y_{N-1}) \in C^\infty([0,1])^N; \quad (BP^k Y^{x,l})(0) = 0,\ \forall k \in \mathbb{N},\ \forall l = 0, \dots, N-1 \Big\}$$

where we have denoted $Y^{x,l} := (y_l, \dots, \partial_x^{M-1} y_l)$ as in (1.4) *.*

Proof. Using Euclidian division, we are led to compute the application J_{Nk+l} defined in Lemma (1.1) for any $k \in \mathbb{N}$ and $l = 0, \dots, N-1$.
We infer from (1.1) that $\partial_t^{Nk+l} \partial_x^i y = P^k \partial_t^l \partial_x^i y$ for any $k \in \mathbb{N}$, $l = 0, ..., N-1$ and $i \in \mathbb{N}$. In particular, $\partial_t^{Nk+l} Y^x = P^k \partial_t^l Y^x$ for any $k \in \mathbb{N}$, $l = 0, ..., N-1$, we can define a linear map $J_{Nk+l} : (\mathbb{R}^N)^{(k+1)M} \to \mathbb{R}^M$ such that

$$J_{Nk+l}(Y_0(0), \cdots, \partial_x^{(k+1)M-1} Y_0(0)) = (P^k y_l, P^k \partial_x y_l, \dots, P^k \partial_x^{M-1} y_l)(0),$$

for any $Y_0 = (y_0, y_1, ..., y_{N-1}) \in C^\infty([0,1])^N$ (denoting $y_0 = y$, $y_l = \partial_t^l y_0$ for $1 \le l \le N-1$). Moreover, for a solution of the equation with initial datum Y_0, we have

$$(P^k y_l, P^k \partial_x y_l, \dots, P^k \partial_x^{M-1} y_l)(0) = P^k Y^{x,l}(0). \tag{2.19}$$

The previous computation gives $\partial_t^{Nk+l} Y^x(0) = J_{Nk+l}(Y_0(0), \cdots, \partial_x^{(k+1)M-1} Y_0(0))$. Therefore, the application J_{Nk+l} by (2.19) satisfies the property (1.1) for all solutions. Then, using the uniqueness of the operators J_l (up to adding unnecessary variables) proved in Lemma 4.17, we conclude that it is the expected application.

In particular, $BJ_{Nk+l}(Y_0(0), \cdots, \partial_x^{(k+1)M-1} Y_0(0)) = 0$ is equivalent to $BP^k Y^{x,l}(0) = 0$ for any $k \in \mathbb{N}$ and $l = 0, \dots, N-1$. □

3. Cauchy problem in the space variable

3.1. **Statement of the global wellposedness result.**

Let $f = f(x, y_0, y_1, \cdots, y_{M-1})$ be as in (1.7)-(1.10). We are concerned with the wellposedness of the Cauchy problem:

$$\begin{cases} \partial_t^N y = Py + f(x, y, ..., \partial_x^{M-1} y), & x \in [-1,1],\ t \in [t_1, t_2], \\ \partial_x^i y(0,t) = k_i(t), & 0 \le i \le M-1, \quad t \in [t_1, t_2] \end{cases} \tag{3.1}$$

for some given functions $k_0, \dots, k_{M-1} \in G^\lambda([t_1,t_2])$. We denote $K_0 = (k_0, ..., k_{M-1})$. Note that the initial conditions of (3.1) can be written as $Y^x(0,t) = K_0(t)$.

The goal of this section is to prove the following result.

Theorem 3.1. *Let P be as in* (1.6) *and $f = f(x, \vec{y})$ be as in* (1.7)-(1.10). *Let $-\infty < t_1 < t_2 < +\infty$ and $R > (4N)^\lambda$, where $\lambda = M/N$. Then there exists some numbers $C > 0$, $Q > 0$, R_1, R_2 with $4Me^{-1/M} < R_1 < R_2$ satisfying that for all $K_0 = (k_0, \dots, k_{M-1}) \in G^\lambda([t_1,t_2])^M$ with*

$$|k_i^{(n)}(t)| \le C \left(\frac{|\zeta_M|^{1/N}}{R} \right)^n (n!)^\lambda, \quad i = 0, 1, \dots, M-1, \ n \ge 0, \ t \in [t_1, t_2], \tag{3.2}$$

there exists a solution $y \in G^{1,\lambda}([-1,1] \times [t_1,t_2])$ of (3.1) *satisfying,*

$$|\partial_x^{p_1} \partial_t^{p_2} y(x,t)| \le Q \frac{(p_1 + \lambda p_2)!}{R_1^{p_1} R_2^{\lambda p_2}} |\zeta_M|^{p_2/N}, \quad (x,t) \in [-1,1] \times [t_1,t_2], \ (p_1,p_2) \in \mathbb{N}^2. \tag{3.3}$$

The proof of Theorem 3.1 will be given after that some preliminary results are established. We use the notation $x! = \Gamma(x+1)$ even if x is not an integer.

Remark 4. *It is sufficient to prove Theorem 3.1 for the unidimensional system* (3.1), *i.e Considering $|\zeta_M| = 1$ and $[t_1,t_2] = [0,t_2]$. Indeed, the equation $\partial_t^N y = Py + f(x, y, \dots, \partial_x^{M-1} y)$ is invariant by translation in time, so that we can assume that $[t_1,t_2] = [0,t_2]$. On the other hand, if $|\zeta_M| \in (0,+\infty) \setminus \{1\}$, we can use the following scaling argument. Set $\widetilde{\zeta}_M := \zeta_M / |\zeta_M|$, $\widetilde{P} := |\zeta_M|^{-1} P$ and $\widetilde{f} := |\zeta_M|^{-1} f$. Note that $\widetilde{P}$ and $\widetilde{f}$ satisfy the expected assumptions with $|\widetilde{\zeta}_M| = 1$. For K_0 satisfying* (3.2) *on $[0,t_2]$, define $\widetilde{K}_0(t) := K_0(|\zeta_M|^{-1/N} t)$. Then $\widetilde{K}_0$ satisfies* (3.2) *with $|\widetilde{\zeta}_M| = 1$, that is $|\widetilde{k}_i^{(n)}(t)| \le C \frac{(n!)^\lambda}{R^n}$ on $[0, |\zeta_M|^{1/N} t_2]$. This allows to define a solution $\widetilde{y}(x,t)$ of* (3.1) *for $x \in [-1,1]$ and $t \in [0, |\zeta_M|^{-1/N} t_2]$ associated with $\widetilde{P}$, $\widetilde{f}$ and $\widetilde{K}_0$. Then the function*

$$y(x,t) := \widetilde{y}(x, |\zeta_M|^{\frac{1}{N}} t), \qquad x \in [-1,1], \ t \in [0,t_2]$$

is a solution of (3.1) *associated with P, f and K_0.*

3.2. **Abstract existence theorem.** We consider a family of Banach spaces $(X_s)_{s \in [0,1]}$ satisfying for $0 \le s' \le s \le 1$,

$$X_s \subset X_{s'}, \tag{3.4}$$

$$\|f\|_{X_{s'}} \le \|f\|_{X_s}; \tag{3.5}$$

that is, the embedding $X_s \subset X_{s'}$ for $s' \le s$.

We are concerned with an abstract Cauchy problem:

$$\begin{cases} \partial_x U(x) = \mathrm{T}(x) U(x), & -1 \le x \le 1, \\ U(0) = U^0 \end{cases}$$

where $U^0 \in X_1$ and $\big(\mathrm{T}(x)\big)_{x \in [-1,1]}$ is a family of nonlinear operators with possible *loss of derivatives*.

The following result, taken from [20, Theorem 2.2], is a global wellposedness result. It extends the abstract result in [28, 29] which gives only *local* solutions.

Theorem 3.2. *Let $\varepsilon \in (0,1/4)$, $D>0$ and a family $(\mathrm{T}(x))_{x\in[-1,1]}$ of nonlinear maps from X_s to $X_{s'}$ for $0\le s'<s\le 1$ satisfying*

$$\begin{aligned} \|\mathrm{T}(x)U\|_{X_{s'}} &\le \frac{\varepsilon}{s-s'}\|U\|_{X_s}, &(3.6)\\ \|\mathrm{T}(x)U-\mathrm{T}(x)V\|_{X_{s'}} &\le \frac{\varepsilon}{s-s'}\|U-V\|_{X_s} &(3.7)\end{aligned}$$

for $0\le s'<s\le 1$, $x\in[-1,1]$ and U,$V\in X_s$ with $\|U\|_{X_s}\le D$, $\|V\|_{X_s}\le D$. Then there exists a number $0<\eta\le D$ such that for any $U^0\in X_1$ with $\left\|U^0\right\|_{X_1}\le\eta$, there exists a solution $U\in C([-1,1],X_{s_0})$ for some $s_0\in(0,1)$ of the integral equation

$$U(x)=U^0+\int_0^x \mathrm{T}(\tau)U(\tau)d\tau. \tag{3.8}$$

Moreover, we have the estimate

$$\|U(x)\|_{X_s}\le C_1\left(1-\frac{\alpha|x|}{a_\infty(1-s)}\right)^{-1}\left\|U^0\right\|_{X_1},\quad \text{for } 0\le s<1,\ |x|<\frac{a_\infty}{\alpha}(1-s),$$

where $\alpha\in(0,1)$, $a_\infty\in(\alpha,1)$ and $C_1>0$ are some constants. In particular, we have

$$\|U(x)\|_{X_s}\le C_1\left(1-\frac{2}{\frac{a_\infty}{\alpha}+1}\right)^{-1}\left\|U^0\right\|_{X_1},\quad \text{for } 0\le s\le s_0=\frac12\left(1-\frac{\alpha}{a_\infty}\right),\ |x|\le 1.$$

If, in addition, we assume that

$$\text{for all } U_0\in X_s \text{ with } \|U_0\|_{X_s}\le D,\ \text{the map } \tau\in[-1,1]\to \mathrm{T}(\tau)U_0\in X_{s'} \text{ is continuous}, \tag{3.9}$$

then U is the classical solution of

$$\begin{cases} \partial_x U(x) = \mathrm{T}(x)U(x), & -1\le x\le 1,\\ U(0)=U^0. \end{cases} \tag{3.10}$$

Note that we slightly changed the order of the quantifiers for D to the original statement in [20, Theorem 2.2]. The result is a direct consequence of [20, Proposition 2.3.] where the quantifiers are written this way.

3.3. **Gevrey type functional spaces.** We define several λ Gevrey spaces for $\lambda>1$ (see [16, 35]) and we follow closely the ideas developed in [20] for the heat equation. We shall take $\lambda=M/N$, but for the moment we stay in the generality.

We introduce a variant of the Gamma function of Euler with a parameter $a\in\mathbb{R}$ given by

$$\Gamma_{\lambda,a}(k)=\begin{cases}2^{-5}(\Gamma(k+1-a))^\lambda(1+k)^{-2}, & k\in\mathbb{N},\ k>|a|+1,\\ \Gamma_\lambda(k), & k\in\mathbb{N},\ 0\le k\le |a|+1\end{cases} \tag{3.11}$$

with

$$\Gamma_\lambda(k)=2^{-5}(k!)^\lambda(1+k)^{-2}, \tag{3.12}$$

and Γ being the usual Gamma function of Euler which is increasing on $[2,+\infty)$.

Clearly, $\Gamma_{\lambda,0} = \Gamma_\lambda$. Note that for $k > |a| + 1$, we have $k + 1 - a \geq 2$ and $k + 1 \geq 2$, so we are in an interval where Γ is increasing. Thus we have for all $k \in \mathbb{N}$

$$\Gamma_{\lambda,a}(k) \leq \Gamma_\lambda(k), \quad \text{if } a \geq 0, \tag{3.13}$$

$$\Gamma_\lambda(k) \leq \Gamma_{\lambda,a}(k), \quad \text{if } a \leq 0. \tag{3.14}$$

For any $L > 0$, we consider the intermediate space of functions in $C^\infty(K)$ (where $K = [t_1, t_2]$ with $-\infty < t_1 < t_2 < \infty$) such that

$$|u|_{L,a} := \sup_{t \in K, k \in \mathbb{N}} \frac{\left|u^{(k)}(t)\right|}{L^{|k-a|}\Gamma_{\lambda,a}(k)} < \infty.$$

Note that for $a = 0$, we recover the spaces defined earlier in [35], and $|u|_{L,0} = |u|_L$.

Definition 2. We consider the norm defined in [35] by Yamanaka

$$\|u\|_L := \max\left\{2^6 \|u\|_{L^\infty(K)}, 2^3 L^{-1} \left|u'\right|_L\right\},$$

and similarly, we define for $a \in \mathbb{R}$

$$\|u\|_{L,a} := \max\left\{2^6 \|u\|_{L^\infty(K)}, 2^3 L^{-1} \left|u'\right|_{L,a}\right\}.$$

For $L > 0$ and $0 \leq a_1 < a_2$, we have

$$\|u\|_{L,a_1} \leq C(L, a_1, a_2, \lambda) \|u\|_{L,a_2} \quad \forall u \in G^\lambda_{L,a_2}. \tag{3.15}$$

Indeed, for $k > a_2 + 1$, we have $k + 1 - a_1 \geq k + 1 - a_2 \geq 2$ where Γ is increasing so that $\Gamma(k + 1 - a_2) \leq \Gamma(k + 1 - a_1)$, and therefore $L^{|k-a_2|}\Gamma_{\lambda,a_2}(k) \leq L^{a_1-a_2} L^{|k-a_1|}\Gamma_{\lambda,a_1}(k)$. We can obtain a similar inequality for $k \leq a_2 + 1$ with different constant which gives then (3.15).

We define the Banach spaces $G^\lambda_{L,a}$ and G^λ_L as

$$G^\lambda_{L,a} := \{u \in C^\infty(K) \quad \text{such that} \quad \|u\|_{L,a} < \infty\} \tag{3.16}$$

and

$$G^\lambda_L := \{u \in C^\infty(K) \quad \text{such that} \quad \|u\|_L < \infty\}. \tag{3.17}$$

The space $G^\lambda_{L,a}$ can be seen as the space of functions Gevrey λ with radius L^{-1} with a derivatives. Roughly, we could think that $u \in G^\lambda_{L,a}$ if $D^a u \in G^\lambda_L$, even if it is not completely true if $a \notin \mathbb{N}$. Note that, as a direct consequence of (3.13)-(3.14), we have the embeddings $G^\lambda_{L,a} \subset G^\lambda_L$ if $a \geq 0$ and $G^\lambda_L \subset G^\lambda_{L,a}$ if $a \leq 0$, together with the inequalities

$$\|u\|_L \leq \max(L^a, L^{-a}) \|u\|_{L,a}, \text{ if } a \geq 0, \tag{3.18}$$

$$\|u\|_{L,a} \leq \max(L^a, L^{-a}) \|u\|_L, \text{ if } a \leq 0. \tag{3.19}$$

Furthermore, for any $a \in \mathbb{R}$ and $0 < L < L'$, we have the embedding $G^\lambda_{L,a} \subset G^\lambda_{L',a}$ with

$$\|u\|_{L',a} \leq \|u\|_{L,a}. \tag{3.20}$$

The following result [36, Theorem 5.4] will be used several times in the sequel.

Lemma 3.3. *(Algebra property) For $L>0$*

$$\|uv\|_L \leq \|u\|_L \|v\|_L \quad \forall u,v \in G^\lambda_L. \tag{3.21}$$

The following result [20, Lemma 2.6] is a variant of [16, Proposition 2.3] with spaces containing non-integer "derivatives".

Lemma 3.4 (Cost of derivatives for Gevrey spaces containing derivatives)**.** *Let $\lambda>0$ and $\delta>0$. Let $q\in\mathbb{N}$ and $a,b\in\mathbb{R}$ with $d=q-a+b>0$. Then there exists some number $C=C(\lambda,\delta,a,b,q)>0$ such that for all $L>0$, $\alpha>1$ and $u\in G^\lambda_{L,a}$, we have*

$$\left|u^{(q)}\right|_{\alpha L,b} \leq \left(C(L^{-d}+L^d)+(1+\delta)\alpha^b L^d\left(\frac{\lambda d}{e\ln\alpha}\right)^{\lambda d}\right)|u|_{L,a} \tag{3.22}$$

and hence

$$\left\|u^{(q)}\right\|_{\alpha L,b} \leq \left(C(L^{-d}+\langle L\rangle^C)+(1+\delta)\alpha^b L^d\left(\frac{\lambda d}{e\ln\alpha}\right)^{\lambda d}\right)\|u\|_{L,a}. \tag{3.23}$$

where we denote $\langle x\rangle := \sqrt{1+x^2}$ for $x\in\mathbb{R}$.

3.4. **Application to the semi-linear PDE.** We write our system in the equivalent form

$$\begin{cases}\partial_x^M u = \frac{1}{\zeta_M}\left(\partial_t^N u - \sum_{j=0}^{M-1}\zeta_j\partial_x^j u - f(x,u,\partial_x u,...,\partial_x^{M-1}u)\right), & x\in[-1,1],\ t\in[t_1,t_2],\\ U^x(0,t)=K_0(t), & t\in[t_1,t_2],\end{cases} \tag{3.24}$$

recalling $U^x(x,t)=\big(u(x,t),\partial_x u(x,t),...,\partial_x^{M-1}u(x,t)\big)$ and $K_0:=(k_0(t),\dots,k_{M-1}(t))$. $|\zeta_M|=1$ will be considered in this section, for more detailed see Remark 4.

We write (3.24) as a first-order system

$$\partial_x U = AU + F(x,U), \tag{3.25}$$
$$U(0) = K_0 \tag{3.26}$$

with $U=U^x=(u,\partial_x u,\cdots,\partial_x^{M-1}u)$,

$$A=\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & 1\\ \zeta_M^{-1}(\partial_t^N-\zeta_0) & -\zeta_M^{-1}\zeta_1 & \cdots & -\zeta_M^{-1}\zeta_{M-2} & -\zeta_M^{-1}\zeta_{M-1}\end{pmatrix}, \tag{3.27}$$

and

$$F(x,\vec{u}) = \begin{pmatrix} 0\\ \vdots\\ -\zeta_M^{-1}f(x,\vec{u})\end{pmatrix},$$

where the current vector $\vec{u} := (u_0, u_1, \dots, u_{M-1})$ will contain the derivatives. We decompose A as

$$\begin{aligned} A &= A_0 + A_R \\ &= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \zeta_M^{-1}\partial_t^N & 0 & \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ -\zeta_M^{-1}\zeta_0 & -\zeta_M^{-1}\zeta_1 & \cdots & -\zeta_M^{-1}\zeta_{M-2} & -\zeta_M^{-1}\zeta_{M-1} \end{pmatrix}. \end{aligned}$$

Let $L > 0$, we define the space

$$\mathcal{X}_L := \{U = (u_0, u_1, \dots, u_{M-1}) \in G^\lambda_{L, \frac{M-1}{\lambda}} \times \dots \times G^\lambda_{L, \frac{1}{\lambda}} \times G^\lambda_L\} \tag{3.28}$$

with the norm

$$\|U\|_{\mathcal{X}_L} = \|(u_0, u_1, \dots, u_{M-1})\|_{\mathcal{X}_L} = \|u_0\|_{L, \frac{M-1}{\lambda}} + \dots + \|u_{M-1}\|_L = \sum_{j=0}^{M-1} \|u_j\|_{L, \frac{M-j-1}{\lambda}},$$

where the norms are those defined in Definition 2 with $\lambda = M/N$. Note that u_0 is more regular than u_1 of "$1/\lambda$ derivative". In particular, using that $|\zeta_M| = 1$, we have that

$$\|A_0 U\|_{\mathcal{X}_L} = \sum_{j=1}^{M-1} \|u_j\|_{L, \frac{M-j}{\lambda}} + \left\|\partial_t^N u_0\right\|_L.$$

In the following result, L_1 stands for the inverse of the radius R of the initial datum.

Theorem 3.5. *Pick any L_1 with $0 < L_1 < \frac{1}{(4N)^\lambda}$. Then there exists a number $\eta > 0$ such that for any $K_0 \in \mathcal{X}_{L_1}$ with $\|K_0\|_{\mathcal{X}_{L_1}} \le \eta$, there exists a solution to (3.24)in $C([-1,1], \mathcal{X}_{L_0})$ for some $L_0 > 0$.*

Proof. In order to apply Theorem 3.2, we introduce a scale of Banach spaces $(X_s)_{s \in [0,1]}$ as follows, for $s \in [0,1]$, we set

$$\|U\|_{X_s} = e^{-\tau(1-s)} \|U\|_{\mathcal{X}_{L(s)}} \quad \text{for } U \in X_s := \mathcal{X}_{L(s)} \tag{3.29}$$

$$L(s) = e^{r(1-s)} L_1, \tag{3.30}$$

where

$$r = 1/N$$

and $\tau > 0$ will be chosen thereafter. Note that (3.5) is satisfied from (3.20) and the fact that $L(s') > L(s)$ for $s' < s$. Additionally, we have that

$$\|U\|_{X_{s'}} \le e^{-\tau(s-s')} \|U\|_{X_s}. \tag{3.31}$$

The use of Lemmas 3.6, 3.7 and 3.8 will allow us to select the parameters such that T $= A + F$ satisfies the assumptions of Theorem 3.2. Then, we only need to notice that $\|K_0\|_{X_1} = \|K_0\|_{\mathcal{X}_{L_1}} \le D$ for $\eta = D$ small. ☐

Remark 5. *It is interesting to notice that for Theorem 3.5, we use the analytic regularity of f in the variables $u_0, \dots, u_{M-1}$, but only the continuity of f in x. The analyticity of f in $x, u_0, \dots, u_{M-1}$ will be used to prove the additional regularity of the solution in the variable x. Also, as noticed in Section 7 concerning the Ginzburg-Landau equation, the same result holds for a polynomial function of $u_0, \overline{u_0} \dots, u_{M-1}, \overline{u_{M-1}}$. The crucial part for the existence is the composition of Gevrey functions.*

Lemma 3.6. *Let $L_1 < \frac{1}{(4N)^\lambda}$. There exist $\tau_0 > 0$ (large enough) and $\varepsilon_0 < 1/4$ such that we have the estimates*

$$\|A_0 U\|_{X_{s'}} \leq \frac{\varepsilon_0}{s - s'} \|U\|_{X_s}, \quad \forall U \in X_s,$$

for all $\tau \geq \tau_0$ (as in (3.29)) and all s, s' with $0 \leq s' < s \leq 1$.

Proof. By assumption, $NL_1^{1/\lambda} < 1/4$. Pick $\delta > 0$ small enough such that

$$(1+\delta) N L_1^{1/\lambda} < 1/4, \tag{3.32}$$

applying Lemma 3.4 to the $M-1$-first terms of $A_0 U$ (namely $u_1, ..., u_{M-1}$) for $\lambda = M/N$ and taking $q = 0$, $b = \frac{M-j}{\lambda}$ and $a = \frac{M-j-1}{\lambda}$, so that $d = \frac{1}{\lambda} > 0$, we obtain the existence of some number $C = C_\delta > 0$ such that for $j = 1, ..., M-1$

$$\|u_j\|_{\alpha L, \frac{M-j}{\lambda}} \leq \left(C(L^{-\frac{1}{\lambda}} + \langle L \rangle^C) + \frac{1+\delta}{e \ln \alpha} \alpha^{\frac{M-1}{\lambda}} L^{1/\lambda} \right) \|u_j\|_{L, \frac{M-j-1}{\lambda}}, \quad \text{for } \alpha > 1 \quad \text{and} \quad L > 0.$$

For the last term of $A_0 U$ (namely $\zeta_M^{-1} \partial_t^N u_0$) with $\lambda = M/N$ and $\delta > 0$, (3.32) is satisfied, and considering now $q = N$, $b = 0$, $a = \frac{M-1}{\lambda}$, so $d = \frac{1}{\lambda} > 0$, we obtain the existence of some number $C = C_\delta > 0$ such that

$$\left\|\partial_t^N u_0\right\|_{\alpha L} \leq \left(C(L^{-\frac{1}{\lambda}} + \langle L \rangle^C) + \frac{1+\delta}{e \ln \alpha} L^{1/\lambda} \right) \|u_0\|_{L, \frac{M-1}{\lambda}}.$$

It gives after summation

$$\|A_0 U\|_{\mathcal{X}_{\alpha L}} \leq \left(C(L^{-\frac{1}{\lambda}} + \langle L \rangle^C) + \frac{1+\delta}{e \ln \alpha} \alpha^{\frac{M-1}{\lambda}} L^{1/\lambda} \right) \|U\|_{\mathcal{X}_L}, \tag{3.33}$$

uniformly for $\alpha > 1$ and $L > 0$.

Therefore, from equation (3.29), (3.30), (3.31) and considering the estimate (3.33) with $L = L(s)$, $\alpha = \frac{L(s')}{L(s)} = e^{r(s-s')} > 1$ and $s' < s$. Hence, for $0 \leq s' < s \leq 1$,

$$\begin{aligned}
\|A_0 U\|_{X_{s'}} &\leq e^{-\tau(s-s')} \left(C(L_1^{-\frac{1}{\lambda}} + \langle e^r L_1 \rangle^C) + (1+\delta) \frac{e^{\frac{M-1}{\lambda} r(s-s')} e^{r\frac{1-s}{\lambda}} L_1^{1/\lambda}}{er(s-s')} \right) \|U\|_{X_s} \\
&\leq \left(Ce^{-\tau(s-s')} (L_1^{-\frac{1}{\lambda}} + e^{rC}) + (1+\delta) e^{rN} \frac{L_1^{1/\lambda}}{er(s-s')} \right) \|U\|_{X_s} \\
&\leq \left(\frac{e^{-1}}{\tau(s-s')} C(L_1^{-\frac{1}{\lambda}} + e^{rC}) + (1+\delta) \frac{e^{rN} L_1^{1/\lambda}}{er(s-s')} \right) \|U\|_{X_s}
\end{aligned} \tag{3.34}$$

where we have used $0 < s - s' \leq 1$, $0 < L_1 < 1/4$ and

$$e^{-\tau(s-s')} = \frac{\tau(s-s')e^{-\tau(s-s')}}{\tau(s-s')} \leq \frac{e^{-1}}{\tau(s-s')}, \tag{3.35}$$

since $te^{-t} \leq e^{-1}$ for $t \geq 0$. Minimizing the constant in the second term of the right hand side of (3.34) leads to the choice $r = 1/N$. (Note that the initial space $X_1 = \mathcal{X}_{L_1}$ is independent of the choice of r.) We arrive at the estimate

$$\|A_0 U\|_{X_{s'}} \leq \left(\frac{Ce^{-1}(L_1^{-\frac{1}{\lambda}} + e^{C/N})}{\tau} + (1+\delta)NL_1^{1/\lambda} \right) \frac{1}{s-s'} \|U\|_{X_s} .$$

By (3.32), selecting τ_0 large enough so that $\varepsilon_0 := \frac{Ce^{-1}(L_1^{-\frac{1}{\lambda}} + e^{C/N})}{\tau_0} + (1+\delta)NL_1^{1/\lambda} < 1/4$. This completes the proof of Lemma 3.6. □

Lemma 3.7. *Let $\varepsilon > 0$, $r = \frac{1}{N}$ and $L_1 > 0$. There exists $\tau_0 > 0$ such that we have the estimates*

$$\|A_R U\|_{X_{s'}} \leq \frac{\varepsilon}{s-s'} \|U\|_{X_s} \quad \forall U \in X_s,$$

for all $\tau \geq \tau_0$ and all s, s' with $0 \leq s' < s \leq 1$.

Proof. Using (3.18), we first get that there exists $C > 0$ (depending on all the previous constants L_1, M,...) such that for $L \in [L_1, e^r L_1]$, ($|\zeta_M| = 1$),

$$\|A_R U\|_{\mathcal{X}_L} = |\zeta_M|^{-1} \left\| \sum_{j=0}^{M-1} \zeta_j u_j \right\|_L \leq \sum_{j=0}^{M-1} \|\zeta_j u_j\|_L \leq C \sum_{j=0}^{M-1} \|u_j\|_{L, \frac{M-j-1}{\lambda}} = C \|U\|_{\mathcal{X}_L} . \tag{3.36}$$

Applying the previous estimate to $L = L(s')$ and using (3.29) and (3.35), we obtain

$$\|A_R U\|_{X_{s'}} = e^{-\tau(1-s')} \|A_R U\|_{\mathcal{X}_{L(s')}} \leq$$
$$Ce^{-\tau(1-s')} \|U\|_{\mathcal{X}_{L(s')}} = Ce^{-\tau(s-s')} \|U\|_{X_{s'}} \leq C \frac{e^{-1}}{\tau(s-s')} \|U\|_{X_s} .$$

It gives the result for τ_0 large enough. □

Lemma 3.8. *Let f be as in (1.7)-(1.10), and let $F(x,U) = \begin{pmatrix} 0 \\ -f(x, u_0, u_1, \ldots, u_{M-1}) \end{pmatrix}$ for $x \in [-1,1]$ and $U = (u_0, u_1, \ldots, u_{M-1}) \in L^\infty(K)^M$ with $\sup_{i=0,\ldots,M-1}(\|u_i\|_{L^\infty(K)}) < 4$. Let $r = 1/N$, $L_1 > 0$, and $\varepsilon > 0$. Then there exists $\tau_0 > 0$ (large enough) such that for any $\tau \geq \tau_0$, there exists $D > 0$ (small enough) such that we have the estimates*

$$\|F(x,U)\|_{X_{s'}} \leq \frac{\varepsilon}{s-s'} \|U\|_{X_s} , \tag{3.37}$$

$$\|F(x,U) - F(x,V)\|_{X_{s'}} \leq \frac{\varepsilon}{s-s'} \|U - V\|_{X_s} \tag{3.38}$$

for $0 \leq s' < s \leq 1$, and $U = (u_0, u_1, \ldots, u_{M-1}) \in X_s, V = (v_0, v_1, \ldots, v_{M-1}) \in X_s$ with

$$\|U\|_{X_s} \leq D, \ \|V\|_{X_s} \leq D. \tag{3.39}$$

Furthermore, for $0 \le s \le 1$ and $U \in X_s$ with $\|U\|_{X_s} \le D$, the map $x \in [-1,1] \to F(x,U) \in X_s$ is continuous.

Proof. The assumption (1.7) gives $F(x,0)=0$ and therefore (3.37) follows from (3.38). Thus it is sufficient to prove (3.38). Pick $0 \le s' < s \le 1$, $D>0$ and $U,V \in X_s$ satisfying (3.39). Then, the definition (1.11) of f gives

$$\begin{aligned}
\|F(x,U)-F(x,V)\|_{X_{s'}} &= \left\| -\begin{pmatrix} 0 \\ f(x,U)-f(x,V) \end{pmatrix} \right\|_{X_{s'}} \\
&= e^{-\tau(1-s')}\|f(x,U)-f(x,V)\|_{L(s')} \\
&\le e^{-\tau(1-s')} \sum_{|\vec{p}|>0} \|A_{\vec{p}}(x)[\prod_{j=0}^{M-1} u_j^{p_j} - \prod_{j=0}^{M-1} v_j^{p_j}]\|_{L(s')} \\
&\le e^{-\tau(1-s')} \sum_{|\vec{p}|>0} |A_{\vec{p}}(x)| \sum_{j=0}^{M-1} \|u_j^{p_j}-v_j^{p_j}\|_{L(s')} \prod_{i\neq j} \left(\|u_i\|_{L(s')}^{p_i} + \|v_i\|_{L(s')}^{p_i} \right)
\end{aligned}$$

where we used the triangle inequality, Lemma 3.3 and an iteration argument. Note that, by (3.18), we have for a constant $\widehat{C} = \widehat{C}(L_1, M) \ge 1$ and any $0 \le s' < 1$

$$\sum_{i=0}^{M-1} \|u_i\|_{L(s')} \le \widehat{C} \sum_{i=0}^{M-1} \|u_i\|_{L(s'),\frac{M-i-1}{\lambda}} \le \widehat{C} e^{\tau(1-s')} \|U\|_{X_{s'}} \le \widehat{C} D e^{\tau}, \tag{3.40}$$

and similarly $\sum_{i=0}^{M-1} \|v_i\|_{L(s')} \le \widehat{C} D e^{\tau}$. Using again Lemma 3.3, for $j=0,\dots,M-1$, we obtain

$$\begin{aligned}
\|u_j^{p_j}-v_j^{p_j}\|_{L(s')} &= \|(u_j-v_j)(u_j^{p_j-1}+u_j^{p_j-2}v_j+\cdots+v_j^{p_j-1})\|_{L(s')} \\
&\le \|u_j-v_j\|_{L(s')} \left(\|u_j\|_{L(s')}^{p_j-1} + \|u_j\|_{L(s')}^{p_j-2}\|v_j\|_{L(s')} + \cdots + \|v_j\|_{L(s')}^{p_j-1} \right) \\
&\le p_j (\widehat{C} D e^{\tau})^{p_j-1} \|u_j-v_j\|_{L(s')}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|F(x,U)-F(x,V)\|_{X_{s'}} &\le 2^{M-1} e^{-\tau(1-s')} \sum_{|\vec{p}|>0} |A_{\vec{p}}(x)| \sum_{j=0}^{M-1} p_j \|u_j-v_j\|_{L(s')} (\widehat{C} D e^{\tau})^{|\vec{p}|-1} \\
&\le C(L_1,N,M)\|U-V\|_{X_{s'}} \sum_{|\vec{p}|>0} |A_{\vec{p}}(x)| \sum_{j=0}^{M-1} p_j (\widehat{C} D e^{\tau})^{|\vec{p}|-1} \\
&=: C(L_1,N,M)\|U-V\|_{X_{s'}} S.
\end{aligned} \tag{3.41}$$

where we have used (3.18). Let us estimate the term

$$S := \sum_{|\vec{p}|>0} |A_{\vec{p}}(x)| \sum_{j=0}^{M-1} p_j (\widehat{C} D e^{\tau})^{|\vec{p}|-1},$$

set $C_a' := C_a/(1-b_2^{-1})$. Estimate (1.12) becomes

$$|A_{\vec{p}}(x)| \leq \frac{C_a}{b^{|\vec{p}|}} \frac{1}{1-\frac{|x|}{b_2}} \leq \frac{C_a'}{b^{|\vec{p}|}}, \quad \text{for } |x| \leq 1,$$

so that we have

$$S \leq \sum_{|\vec{p}|>0} \frac{C_a'}{b^{|\vec{p}|}} |\vec{p}| (CDe^{\tau})^{|\vec{p}|-1} \leq \sum_{R=1}^{+\infty} \sum_{|\vec{p}|=R} \frac{C_a'}{b^R} R (CDe^{\tau})^{R-1}.$$

Using the fact that $\sum_{|\vec{p}|=R} 1 \leq C(R+1)^{M-1}$ and that for $0<\rho<1$,

$$\sum_{R=0}^{\infty} (R+1)\cdots(R+M)\rho^R = \frac{d^M}{d\rho^M} \sum_{R=0}^{\infty} \rho^{R+M} = \frac{d^M}{d\rho^M} \frac{\rho^M}{1-\rho} = P(\rho)(1-\rho)^{-M-1},$$

for some $P \in \mathbb{R}[X]$. We obtain

$$\begin{aligned} S &\leq C(C_a', b, M) \sum_{R=1}^{+\infty} (R+1)^{M-1} R \left(\frac{\widehat{C} D e^{\tau}}{b} \right)^{R-1} \\ &\leq C(C_a', b, M) \sum_{R=0}^{+\infty} (R+1)\cdots(R+M) \left(\frac{\widehat{C} D e^{\tau}}{b} \right)^{R} \\ &\leq C(C_a', b, M) \left(1 - \frac{\widehat{C} D e^{\tau}}{b} \right)^{-M-1} \leq C(C_a', b, M), \end{aligned}$$

provided that

$$D \leq \frac{b e^{-\tau}}{2\widehat{C}(L_1, M)}. \tag{3.42}$$

Therefore, using (3.31), (3.35) and (3.41), we infer that (3.42) implies

$$\begin{aligned} \|F(x,U) - F(x,V)\|_{X_{s'}} &\leq C(C_a', b, M, N, L_1) \|U-V\|_{X_{s'}} \\ &\leq C(C_a', b, M, N, L_1) e^{-\tau(s-s')} \|U-V\|_{X_s} \\ &\leq \frac{C(C_a', b, M, N, L_1)}{e} \frac{1}{\tau(s-s')} \|U-V\|_{X_s}. \end{aligned}$$

To complete the proof of (3.38), it is sufficient to pick $\tau \geq \tau_0$ with τ_0 such that $\frac{C(C_a', b, M, N, L_1)}{e\tau_0} \leq \epsilon$, and D as in (3.42).

For given $0 \leq s \leq 1$ and $U = (u_0, u_1, ..., u_{M-1}) \in X_s$ with $\|U\|_{X_s} \leq D$, let us prove that the map $x \in [-1,1] \to F(x,U) \in X_s$ is continuous. Pick any $x, x' \in [-1,1]$. From the mean value theorem, we have for $r \in \mathbb{N}$ such that $|x^R - x'^R| \leq R|x-x'|$ with $R \in \mathbb{N}$,

$$|A_{\vec{p}}(x) - A_{\vec{p}}(x')| \leq |x-x'| \sum_{R\in\mathbb{N}} \frac{R C_a}{b^{|\vec{p}|} b_2^R} = \frac{C_a}{b^{|\vec{p}|} b_2} \left(1 - \frac{1}{b_2}\right)^{-2} |x-x'|.$$

We infer that

$$\begin{aligned}\|F(x,U)-F(x',U)\|_{X_s} &= e^{-\tau(1-s)}\|f(x,u_0,\cdots,u_{M-1})-f(x',u_0,\cdots,u_{M-1})\|_{L(s)}\\ &\le \sum_{|\vec{p}|>0}|A_{\vec{p}}(x)-A_{\vec{p}}(x')|\|u_0^{p_0}u_1^{p_1},\cdots u_{M-1}^{p_{M-1}}\|_{L(s)}\\ &\le \frac{C_a}{b_2}\left(1-\frac{1}{b_2}\right)^{-2}|x-x'|\sum_{|\vec{p}|>0}\frac{\left(\widehat{C}(L_1,M)De^{\tau}\right)^{|\vec{p}|}}{b^{|\vec{p}|}},\end{aligned}$$

due to Lemma 3.3 and (3.40), the last series being convergent when (3.42) is fulfilled. This proves the continuity of the map $x\in[-1,1]\to F(x,U)\in X_s$. □

We are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. By Remark 4, we can assume $|\zeta_M|=1$. Let $f=f(x,\vec{y})$ be as in (1.7)-(1.10), $-\infty<t_1<t_2<+\infty$ and $R>(4N)^\lambda$. Pick $k_0,k_1,\cdots,k_{M-1}\in G^\lambda([t_1,t_2])$ such that (3.2) holds. We will show that Theorem 3.5 can be applied provided that C is small enough. Pick $L_1\in(1/R,1/(4N)^\lambda)$. Let $\eta=\eta(L_1)>0$ be as in Theorem 3.5. Let $K_0=(k_0,k_1,\cdots,k_{M-1})$. We have to show that

$$\|K_0\|_{\mathcal{X}_{L_1}}=\sum_{j=0}^{M-1}\|k_i\|_{L_1,\frac{M-j-1}{\lambda}}\le\eta,$$

for C small enough. Thanks to (3.15) and up to a change of $\eta(L_1)$ by a smaller constant, it is sufficient to have for any $i=0,\cdots,M-1$,

$$\|k_i\|_{L_1,\frac{M-1}{\lambda}}\le\frac{\eta}{2}. \tag{3.43}$$

Recall that

$$\|f\|_{L_1,\frac{M-1}{\lambda}}=\max\left(2^6\|f\|_{L^\infty([t_1,t_2])},2^3L_1^{-1}\sup_{t\in[t_1,t_2],n\in\mathbb{N}}\frac{|f^{(n+1)}(t)|}{L_1^{|n-\frac{M-1}{\lambda}|}\Gamma_{\lambda,\frac{M-1}{\lambda}}(n)}\right), \tag{3.44}$$

where

$$\Gamma_{\lambda,\frac{M-1}{\lambda}}(n)=\begin{cases}2^{-5}\big(\Gamma(n+1-\frac{M-1}{\lambda})\big)^\lambda(1+n)^{-2}, & \text{if } n>\frac{M-1}{\lambda}+1,\\ 2^{-5}(n!)^\lambda(1+n)^{-2}, & \text{if } 0\le n\le\frac{M-1}{\lambda}+1.\end{cases}$$

Then, if follows that (3.43) is satisfied provided that

$$\|k_i\|_{L^\infty([t_1,t_2])}\le 2^{-7}\eta, \tag{3.45}$$

$$\|k_i^{(n+1)}\|_{L^\infty([t_1,t_2])}\le 2^{-4}\eta L_1^{1+|n-\frac{M-1}{\lambda}|}\Gamma_{\lambda,\frac{M-1}{\lambda}}(n),\quad\forall n\in\mathbb{N}. \tag{3.46}$$

Since $\Gamma(n+1-\frac{M-1}{\lambda})\sim\Gamma(n+1)/n^{\frac{M-1}{\lambda}}\sim n!/n^{\frac{M-1}{\lambda}}$ as $n\to+\infty$, we have that $\big(\Gamma(n+1-\frac{M-1}{\lambda})\big)^\lambda\sim(n!)^\lambda/n^{M-1}$. Thus, the r.h.s. of (3.46) is equivalent to $2^{-9}\eta L_1^{n+1-\frac{M-1}{\lambda}}(n!)^\lambda n^{-(M+1)}$ as $n\to+\infty$. Using (3.2) and $L_1>1/R$, we have that (3.46) holds if C is small enough. The same is true for (3.45).

We infer from Theorem 3.5 the existence of a solution $U = (y, \partial_x y, \cdots, \partial_x^{M-1} y) \in C([-1,1], X_{s_0})$ (for some $s_0 \in (0,1)$) of (3.24). Let us check that $y \in C^\infty([-1,1] \times [t_1,t_2])$. To this end, we prove by induction on $n \in \mathbb{N}$ the following statement

$$U \in C^n([-1,1], C^k([t_1,t_2])^M), \quad \forall k \in \mathbb{N}. \tag{3.47}$$

The assertion (3.47) is true for $n = 0$, since $X_{s_0} \subset C^k([t_1,t_2])^M$ for all $k \in \mathbb{N}$. Assume that (3.47) is true for some $n \in \mathbb{N}$. Since A is a continuous linear map from $C^{k+N}([t_1,t_2])^M$ into $C^k([t_1,t_2])^M$ for all $k \in \mathbb{N}$, we have that

$$AU \in C^n([-1,1], C^k([t_1,t_2])^M), \quad \forall k \in \mathbb{N}.$$

On the other hand, as f is analytic and hence of class C^∞, we infer from (3.47) that $F(x,U) \in C^n([-1,1], C^k([t_1,t_2])^M)$ for all $k \in \mathbb{N}$. Since $\partial_x U = AU + F(x,U)$, we obtain that (3.47) is true with n replaced by $n+1$. Therefore, $y \in C^\infty([-1,1] \times [t_1,t_2])$. Finally, the proof that $y \in G^{1,\lambda}([-1,1] \times [t_1,t_2])$, is given in Appendix 6.1, which uses some estimates of the next section, with eventually a stronger smallness assumption on the initial data. □

4. Correspondence between the space derivatives and the time derivatives

We would like to know the relationship between the time derivatives and the space derivatives of any solution of a general nonlinear equation given by

$$\partial_t^N y = Py + f(x, Y^x) \tag{4.1}$$

where $f = f(x, Y^x)$ is of class C^∞ on $\mathbb{R}^{M+1}$.

When $f = 0$ and $Py = \partial_x^M y$, then it is easy to see that

$$\partial_t^{nN+j} Y^x = \partial_x^{nM} \partial_t^j Y^x, \quad \forall j \in \{0, ..., N-1\}, \ \forall n \in \mathbb{N}. \tag{4.2}$$

It follows that for any (x_0,t_0) the determination of the jet $(\partial_t^n Y^x(x_0,t_0))_{n\geq 0}$ is equivalent to the determination of the jet $(\partial_x^n Y^t(x_0,t_0))_{n\geq 0}$. In the general case ($f = f(x,Y^x)$ and $Py = \sum_{j=0}^M \zeta_j \partial_x^j y$), the relation (4.2) may not be true. Nevertheless, there is still a one-to-one correspondence between the jet $(\partial_t^n Y^x(x_0,t_0))_{n\geq 0}$ and the jet $(\partial_x^n Y^t(x_0,t_0))_{n\geq 0}$.
Introduce some notations. For given $-\infty < t_1 \leq \tau \leq t_2 < +\infty$, we set

$$\mathcal{S} := \{y \in C^\infty([-1,1] \times [t_1,t_2]) : \ y \text{ satisfies (4.1) on } [-1,1] \times [t_1,t_2]\}, \tag{4.3}$$

$$\mathcal{J}^t := \{(\partial_t^n Y^x(0,\tau))_{n\geq 0} : \ Y^x = (y, \partial_x y, ..., \partial_x^{M-1} y), \ \ y \in \mathcal{S}\} \subset (\mathbb{R}^M)^{\mathbb{N}}, \tag{4.4}$$

$$\mathcal{J}^x := \{(\partial_x^n Y^t(0,\tau))_{n\geq 0} : \ Y^t = (y, \partial_t y, ..., \partial_t^{N-1} y), \ \ y \in \mathcal{S}\} \subset (\mathbb{R}^N)^{\mathbb{N}}. \tag{4.5}$$

The set $\mathcal{J}^t$ (resp. $\mathcal{J}^x$), which stands for the set of sequences of *time derivatives* (resp. *space derivatives*) at $(0,\tau)$ of Y^x (resp. Y^t) for smooth solutions y of (4.1), is a subset of $(\mathbb{R}^M)^{\mathbb{N}}$ (resp. $(\mathbb{R}^N)^{\mathbb{N}}$) that we will not determine explicitly.

Proposition 4.1. *Let $-\infty < t_1 \leq \tau \leq t_2 < +\infty$ and assume that $f \in C^\infty(\mathbb{R}^{M+1})$. Then there exists a map $\Lambda : (\mathbb{R}^N)^{\mathbb{N}} \to (\mathbb{R}^M)^{\mathbb{N}}$ whose restriction (still denoted by Λ) $\Lambda : \ \mathcal{J}^x \to \mathcal{J}^t$ is a bijection such that for any $y \in C^\infty([-1,1] \times [t_1,t_2])$ satisfying (4.1) on $[-1,1] \times [t_1,t_2]$, we have $(\partial_t^n Y^x(0,\tau))_{n\geq 0} = \Lambda\left((\partial_x^n Y^t(0,\tau))_{n\geq 0}\right)$, where $Y^x = (y, \partial_x y, ..., \partial_x^{M-1} y)$ and $Y^t = (y, \partial_t y, ..., \partial_t^{N-1} y)$.*

Proof. Proposition 4.1 is a consequence of Lemma 4.2 (see below) which, roughly speaking, consists in taking sufficiently many derivatives in (4.1). □

Notation 4.1. *The space* $(\mathbb{R}^{q+1})^{p+1}$ *will be denoted* $E_{p,q}$. *The current vector in* $E_{p,q}$ *will be denoted* $Y_{p,q} \in E_{p,q}$ *when a confusion may occur, but very often merely* Y *to make notations easier.*
For $y \in C^\infty([-1,1]\times[t_1,t_2])$ *and* $p,q \in \mathbb{N}$, *we denote the vector* $Y^{x,t}_{p,q}(y) := (Y^t_q, \partial_x Y^t_q, \dots, \partial_x^p Y^t_q) \in E_{p,q}$ *with* $Y^t_q(x,t) = (y(x,t), \partial_t y(x,t), ..., \partial_t^q y(x,t))$ *as it was defined in* (1.5). *Most of the time, when only one function* y *is concerned, we will write* $Y^{x,t}_{p,q}$.

Lemma 4.2. *Let* $f \in C^\infty(\mathbb{R}^{M+1})$ *and* $l,k \in \mathbb{N}$ *with* $l = Nn + j$ *for some* $0 \le j < N$ *and* $n \in \mathbb{N}$. *Then there exists a smooth function* $H^k_l : \mathbb{R} \times E_{Mn+k-1,N-1} \to \mathbb{R}$ *such that any solution* $y \in C^\infty([0,1]\times[t_1,t_2])$ *of* (4.1) *satisfies*

$$\partial_t^l \partial_x^k y = P^n \partial_t^j \partial_x^k y + H^k_l(x, Y^{x,t}_{Mn+k-1,N-1}) \tag{4.6}$$

where we have used the Notations 4.1.

We introduce first some definitions, notations, and lemmas that will be needed for the proof of Lemma 4.2. To apply Leibniz formula for x in a formal way, we have to see how the derivations ∂_x and ∂_t operate in $E_{p,q}$. This leads us to define the following operators.

Notation 4.2. *There is a linear operator* $\widetilde{D}_t$ *from* $E_{p,q+1}$ *to* $E_{p,q}$ *such that we can write* $\partial_t Y^{x,t}_{p,q} = \widetilde{D}_t(Y^{x,t}_{p,q+1})$ *for any smooth function.*

Similarly, we define the operator $\widetilde{D}_x$ *from* $E_{p+1,q}$ *to* $E_{p,q}$ *by the shift* $\widetilde{D}_x(Y_0, Y_1, \dots, Y_{p+1}) = (Y_1, \dots, Y_{p+1})$ *so that for any* $y \in C^\infty([-1,1]\times[t_1,t_2])$, $Y^{x,t}_{p,q}$ *being as in Notation 4.1, we have*

$$\widetilde{D}_x Y^{x,t}_{k+1,N-1} = \partial_x Y^{x,t}_{k,N-1}. \tag{4.7}$$

Note that $\widetilde{D}_t$ can also be seen as a shift, but after a proper identification between $E_{p,q}$ and $E_{q,p}$. The operator $\widetilde{D}_x$ depends of course on p and q but, since the definition is similar for each p, q, it should not lead to any confusion.

Notation 4.3. *For* $Y = (Y^0, \dots, Y^i, \dots, Y^k) \in E_{k,N-1}$, *we denote*

$$I(Y) := (Y^0, \dots, (-1)^i Y^i, \dots, (-1)^k Y^k).$$

Strictly speaking, the operator I *depends on* k, *but since it takes the same form on each space, we will keep the same notation. The interest of this operator is that for* $y \in C^\infty([-1,1]\times[t_1,t_2])$ *and* $Y = Y^{x,t}_{k,N-1}(y)$ *as in Notation 4.1, we have*

$$I(Y(y)) = Y(y_-)(-x), \tag{4.8}$$

where y_- *is the reflected function* $y_-(t,x) := y(t,-x)$.

We notice that

$$\widetilde{D}_t I(Y) = I(\widetilde{D}_t Y), \tag{4.9}$$

$$\widetilde{D}_x I(Y) = -I(\widetilde{D}_x Y). \tag{4.10}$$

Lemma 4.3. *Let $p, q \in \mathbb{N}$ and let $M : \mathbb{R} \times E_{p,q} \to \mathbb{R}$ be a smooth function. Then there exist two smooth functions $M_t : [-1,1] \times E_{p,q+1} \to \mathbb{R}$ and $M_x : [-1,1] \times E_{p+1,q} \to \mathbb{R}$ such that for any $y \in C^\infty([-1,1] \times [t_1, t_2])$ (not necessarily solution of (4.1)), $Y^{x,t}_{p,q}$ being as in Notation 4.1, we have*

$$\partial_t M(x, Y^{x,t}_{p,q}) = M_t(x, Y^{x,t}_{p,q+1}), \tag{4.11}$$

$$\partial_x M(x, Y^{x,t}_{p,q}) = M_x(x, Y^{x,t}_{p+1,q}). \tag{4.12}$$

Moreover, if we assume that for some $\varpi, \sigma \in \{-1, 1\}$, $M(-x, \varpi I(Y)) = \sigma M(x, Y)$, then we have

$$M_t(-x, \varpi I(Y)) = \sigma M_t(x, Y), \tag{4.13}$$

$$M_x(-x, \varpi I(Y)) = -\sigma M_x(x, Y). \tag{4.14}$$

Proof. By the chain rule, we have

$$\partial_t M(x, Y^{x,t}_{p,q}) = \nabla M(x, Y^{x,t}_{p,q}) \cdot \begin{pmatrix} 0 \\ \partial_t Y^{x,t}_{p,q} \end{pmatrix}. \tag{4.15}$$

Using the operator $\widetilde{D}_t$ introduced in Notation 4.2, we can define M_t as

$$M_t(x, Y_{p,q+1}) := \nabla M(x, Y_{p,q}) \cdot \begin{pmatrix} 0 \\ \widetilde{D}_t(Y_{p,q+1}) \end{pmatrix}, \quad \forall x \in [-1,1],\ \forall Y_{p,q+1} \in E_{p,q+1}. \tag{4.16}$$

For $Y_{p,q+1} = (Y_0, Y_1, ..., Y_p) \in E_{p,q+1}$, we have denoted $Y_{p,q}$ the vector in $E_{p,q}$ obtained by selecting the $q+1$ first components of each vector Y_i for $0 \le i \le p$. With this definition, (4.11) is true for any smooth function y.

Similarly, we define the function M_x by

$$M_x(x, Y_{p+1,q}) := \nabla M(x, Y_{p,q}) \cdot \begin{pmatrix} 1 \\ \widetilde{D}_x(Y_{p+1,q}) \end{pmatrix}, \quad \forall x \in [-1,1],\ \forall Y_{p+1,q} \in E_{p+1,q}, \tag{4.17}$$

and it can be seen that (4.12) is true for any smooth function y.
To prove (4.13), we take the derivative w.r.t. Y in the relation $M(-x, \varpi I(Y)) = \sigma M(x, Y)$ to obtain for any $Z \in E_{p,q}$,

$$\nabla M(-x, \varpi I(Y)) \cdot \begin{pmatrix} 0 \\ \varpi I(Z) \end{pmatrix} = \sigma \nabla M(x, Y) \cdot \begin{pmatrix} 0 \\ Z \end{pmatrix}.$$

Let $Y_{p,q+1} \in E_{p,q+1}$. Taking $Y = Y_{p,q}$ and $Z = \widetilde{D}_t(Y_{p,q+1})$ and noticing that $I(\widetilde{D}_t(Y_{p,q+1})) = \widetilde{D}_t(I(Y_{p,q+1}))$ by (4.9), we obtain

$$\nabla M(-x, \varpi I(Y_{p,q})) \cdot \begin{pmatrix} 0 \\ \varpi \widetilde{D}_t(I(Y_{p,q+1})) \end{pmatrix} = \sigma \nabla M(x, Y_{p,q}) \cdot \begin{pmatrix} 0 \\ \widetilde{D}_t(Y_{p,q+1}) \end{pmatrix},$$

which is exactly (4.13). The proof of (4.14) is similar and is omitted. ☐

Proof of Lemma 4.2. We will actually prove the slightly stronger result that for $k \in \mathbb{N}$ and $l = Nn + j$ for some $0 \le j < N$ and $n \in \mathbb{N}$, each H^k_l is actually a function of x and $Y \in E_{Mn+k-1,j}$

so that (4.6) is satisfied with $H_l^k(x, Y^{x,t}_{Mn+k-1,N-1})$ replaced by $H_l^k(x, Y^{x,t}_{Mn+k-1,j})$.

The case $n=0$, $0 \le j < N$ is trivial since we can take $H_l^k = 0$.

For some technical reasons, we will also need to deal with the case $n=1$. Letting $f_0(x, Y^x_{M-1}) := f(x, Y^x_{M-1})$, we apply the operator ∂_t^j in (4.1) for $0 \le j < N$ to get

$$\partial_t^{N+j} y = P\partial_t^j y + \partial_t^j f(x, Y^{x,t}_{M-1,0}) \tag{4.18}$$

We want to define functions f_j so that for any $y \in C^\infty([-1,1] \times [t_1,t_2])$, we have

$$f_j(x, Y^{x,t}_{M-1,j}) = \partial_t^j f_0(x, Y^{x,t}_{M-1,0}) \text{ for } 0 < j < N. \tag{4.19}$$

Using Notation 4.2 we can define f_j iteratively by

$$f_j(x, Y_{M-1,j}) := \nabla f_{j-1}(x, Y_{M-1,j-1}) \cdot \begin{pmatrix} 0 \\ \widetilde{D}_t(Y_{M-1,j}) \end{pmatrix}, \tag{4.20}$$

so that by Lemma 4.3, (4.19) is true for any $y \in C^\infty([-1,1] \times [t_1,t_2])$. Now that the f_j are defined, we see that any solution y of (4.1) satisfies (4.18) and also

$$\partial_t^{N+j} y = P\partial_t^j y + f_j(x, Y^{x,t}_{M-1,j}). \tag{4.21}$$

In particular, defining $H^0_{N+j} := f_j$, we see that the case $k=0$, $l = N+j$ with $0 \le j < N$ is treated.

Applying ∂_x^k in (4.21) and using Lemma 4.3, we can find some smooth functions H^k_{N+j} such that

$$\partial_t^{N+j}\partial_x^k y = P\partial_t^j \partial_x^k y + H^k_{N+j}(x, Y^{x,t}_{M-1+k,j}). \tag{4.22}$$

The H^k_{N+j} are defined by the iteration formula

$$H^k_{N+j}(x, Y_{M-1+k,j}) := \nabla H^{k-1}_{N+j}(x, Y_{M-1+k-1,j}) \cdot \begin{pmatrix} 1 \\ \widetilde{D}_x(Y_{M-1+k,j}) \end{pmatrix}, \tag{4.23}$$

this is the case $n=1$ of the Lemma.

Now, we construct the functions H^k_{Nn+j} by induction on n. Assume that the (4.6) is satisfied for some $n \in \mathbb{N}^*$, for all $l = Nn+j$ with $0 \le j < N$ and all $k \in \mathbb{N}$. Applying the operator ∂_t^N in (4.6) yields

$$\partial_t^{l+N}\partial_x^k y = P^n \partial_t^j \partial_x^k \partial_t^N y + \partial_t^N H_l^k(x, Y^{x,t}_{Mn+k-1,j}). \tag{4.24}$$

Using equation (4.1), we obtain

$$\partial_t^{l+N}\partial_x^k y = P^{n+1}\partial_t^j\partial_x^k y + P^n \partial_x^k \partial_t^j f(x, Y^x_{M-1}) + \partial_t^N H_l^k(x, Y^{x,t}_{Mn+k-1,j}). \tag{4.25}$$

So, we are led to prove that the last two terms $P^n\partial_x^k\partial_t^j f(x, Y^x_{M-1}) + \partial_t^N H_l^k(x, Y^{x,t}_{Mn+k-1,j})$ can be written as $H^k_{l+N}(x, Y^{x,t}_{M(n+1)+k-1,j})$. Concerning the first one, due to (4.19) we can write

$$P^n\partial_x^k\partial_t^j f(x, Y^x_{M-1}) = P^n \partial_x^k f_j(x, Y^{x,t}_{M-1,j}). \tag{4.26}$$

Since $P^n\partial_x^k$ is a differential operator of order $Mn+k$ in x, we see by successive applications of Lemma 4.3 that the previous term can be written as a smooth function of x and $Y^{x,t}_{M(n+1)+k-1,j}$.

By iterative applications of Lemma 4.3, the second term $\partial_t^N H_l^k(x, Y^{x,t}_{Mn+k-1,j})$ can be written as $F(x, Y^{x,t}_{Mn+k-1,j+N})$ for some smooth function F. But thanks to the case $n=1$, namely (4.22), for each $0 \le p \le Mn+k-1$, $\partial_t^{N+j}\partial_x^p y$ can be written as $J^p_{N+j}(x, Y^{x,t}_{M+p,j})$ for some smooth function J^p_{N+j}. In particular, $Y^{x,t}_{Mn+k-1,N+j}$ can be written as a smooth function of x and $Y^{x,t}_{M+Mn+k-1,j}$. It follows that $\partial_t^N H_l^k(x, Y^{x,t}_{Mn+k-1,j}) = F(x, Y^{x,t}_{Mn+k-1,j+N})$ can be written as a smooth function of x and $Y^{x,t}_{M(n+1)+k-1,j}$. Going back to (4.25) and summing up the expression of the last two terms as functions of x and $Y^{x,t}_{M(n+1)+k-1,j}$, we can write

$$\partial_t^{N(n+1)+j}\partial_x^k y \quad = \quad P^{n+1}\partial_t^j\partial_x^k y + H^k_{N(n+1)+j}(x, Y^{x,t}_{M(n+1)+k-1,j}) \tag{4.27}$$

for some smooth function $H^k_{N(n+1)+j}$. This is the expected result at step $n+1$. □

We present a few consequences of Lemma 4.2.

Notation 4.4. *Let $k \in \mathbb{N}$ and $l = Nn + j$ for some $0 \le j < N$ and $n \in \mathbb{N}$. Noticing that $P^n\partial_t^j\partial_x^k y$ can be expressed as a linear combination of variables in $Y^{x,t}_{Mn+k,N-1}$, we can define a smooth function $J_l^k : [-1,1] \times E_{Mn+k,N-1} \to \mathbb{R}$ such that*

$$J_l^k(x, Y^{x,t}_{Mn+k,N-1}) = \partial_t^l\partial_x^k y = P^n\partial_t^j\partial_x^k y + H_l^k(x, Y^{x,t}_{Mn+k-1,N-1}) \tag{4.28}$$

for any solution y of (4.1). *We define also the vector-valued functions*

$$J_l : [-1,1] \times E_{Mn+M-1,N-1} \to \mathbb{R}^M,$$

with $J_l = (J_l^0, J_l^1, \ldots, J_l^{M-1})$.

These definitions will mainly be used at $x = 0$ and $t = 0$. Since the knowledge of the initial datum Y_0 and all its x-derivatives are sufficient to know $Y^{x,t}_{Mn+k,N-1}$ for $t = 0$, J_l has to be thought as the function that, from a sufficient amount of x-derivatives of the initial datum, provides $(\partial_t^l Y^x)(0,0)$, that is the l time derivative of the boundary data. More precisely, if y is a solution of (4.1), we have for any t, x

$$\partial_t^l Y^x = J_l(x, Y^{x,t}_{Mn+M-1,N-1}). \tag{4.29}$$

In particular, this definition of J_l provides a proof for Lemma 1.1 with the appropriate choice of $m(l) = Mn + M - 1$ if $l = Nn + j$ for some $0 \le j < N$ and $n \in \mathbb{N}$.

The two following Lemmas are almost tautological with the definitions, but they are important to justify the relevance of the set $\mathcal{C}$.

Lemma 4.4. *Assume that y is a smooth solution of* (1.1)-(1.3). *Then $Y^t(.,t) \in \mathcal{C}$ for all $t \in [0,T]$.*

Proof. From (1.2), we have $BY^x(0,t) = 0$ for all $t \in [0,T]$. Applying the operator ∂_t^l in that equation yields $B\partial_t^l Y^x(0,t) = 0$ for all $t \in [0,T]$. Writing $l = Nn + j$, with $0 \leq j < N$, $n \in \mathbb{N}$, and using the fundamental property (4.29) of the function J_l, we obtain $BJ_l(x, Y^t, \partial_x Y^t, ..., \partial_x^{Mn+M-1} Y^t)_{x=0}$. It means that $Y^t(.,t) \in \mathcal{C}$ for all $t \in [0,T]$. □

Lemma 4.5. *Let y be a smooth solution to $\partial_t^N y = Py + f(x, y, \partial_x y, ..., \partial_x^{M-1} y)$ such that $Y^x(0,t) \in \mathcal{C}$ for some $t \in [0,T]$. Then Y^x satisfies the boundary condition $BY^x(0,t) = 0$.*

Proof. We have $Y^x(0,t) \in \mathcal{C}$, which implies, with the choice $l = 0$

$$BJ_0(x, Y^t, \partial_x Y^t, ..., \partial_x^{Mn+M-1} Y^t)_{x=0} = 0.$$

Using property (4.29) at time t and with $x = 0$ and $l = n = 0$, one obtains $BY^x(0,t) = 0$. □

The following Lemma is needed to prove Proposition 1.3.

Lemma 4.6. *Assume that M is even and that*

$$P = \sum_{j=0}^{M/2} \zeta_{2j} \partial_x^{2j}. \tag{4.30}$$

(1) If (1.18) *holds, then for all $l, k \in \mathbb{N}$ we have*

$$H_l^k(-x, -I(Y)) = (-1)^{k+1} H_l^k(x, Y), \quad \forall x \in [-1,1],\ \forall Y \in E_{Mn+k-1,N-1}, \tag{4.31}$$

$$J_l^k(-x, -I(Y)) = (-1)^{k+1} J_l^k(x, Y), \quad \forall x \in [-1,1],\ \forall Y \in E_{Mn+k,N-1}. \tag{4.32}$$

(2) If (1.19) *holds, then for all $l, k \in \mathbb{N}$ we have*

$$H_l^k(-x, I(Y)) = (-1)^k H_l^k(x, Y), \quad \forall x \in [-1,1],\ \forall Y \in E_{Mn+k-1,N-1}, \tag{4.33}$$

$$J_l^k(-x, I(Y)) = (-1)^k J_l^k(x, Y), \quad \forall x \in [-1,1],\ \forall Y \in E_{Mn+k,N-1}. \tag{4.34}$$

Proof. To treat both cases simultaneously, we define ϖ as $\varpi = -1$ (resp $\varpi = 1$) if (1.18) holds (resp. (1.19) holds). Therefore, we want to prove

$$H_l^k(-x, \varpi I(Y)) = \varpi(-1)^k H_l^k(x, Y) \quad \forall Y \in E_{Mn+k-1,N-1}, \tag{4.35}$$

$$J_l^k(-x, \varpi I(Y)) = \varpi(-1)^k J_l^k(x, Y) \quad \forall Y \in E_{Mn+k,N-1}. \tag{4.36}$$

We still denote $l = Nn + j$, where $n \in \mathbb{N}$ and $0 \leq j < N$. We first prove (4.35) by induction on n. If $n = 0$, then (4.35) is obvious since $H_l^k = 0$.
Assume that $n = 1$ so that $l = N + j$. Assume first that $k = 0$. We claim that

$$f_j(-x, \varpi I(Y_{M-1,j})) = \varpi f_j(x, Y_{M-1,j}), \quad \forall Y_{M-1,j} \in E_{M-1,j}. \tag{4.37}$$

We proceed by induction. For $j = 0$, $f_0 = f$, so it follows from assumption (1.18) or (1.19) thanks to the choice of ϖ. If (4.37) is true for $j - 1$, taking derivatives with respect to Y, we also have, for any $Z \in E_{M-1,j-1}$,

$$\varpi \nabla f_{j-1}(-x, \varpi I(Y_{M-1,j-1})) \cdot \begin{pmatrix} 0 \\ I(Z) \end{pmatrix} = \varpi \nabla f_{j-1}(x, Y_{M-1,j-1}) \cdot \begin{pmatrix} 0 \\ Z \end{pmatrix}. \tag{4.38}$$

By (4.9), (4.20) and (4.38), we have

$$\begin{aligned} f_j(-x,\varpi I(Y_{M-1,j})) &= \nabla f_{j-1}(-x,\varpi I(Y_{M-1,j-1}))\cdot\begin{pmatrix}0\\ \widetilde{D}_t(\varpi I(Y_{M-1,j}))\end{pmatrix}\\ &= \varpi\nabla f_{j-1}(-x,\varpi I(Y_{M-1,j-1}))\cdot\begin{pmatrix}0\\ I(\widetilde{D}_t(Y_{M-1,j}))\end{pmatrix}\\ &= \varpi\nabla f_{j-1}(x,Y_{M-1,j-1})\cdot\begin{pmatrix}0\\ \widetilde{D}_t(Y_{M-1,j})\end{pmatrix}\\ &= \varpi f_j(x,Y_{M-1,j}). \end{aligned}$$

Assume now that (4.35) is true for $0\le j<N$ and at step $k-1$, i.e.

$$H^{k-1}_{N+j}(-x,\varpi I(Y_{M-1+k-1,j}))=\varpi(-1)^{k-1}H^{k-1}_{N+j}(x,Y_{M-1+k-1,j}).$$

Taking derivatives with respect to x and Y, it gives for any $Z\in E_{M-1+k-1,j}$,

$$\nabla H^{k-1}_{N+j}(-x,\varpi I(Y_{M-1+k-1,j}))\cdot\begin{pmatrix}-1\\ \varpi I(Z)\end{pmatrix}=\varpi(-1)^{k-1}\nabla H^{k-1}_{N+j}(x,Y_{M-1+k-1,j})\cdot\begin{pmatrix}1\\ Z\end{pmatrix}.$$

Combined with (4.23) and (4.10), this gives

$$\begin{aligned} H^k_{N+j}(-x,\varpi I(Y_{M-1+k,j})) &= \nabla H^{k-1}_{N+j}(-x,\varpi I(Y_{M-1+k-1,j}))\cdot\begin{pmatrix}1\\ \widetilde{D}_x(\varpi I(Y_{M-1+k,j}))\end{pmatrix}\\ &= \nabla H^{k-1}_{N+j}(-x,\varpi I(Y_{M-1+k-1,j}))\cdot\begin{pmatrix}1\\ -\varpi I(\widetilde{D}_x(Y_{M-1+k,j}))\end{pmatrix}\\ &= \varpi(-1)^k\nabla H^{k-1}_{N+j}(x,Y_{M-1+k-1,j})\cdot\begin{pmatrix}1\\ \widetilde{D}_x(Y_{M-1+k,j})\end{pmatrix}\\ &= \varpi(-1)^kH^k_{N+j}(x,Y_{M-1+k,j}). \end{aligned}$$

Thus (4.35) is proved for $n=1$. Assume that (4.35) is true for $l=Nn+j$, with $0\le j<N$ and $n\in\mathbb{N}^*$, and for $k\in\mathbb{N}$. Let us prove that (4.35) is also true for $l+N$ and k. From (4.25)-(4.26), we have that

$$H^k_{l+N}(x,Y^{x,t}_{M(n+1)+k-1,j})=P^n\partial^k_x f_j(x,Y^{x,t}_{M-1,j})+\partial^N_t H^k_l(x,Y^{x,t}_{M(n+1)+k-1,j}). \tag{4.39}$$

By Lemma 4.3, (4.30) and (4.37), we infer that the first term $P^n\partial^k_x f_j(x,Y^{x,t}_{M-1,j})$ can be written as $G(x,Y^{x,t}_{M(n+1)+k-1,j})$ where G satisfies $G(-x,\varpi I(Y))=(-1)^k\varpi G(x,Y)$.
The second term $\partial^N_t H^k_l(x,Y^{x,t}_{Mn+k-1,j})$, by an application of Lemma 4.3, can be written as $F(Y^{x,t}_{Mn+k-1,N+j})$ for a smooth function F that satisfies the same parity property as H^k_l, that is $F(-x,\varpi I(Y))=\varpi(-1)^kF(x,Y)$.

But the case $n=1$ (see (4.22)) gives that, for each $0\le p\le Mn+k-1$, $\partial^{N+j}_t\partial^p_x y$ can be written as $J^p_{N+j}(Y^{x,t}_{M+p,j})$ for some smooth function J^p_{N+j} that satisfies $J^p_{N+j}(-x,\varpi I(Y))=\varpi(-1)^pJ^p_{N+j}(x,Y)$. In particular, $Y^{x,t}_{Mn+k-1,N+j}$ can be written as $K(x,Y^{x,t}_{M+Mn+k-1,N-1})$ (the components of K are the $J^p_{N+j}(x,Y)$). Therefore, the symmetry properties of J^p_{N+j} imply $K(-x,\varpi I(Y)) = \varpi I(K(x,Y))$. In particular, we can write

$$\partial_t^N H_l^k(x, Y^{x,t}_{Mn+k-1,j}) = F(x, Y^{x,t}_{Mn+k-1,j+N}) = F(x, K(x, Y^{x,t}_{M+Mn+k-1,N-1})).$$

Summarizing the symmetry properties of K and F, we obtain

$$F(-x, K(-x,\varpi I(Y))) = F(-x, \varpi I(K(x,Y)) = \varpi(-1)^k F(x, K(x,Y)).$$

This is the expected result, and it completes the proof of (4.35).
To prove (4.36), we use (4.28) and (4.35). Thus it remains to establish the symmetry property for the term $P^n\partial_t^j\partial_x^k y$ for any smooth function y. This follows at once from (4.8) and (4.30). □

Next, we relate the behaviors as $n\to+\infty$ of the jets $(\partial_x^n Y^t(0,\tau))_{n\ge0}$ and $(\partial_t^n Y^x(0,\tau))_{n\ge0}$. To do that, we assume that in (4.1) the nonlinear term reads

$$f(x,y_0,y_1,...,y_{M-1}) = \sum_{(\vec p,r)\in\mathbb{N}^{M+1}} a_{\vec p,r}\, y_0^{p_0} y_1^{p_1}\dots y_{M-1}^{p_{M-1}} x^r \quad \forall (x,y_0,...,y_{M-1})\in(-4,4)^{M+1}, \tag{4.40}$$

where the coefficients $a_{\vec p,r}$, $(\vec p,r)\in\mathbb{N}^{M+1}$, satisfy (1.9)-(1.10).
For $x\in(-1,+\infty)$, we denote $x! = \Gamma(x+1)$, where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the Gamma function. Then $(x+1)! = (x+1)(x!)$ for $x>-1$. We also set $\begin{pmatrix} y \\ x\end{pmatrix} = \frac{y!}{x!(y-x)!}$ for $y\ge x\ge 0$.

Proposition 4.7. *Let $-\infty < t_1 \le \tau \le t_2 < +\infty$ and $f = f(x,y_0,y_1,...,y_{M-1})$ be as in* (1.7)-(1.8) *with the coefficients $a_{\vec p,r}$, $(\vec p,r)\in\mathbb{N}^{M+1}$, satisfying* (1.9)-(1.10). *Assume that $|\zeta_M| = 1$. Let $\widetilde R > 4$, $R, R'\in\mathbb{R}$ with $4 < R' < R < \min(\widetilde R, b_2)$, and $\mu > M+1$. Then there exists some number $\widetilde C > 0$ such that for any $C\in(0,\widetilde C]$, one can find a number $C' = C'(C,R,R',\mu) > 0$ with $\lim_{C\to0^+} C'(C,R,R',\mu) = 0$ such that*

(1) for any function $y\in C^\infty([-1,1]\times[t_1,t_2])$ satisfying (4.1) *on $[-1,1]\times[t_1,t_2]$ and*

$$Y^t(x,\tau) = Y_0(x) = \sum_{k=0}^\infty A_k\frac{x^k}{k!}, \quad \forall x\in[-1,1] \tag{4.41}$$

for some $Y_0\in(\mathcal{R}_{\widetilde R,C})^N$, we have

$$|\partial_x^k\partial_t^n y(0,\tau)| \le C'\frac{(\lambda n+k)!}{R^k R'^{\lambda n}(\lambda n+k+1)^\mu}\ \forall k,n\in\mathbb{N}; \tag{4.42}$$

(2) there exists an application

$$\Lambda^\infty : (A_k)_{k\ge0}\in(\mathcal{N}_{\tilde R,C})^N \to (d_n^k)_{(n,k)\in\mathbb{N}^2}\in\mathbb{R}^{\mathbb{N}^2}$$

such that if there exists a solution y of (4.1) *on $[-1,1]\times[t_1,t_2]$ with $Y^t(x,\tau) = \sum_{k\ge0}A_k\frac{x^k}{k!}$, then $\partial_x^k\partial_t^n y(0,\tau) = d_n^k$ for all $(n,k)\in\mathbb{N}^2$ (without knowing* a priori *the existence of such solution). Moreover, we have*

$$|d_n^k| \le C'\frac{(\lambda n+k)!}{R^kR'^{\lambda n}(\lambda n+k+1)^\mu} \qquad \forall k,n\in\mathbb{N}. \tag{4.43}$$

(3) *The application Λ^∞ satisfies the following property: Assume y is a smooth solution of (4.1) such that there exists $(d_n^k)_{(n,k)\in\mathbb{N}^2} = \Lambda^\infty((A_k)_{k\geq 0})$ for some $(A_k)_{k\geq 0} \in (\mathcal{N}_{\tilde{R},C})^N$ so that $\partial_x^k\partial_t^n y(0,\tau) = d_n^k$ for $k = 0,\dots,M-1$ and $n \in \mathbb{N}$. Then $\partial_x^k\partial_t^n y(0,\tau) = d_n^k$ for all $(n,k) \in \mathbb{N}^2$.*

We shall need several lemmas and give the proof of Proposition 4.7 later.

Lemma 4.8. *(see* [17, Lemma A.1]*) For all $k, q \in \mathbb{N}$ and $a \in \{0, ..., k+q\}$, we have*

$$\sum_{\substack{j+p=a\\0\leq j\leq k\\0\leq p\leq q}} \begin{pmatrix} k \\ j \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} k+q \\ a \end{pmatrix}.$$

Lemma 4.9. *For all $\lambda \in [1,+\infty)$ and all $k, j, n, i \in \mathbb{N}$ with $k \geq j$ and $n \geq i$, we have*

$$\begin{pmatrix} k \\ j \end{pmatrix}\begin{pmatrix} n \\ i \end{pmatrix} \leq \lambda \begin{pmatrix} k+\lambda n \\ j+\lambda i \end{pmatrix}. \tag{4.44}$$

Proof of Lemma 4.9. Recall the relationship (see e.g. [32]) between the Gamma function Γ and the Beta function B defined by $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ for Re $x > 0$ and Re $y > 0$:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{4.45}$$

In particular, we have for $x, y \in [0,+\infty)$

$$\begin{aligned}\begin{pmatrix} x+y \\ x \end{pmatrix} &= \frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)}\\ &= \frac{\Gamma(x+y+1)}{\Gamma(x+y+2)}B(x+1,y+1)^{-1} = \left((x+y+1)\int_0^1 t^x(1-t)^y dt\right)^{-1}.\end{aligned}$$

Taking $x = j+\lambda i$, $y = k-j+\lambda(n-i)$, this yields

$$(k+\lambda n+1)\begin{pmatrix} k+\lambda n \\ j+\lambda i \end{pmatrix} = \left(\int_0^1 t^{j+\lambda i}(1-t)^{k-j+\lambda(n-i)}dt\right)^{-1}. \tag{4.46}$$

As the right-hand side of (4.46) is a non-decreasing function of λ, we infer that for $\lambda \geq 1$

$$(k+n+1)\begin{pmatrix} k+n \\ j+i \end{pmatrix} \leq (k+\lambda n+1)\begin{pmatrix} k+\lambda n \\ j+\lambda i \end{pmatrix}.$$

Therefore, using Lemma 4.8,

$$\begin{pmatrix} k \\ j \end{pmatrix}\begin{pmatrix} n \\ i \end{pmatrix} \leq \begin{pmatrix} k+n \\ j+i \end{pmatrix} \leq \frac{k+\lambda n+1}{k+n+1}\begin{pmatrix} k+\lambda n \\ j+\lambda i \end{pmatrix} \leq \lambda \begin{pmatrix} k+\lambda n \\ j+\lambda i \end{pmatrix}.$$

□

The following result gives the algebra property for the mixed Gevrey spaces $G^{1,\lambda}([-1,1]\times[t_1,t_2])$.

Lemma 4.10. *Let* $-\infty < t_1 \le t_2 < \infty$, $(x_0,t_0)\in[-1,1]\times[t_1,t_2]$, $R,R'\in(0,+\infty)$, $q\in\mathbb{N}$, $\lambda\in[1,+\infty)$, $\mu\in(q+2,+\infty)$, $k_0,n_0\in\mathbb{N}$, $C_1,C_2\in(0,+\infty)$, *and* $y_1,y_2\in C^\infty([-1,1]\times[t_1,t_2])$ *be such that*

$$|\partial_x^k\partial_t^n y_i(x_0,t_0)| \le C_i\frac{(\lambda n+k+q)!}{R^kR'^{\lambda n}(\lambda n+k+1)^\mu}\quad \forall i=1,2,\quad \forall k\in\{0,...,k_0\},\ \forall n\in\{0,...,n_0\}. \tag{4.47}$$

Then we have

$$|\partial_x^k\partial_t^n (y_1y_2)(x_0,t_0)| \le K_{q,\mu}C_1C_2\frac{(\lambda n+k+q)!}{R^kR'^{\lambda n}(\lambda n+k+1)^\mu}\quad \forall k\in\{0,...,k_0\},\ \forall n\in\{0,...,n_0\}, \tag{4.48}$$

where

$$K_{q,\mu}:=\lambda 2^{\mu-q+1}(1+q)^{2q}\sum_{j\ge0}\sum_{i\ge0}\frac{1}{(\lambda i+j+1)^{\mu-q}}<\infty.$$

Proof of Lemma 4.10: Using $(\lambda n+k+q)^q\le(1+q)^q\,(1+\lambda n+k)^q$, we obtain

$$(\lambda n+k+q)!=(\lambda n+k)!\prod_{j=1}^q(\lambda n+k+j)\le(\lambda n+k)!(\lambda n+k+q)^q\le(1+q)^q(\lambda n+k)!\,(1+\lambda n+k)^q\,.$$

So, denoting $\widetilde{\mu}:=\mu-q>2$ and $\widetilde{C}_i:=(1+q)^qC_i$, we have

$$|\partial_x^k\partial_t^n y_i(x_0,t_0)| \le \widetilde{C}_i\frac{(\lambda n+k)!}{R^kR'^{\lambda n}(\lambda n+k+1)^{\widetilde{\mu}}},\quad \forall i=1,2,\quad \forall k\in\{0,...,k_0\},\ \forall n\in\{0,...,n_0\}. \tag{4.49}$$

We infer from the Leibniz rule that

$$\begin{aligned}
&|\partial_x^k\partial_t^n(y_1y_2)(x_0,t_0)|\\
&\quad=\left|\sum_{0\le j\le k}\ \sum_{0\le i\le n}\binom{k}{j}\binom{n}{i}(\partial_x^j\partial_t^iy_1)(x_0,t_0)(\partial_x^{k-j}\partial_t^{n-i}y_2)(x_0,t_0)\right|\\
&\quad\le\sum_{0\le j\le k}\ \sum_{0\le i\le n}\binom{k}{j}\binom{n}{i}\frac{\widetilde{C}_1(\lambda i+j)!}{R^jR'^{\lambda i}(\lambda i+j+1)^{\widetilde{\mu}}}\frac{\widetilde{C}_2(\lambda(n-i)+k-j)!}{R^{k-j}R'^{\lambda(n-i)}(\lambda(n-i)+k-j+1)^{\widetilde{\mu}}}\\
&\quad=\frac{\widetilde{C}_1\widetilde{C}_2}{R^kR'^{\lambda n}}(\lambda n+k)!\underbrace{\sum_{0\le j\le k}\ \sum_{0\le i\le n}\frac{\binom{k}{j}\binom{n}{i}\binom{\lambda n+k}{\lambda i+j}^{-1}}{(\lambda i+j+1)^{\widetilde{\mu}}(\lambda(n-i)+k-j+1)^{\widetilde{\mu}}}}_{I}\,.
\end{aligned}$$

We infer from Lemma 4.9 that

$$\binom{k}{j}\binom{n}{i}\binom{\lambda n+k}{\lambda i+j}^{-1}\le\lambda.$$

Finally, by the convexity of $x \to x^{\widetilde{\mu}}$ on $[0, +\infty)$, we have that

$$\sum_{0\leq j\leq k}\sum_{0\leq i\leq n}\frac{(\lambda n+k+2)^{\widetilde{\mu}}}{(\lambda i+j+1)^{\widetilde{\mu}}(\lambda(n-i)+k-j+1)^{\widetilde{\mu}}}=$$

$$\sum_{0\leq j\leq k}\sum_{0\leq i\leq n}\left(\frac{1}{\lambda i+j+1}+\frac{1}{\lambda(n-i)+k-j+1}\right)^{\widetilde{\mu}}\leq$$

$$2^{\widetilde{\mu}-1}\sum_{0\leq j\leq k}\sum_{0\leq i\leq n}\left(\frac{1}{(\lambda i+j+1)^{\widetilde{\mu}}}+\frac{1}{(\lambda(n-i)+k-j+1)^{\widetilde{\mu}}}\right)\leq$$

$$2^{\widetilde{\mu}}\sum_{j\geq 0}\sum_{i\geq 0}\frac{1}{(\lambda i+j+1)^{\widetilde{\mu}}}<\infty,$$

where we used the fact that $\widetilde{\mu} = \mu - q > 2$.
It follows that

$$\begin{aligned} I &\leq 2^{\widetilde{\mu}}\lambda\left(\sum_{j\geq 0}\sum_{i\geq 0}\frac{1}{(\lambda i+j+1)^{\widetilde{\mu}}}\right)\frac{1}{(\lambda n+k+2)^{\widetilde{\mu}}} \\ &= 2^{\mu-q}\lambda\left(\sum_{j\geq 0}\sum_{i\geq 0}\frac{1}{(\lambda i+j+1)^{\mu-q}}\right)\frac{(\lambda n+k+2)^{q}}{(\lambda n+k+2)^{\mu}}, \end{aligned}$$

and hence the proof of Lemma 4.10 is complete once we have noticed that $(\lambda n+k)!(\lambda n+k+2)^q \leq 2(\lambda n + k + q)!$. (We used the fact that $(x + 2)^q \leq 2\prod_{j=1}^{q}(x + j)$ for all $x \geq 0$, $q \in \mathbb{N}^*$.) $\square$

Remark 6. *Lemma 4.10 can also be written as the existence of an application* $\pi : \mathbb{R}^{(k_0+1)\times(n_0+1)} \times \mathbb{R}^{(k_0+1)\times(n_0+1)} \mapsto \mathbb{R}^{(k_0+1)\times(n_0+1)}$ *such that, if for some* $d_1, d_2 \in \mathbb{R}^{(k_0+1)\times(n_0+1)}$ *and two smooth functions* y_1, y_2 *satisfying* $\partial_x^k\partial_t^n y_i(x_0, t_0) = d_{n,i}^k$, $i = 1, 2$ *for all* $k \in \{0, ..., k_0\}$, *for all* $n \in \{0, ..., n_0\}$, *then* $\partial_x^k\partial_t^n(y_1y_2)(x_0, t_0) = (\pi(d_1, d_2))_n^k$. *The definition of* $\pi(d_1, d_2)$ *is given inside of the proof by the Leibniz formula. The Lemma gives then that the estimates*

$$\left|d_{n,i}^k\right| \leq C_i\frac{(\lambda n+k+q)!}{R^kR'^{\lambda n}(\lambda n+k+1)^{\mu}} \quad \forall i=1,2, \quad \forall k\in\{0,...,k_0\},\ \forall n\in\{0,...,n_0\}. \tag{4.50}$$

imply

$$\left|\left(\pi(d,\widetilde{d})\right)_n^k\right| \leq K_{q,\mu}C_1C_2\frac{(\lambda n+k+q)!}{R^kR'^{\lambda n}(\lambda n+k+1)^{\mu}} \quad \forall k\in\{0,...,k_0\},\ \forall n\in\{0,...,n_0\}. \tag{4.51}$$

This equivalent way of writing the same result is consistent with the second part of Proposition 4.7.

We are now ready to complete the proof of Proposition 4.7.

Proof of Proposition 4.7. We will prove the first part of the proposition. The construction of the application Λ in the second part of the proposition will appear along the proof.

Pick any number $\mu > M+1$. We shall prove by induction on $n \in \mathbb{N}$ that

$$|\partial_x^k \partial_t^n y(0,\tau)| \le C_n \frac{(\lambda n + k)!}{R^k R'^{\lambda n} (\lambda n + k + 1)^\mu}, \quad \forall k \in \mathbb{N}, \tag{4.52}$$

where $0 < C_n \le C_{n+1} \le C' < +\infty$. The value of the constant C' will appear along the proof. Assume first that $n = 0$. Recall that $Y^t(x,\tau) = Y_0(x) = \sum_{k=0}^\infty A_k \frac{x^k}{k!}$ with $\|A_k\|_\infty \le C \frac{k!}{\widetilde{R}^k}$. Denote $A_k = (A_k^0, ..., A_k^{N-1})$. Using the fact that $R < \widetilde{R}$, we have that for $0 \le n \le N-1$,

$$|\partial_x^k \partial_t^n y(0,\tau)| = |A_k^n| \le C \frac{k!}{\widetilde{R}^k} \le CD \frac{(\lambda n + k)!}{R^k R'^{\lambda n} (\lambda n + k + 1)^\mu}$$

where

$$D := \left(\sup_{k \in \mathbb{N}, 0 \le n \le N-1} (\frac{R}{\widetilde{R}})^k R'^{\lambda n} (\lambda n + k + 1)^\mu \frac{k!}{(\lambda n + k)!} \right) < \infty.$$

Il follows that (4.52) holds for $0 \le n \le N-1$ for some constants $C_0, ..., C_{N-1} \le CD$.
Assume now that (4.52) is true up to the rank $n-1$ for some $n \ge N$. Let us show that (4.52) is also true at the rank n for some constant $C_n > 0$. Then, by (1.1) and (1.11), we have that

$$\begin{aligned} \partial_x^k \partial_t^n y(0,\tau) &= \partial_x^k \partial_t^{n-N} \sum_{j=0}^M \zeta_j \partial_x^j y(0,\tau) + \sum_{\vec{p} \neq 0} \partial_x^k \partial_t^{n-N} \left(A_{\vec{p}}(x) y^{p_0} (\partial_x y)^{p_1} \cdots (\partial_x^{M-1} y)^{p_{M-1}} \right)(0,\tau) \\ &=: I_1 + I_2. \end{aligned} \tag{4.53}$$

Let us estimate I_1 first. For $0 \le j \le M$, we have that

$$\begin{aligned} |\zeta_j \partial_x^{k+j} \partial_t^{n-N} y(0,\tau)| &\le |\zeta_j| C_{n-N} \frac{(\lambda(n-N) + k + j)!}{R^{k+j} R'^{\lambda(n-N)} (\lambda(n-N) + k + j + 1)^\mu} \\ &\le |\zeta_j| C_{n-N} \frac{(\lambda n + k + j - M)!}{R^{k+j} R'^{\lambda n - M} (\lambda n + k + j - M + 1)^\mu}, \end{aligned}$$

where we have used $\lambda N = M$. It follows that

$$\begin{aligned} |I_1| &\le \left[C_{n-N} \left(\frac{R'}{R} \right)^M + \sum_{j=0}^{M-1} \frac{|\zeta_j| C_{n-N}}{(\lambda n + k + j - M + 1) \cdots (\lambda n + k)} \frac{R'^M}{R^j} \left(\frac{\lambda n + k + 1}{\lambda n + k + j - M + 1} \right)^\mu \right] \\ &\quad \times \frac{(\lambda n + k)!}{R^k R'^{\lambda n} (\lambda n + k + 1)^\mu} \\ &\le \left[C_{n-N} \left(\frac{R'}{R} \right)^M + \sum_{j=0}^{M-1} \frac{|\zeta_j| C_{n-N}}{(\lambda n + j - M + 1) \cdots \lambda n} \frac{R'^M}{R^j} (M+1)^\mu \right] \\ &\quad \times \frac{(\lambda n + k)!}{R^k R'^{\lambda n} (\lambda n + k + 1)^\mu}. \end{aligned} \tag{4.54}$$

where we have used $k, j \ge 0$ and $\lambda n + k \ge M$ so that $\frac{\lambda n + k + 1}{\lambda n + k + j - M + 1} \le M + 1$. Let us estimate I_2. Since $A_{\vec{p}}$ does not depend on t, we have that $\partial_x^k \partial_t^m A_{\vec{p}} = 0$ for $m \ge 1$ and $k \ge 0$. Next, for

$k \geq 0$, we have that

$$|\partial_x^k A_{\vec{p}}(0)| = k!\,|a_{\vec{p},k}| \leq \frac{C_a\ k!}{b^{|\vec{p}|} b_2^k} \leq \frac{\overline{C}}{b^{|\vec{p}|}} \frac{k!}{(k+1)^\mu R^k},$$

for some constant $\overline{C} > 0$ depending on R, b_2, μ, since $R < b_2$.
Note that, still by the iteration assumption (4.52) at step $n-1$, for $0 \leq j \leq M-1$ the function $\partial_x^j y$ satisfies the estimate

$$\begin{aligned} |\partial_x^k \partial_t^m (\partial_x^j y)(0,\tau)| &\leq C_m \frac{(\lambda m + k + j)!}{R^{k+j} R'^{\lambda m} (\lambda m + k + j + 1)^\mu} \\ &\leq \frac{C_m}{R^j} \frac{(\lambda m + k + M - 1)!}{R^k R'^{\lambda m} (\lambda m + k + 1)^\mu} \quad \forall k \in \mathbb{N},\ \forall m \in \{0, ..., n-1\}. \end{aligned}$$

Let $q = M - 1$. Since $\mu > M + 1 = q + 2$, it follows from iterated applications of Lemma 4.10 that

$$\begin{aligned} &\left| \partial_x^k \partial_t^{n-N} \big(A_{\vec{p}} y^{p_0} (\partial_x y)^{p_1} \cdots (\partial_x^{M-1} y)^{p_{M-1}} \big)(0,\tau) \right| \\ &\qquad \leq \overline{C} \left(\frac{K C_{n-N}}{b} \right)^{|\vec{p}|} \frac{(\lambda(n-N) + k + M - 1)!}{R^k R'^{\lambda(n-N)} (\lambda(n-N) + k + 1)^\mu} \prod_{j=0}^{M-1} \frac{1}{(R^j)^{p_j}} \end{aligned} \tag{4.55}$$

where $K = K_{q,\mu} > 0$. If, for some number $\delta \in (0,1)$, we have

$$C_{n-N} \leq \delta \min_{0 \leq j \leq M-1} \frac{b R^j}{K}, \tag{4.56}$$

then

$$\sum_{\vec{p} \neq 0} \left(\frac{K C_{n-N}}{b} \right)^{|\vec{p}|} \prod_{j=0}^{M-1} \frac{1}{(R^j)^{p_j}} \leq \frac{K C_{n-N}}{b} \left((1-\delta)^{-1} + \cdots + (1-\delta)^{-M} \right) \leq \frac{MK}{b(1-\delta)^M} C_{n-N}.$$

(We considered the subcases (1) $p_0 \geq 1$ and $p_1 = \ldots = p_{M-1} = 0$; (2) $p_0 \geq 0$, $p_1 \geq 1$ and $p_2 = \ldots = p_{M-1} = 0$; (3) $p_0 \geq 0$, $p_1 \geq 0$, $p_2 \geq 1$ and $p_3 = \ldots = p_{M-1} = 0$ etc.). Gathering the previous estimates and noticing that $\lambda N = M$, it follows that

$$\begin{aligned} |I_2| &\leq \overline{C} \frac{(\lambda n + k - 1)!}{R^k R'^{\lambda n - M} (\lambda n - M + k + 1)^\mu} \frac{MK}{b(1-\delta)^M} C_{n-N} \\ &\leq \left[\frac{\overline{C} M K R'^M}{b(1-\delta)^M (\lambda n + k)} \left(\frac{\lambda n + k + 1}{\lambda n - M + k + 1} \right)^\mu \right] \times \frac{(\lambda n + k)!}{R^k R'^{\lambda n} (\lambda n + k + 1)^\mu} C_{n-N} && (4.57) \\ &\leq \left[\frac{\overline{C} M K R'^M}{b(1-\delta)^M \lambda n} (M+1)^\mu \right] \times \frac{(\lambda n + k)!}{R^k R'^{\lambda n} (\lambda n + k + 1)^\mu} C_{n-N}. && (4.58) \end{aligned}$$

where we have used again $\frac{\lambda n + k + 1}{\lambda n + k - M + 1} \leq M + 1$. We set $C_n := \max(\lambda_n, 1) C_{n-N}$, where

$$\lambda_n := |\zeta_M| \left(\frac{R'}{R} \right)^M + \sum_{j=0}^{M-1} \frac{|\zeta_j|}{(\lambda n + j - M + 1) \ldots \lambda n} \frac{R'^M}{R^j} (M+1)^\mu + \frac{\overline{C} M K R'^M}{b(1-\delta)^M \lambda n} (M+1)^\mu.$$

Then (4.52) holds. Since $\left(\frac{R'}{R}\right)^M < 1$ and $|\zeta_M| = 1$, it is clear that $|\lambda_n| \le 1$ for $n \gg 1$, say for $n \ge n_0 \ge N$. This yields $C_n \le C_{n-N}$ for all $n \ge n_0 \ge N$, provided that (4.56) holds for $n \le n_0$. To ensure (4.56) for $n \le n_0$, it is sufficient to choose C small enough (or, equivalently, $\widetilde{C}$ small enough). The proof by induction of (4.52) is achieved. The proof of the first part of Proposition 4.7 is complete. For the second part of Proposition 4.7, we follow the proof of the first part and define the coefficients d_n^k by induction on n.

For $n = 0, \dots, N-1$ and $k \in \mathbb{N}$, if we denote $A_k = (A_k^0, A_k^1, ..., A_k^{N-1})$, then we have $\partial_x^k \partial_t^n y(0,\tau) = A_k^n$ for any solution satisfying (4.41). So we are led to define $d_n^k := A_k^n$.
For $n \ge N$, following the proof of the previous estimates, we obtain using the notations introduced in (4.53) and Leibniz' rule

$$I_1 = \sum_{j=0}^{M} \zeta_j \partial_x^{k+j} \partial_t^{n-N} y(0,\tau) \tag{4.59}$$

$$I_2 = \sum_{\vec{p}\neq 0} \sum_{k_1+\cdots+k_{M+1}=k} \sum_{n_1+\cdots+n_{M+1}=n-N} \frac{k!}{k_1!\cdots k_{M+1}!} \frac{(n-N)!}{n_1!\cdots n_{M+1}!} \Big(\partial_x^{k_1}\partial_t^{n_1}(A_{\vec{p}})\, \partial_x^{k_2}\partial_t^{n_2}(y^{p_0})\, \partial_x^{k_3}\partial_t^{n_3}(\partial_x y)^{p_1} \cdots \partial_x^{k_{M+1}}\partial_t^{n_{M+1}}(\partial_x^{M-1}y)^{p_{M-1}}\Big)(0,\tau) \tag{4.60}$$

with, for $0 \le i \le M-1$,

$$\partial_x^{k_{i+2}}\partial_t^{n_{i+2}}(\partial_x^i y)^{p_i}(0,\tau) = \sum_{l_1+\cdots+l_{p_i}=k_{i+2}} \sum_{m_1+\cdots+m_{p_i}=n_{i+2}} \frac{k_{i+2}!}{l_1!\cdots l_{p_i}!} \frac{n_{i+2}!}{m_1!\cdots m_{p_i}!} \partial_x^{l_1+i}\partial_t^{m_1} y(0,\tau)\cdots \partial_x^{l_{p_i}+i}\partial_t^{m_{p_i}} y(0,\tau). \tag{4.61}$$

We define some $\widetilde{I}_1$ and $\widetilde{I}_2$ by replacing in (4.59) $\partial_x^{k+j}\partial_t^{n-N} y(0,\tau)$ by d_{n-N}^{k+j}, and in (4.61) $\partial_x^{l_j+i}\partial_t^{m_j} y(0,\tau)$ by $d_{m_j}^{l_j+i}$, where $m_j \le n_{i+2} \le n-N$. For instance, $\widetilde{I}_1$ writes

$$\widetilde{I}_1 = \sum_{j=0}^{M} \zeta_j d_{n-N}^{k+j}$$

and $\widetilde{I}_2$ is defined similarly. We see that

$$d_n^k := \widetilde{I}_1 + \widetilde{I}_2$$

is uniquely defined in terms of the d_m^l's for $m \le n-N$, $l \in \mathbb{N}$. Thus the sequence $(d_n^k)_{(n,k)\in\mathbb{N}^2}$ can be defined by induction on n and the same estimates as before allow us to obtain (4.43), see also Remark 6.
For the third part of Proposition 4.7, we prove by iteration on k that $\partial_x^k\partial_t^n y(0,\tau) = d_n^k$ for all $n \in \mathbb{N}$. By assumption, the result is true for all $k = 0, \dots, M-1$. We assume that the result is true until the rank $k+M-1$ and we prove it at rank $k+M$. Let $n \ge N$. We know that we

have

$$\partial_x^k \partial_t^n y(0,\tau) = \sum_{j=0}^{M} \zeta_j \partial_x^{k+j} \partial_t^{n-N} y(0,\tau) + I_2, \tag{4.62}$$

where I_2 is defined by (4.60) and (4.61). The d_n^k have been defined by iteration on n by the formula

$$d_n^k = \sum_{j=0}^{M} \zeta_j d_{n-N}^{k+j} + \widetilde{I}_2 \tag{4.63}$$

where $\widetilde{I}_2$ has been obtained by replacing $\partial_x^{l_j+i}\partial_t^{m_j} y(0,\tau)$ by $d_{m_j}^{l_j+i}$ in the formula of I_2. Since it only involves some terms with $0 \le i \le M-1$ and $0 \le l_j \le k$, we have $l_j + i \le k + M - 1$ for these terms and the iteration property gives $\partial_x^{l_j+i}\partial_t^{m_j} y(0,\tau) = d_{m_j}^{l_j+i}$. In particular, $\widetilde{I}_2 = I_2$. So, (4.63) can be written as

$$\zeta_M d_{n-N}^{k+M} = d_n^k - \sum_{j=0}^{M-1} \zeta_j d_{n-N}^{k+j} - I_2.$$

Again, for $0 \le j \le M-1$, the iteration assumption gives $d_{n-N}^{k+j} = \partial_x^{k+j}\partial_t^{n-N} y(0,\tau)$ and $d_n^k = \partial_x^k \partial_t^n y(0,\tau)$. So, we obtain

$$\zeta_M d_{n-N}^{k+M} = \partial_x^k \partial_t^n y(0,\tau) - \sum_{j=0}^{M-1} \zeta_j \partial_x^{k+j}\partial_t^{n-N} y(0,\tau) - I_2.$$

After comparison with (4.62) and since $\zeta_M \neq 0$, we obtain $d_{n-N}^{k+M} = \partial_x^{k+M}\partial_t^{n-N} y(0,\tau)$. Since $n \ge N$ is arbitrary, it gives the result at step $k+M$. □

Remark 7. *We note that even if we do not know* a priori *whether Y_0 will give rise to a solution, the algorithm is still well-defined. Our proof will show* a posteriori *that any initial data Y_0 which is analytic (with an appropriate radius) and small enough will produce a solution making this detail not so relevant. But this fact is not obvious at this moment of the proof.*

Note that at that moment, both Proposition 4.1 and Proposition 4.7 seem to give two relations between the space derivatives of Y_0 and the time derivatives of an eventual solution. If there exists a solution y starting from Y_0 at time $t = 0$, that relation should be unique (but this claim is not proved yet).

The following result will show the existence of a solution. It will allow us clarifying the relation between the $d_{n,k}$ and the functions J_n^k in Corollary 4.14. There is likely a direct way to prove this relation, but it might be quite computational. The difference between Lemma 4.2 and Proposition 4.7 is only the order in which we apply time and space derivatives to the equation.

Proposition 4.11 (Existence of solution without boundary condition)**.** *Let $-\infty < t_1 \le \tau \le t_2 < +\infty$ and $f = f(x, y_0, y_1, ..., y_{M-1})$ be as in* (1.7)-(1.8) *with the coefficients $a_{\vec{p},r}$, $(\vec{p}, r) \in \mathbb{N}^{M+1}$, satisfying* (1.9)-(1.10). *Assume in addition that $b_2 > \hat{R} := 4N\lambda e^{(\lambda e)^{-1}}$. Let $\widetilde{R} > \hat{R}$. Then* *there exists some number* $\widetilde{C} > 0$ *such that for any* $C \in (0, \widetilde{C}]$ *and any numbers* R_L *with* $\hat{R} < R_L < \min(\widetilde{R}, b_2)$ *there exists a number* $C'' = C''(C, \widetilde{R}, R_L) > 0$ *with* $\lim_{C \to 0^+} C''(C, \widetilde{R}, R_L) = 0$ *such that for any* $Y_0 \in (\mathcal{R}_{\widetilde{R},C})^N$*, we can pick a function* $y \in G^{1,\lambda}([-1,1] \times [t_1, t_2])$ *satisfying* (4.1) *for* $(x,t) \in [-1,1] \times [t_1, t_2]$ *and*

$$Y^t(x, \tau) = Y_0(x) = \sum_{k=0}^{\infty} A_k \frac{x^k}{k!}, \quad \forall x \in [-1, 1], \tag{4.64}$$

and such that for all $t \in [t_1, t_2]$

$$\|\partial_t^n Y^x(0,t)\|_\infty \le C''(n!)^\lambda \left(\frac{D|\zeta_M|^{1/M}}{R_L} \right)^{n\lambda}, \tag{4.65}$$

with $D := \lambda e^{(\lambda e)^{-1}}$.

Proof. We assume first that $|\zeta_M| = 1$, dealing with the general case at the end of the proof. Note that the scaling in time affects only (4.65).

Let $\hat{R} := 4N\lambda e^{(\lambda e)^{-1}}$, we will need some intermediate radii R, R', R'' with $\hat{R} < R_L < R'' < R' < R < \min(\widetilde{R}, b_2)$. Pick $\widetilde{C}, C$ as in Proposition 4.7, and pick any $Y_0 \in (\mathcal{R}_{\widetilde{R},C})^N$. If a function y as in Proposition 4.11 does exists, then both sequences of numbers

$$d_n^k \;\; := \;\; \partial_t^n \partial_x^k y(0, \tau), \quad n \in \mathbb{N}, \quad k \in \mathbb{N}$$

can be computed inductively in terms of the coefficients $A_k = \partial_x^k Y_0(0)$, $k \in \mathbb{N}$, according to Proposition 4.7, that is $(d_n^k)_{(n,k)\in\mathbb{N}^2} = \Lambda^\infty (A_k)_{k\in\mathbb{N}}$. Note that the sequence $(d_n^k)_{(n,k)\in\mathbb{N}^2}$ can be defined in terms of the coefficients A_k's, even if the existence of the solution y is not yet established, according to Proposition 4.7 (2). Furthermore, it follows from Proposition 4.7 that we have for some constant $C' = C'(C, R, R') > 0$,

$$|d_n^k| \;\; \le \;\; C' \frac{(\lambda n + k)!}{R^k R'^{\lambda n}}, \quad \forall n \in \mathbb{N}, \quad \forall k \in \{0, \dots, M-1\}.$$

Since $R'' \in (\hat{R}, R')$, there exists some constant $P = P(R, R', R'') > 0$ such that we have also

$$|d_n^k| \le C' P \frac{(\lambda n)!}{(R'')^{\lambda n}}, \quad \forall n \in \mathbb{N}, \quad \forall k \in \{0, \dots, M-1\}.$$

The following lemma is a consequence of [25, Proposition 3.6]. The proof that [25, Proposition 3.6] implies Lemma 4.12 will be done later.

Lemma 4.12. *Let* $\lambda > 1$*. Let* $(d_q)_{q \ge 0}$ *be a sequence of real numbers such that*

$$|d_q| \le C H^q (\lambda q)! \quad \forall q \ge 0$$

for some $H > 0$ *and* $C > 0$*. Then for all* $\tilde{H} > e^{e^{-1}} H$ *there exists a function* $f \in C^\infty(\mathbb{R})$ *such that*

$$f^{(q)}(0) \;\; = \;\; d_q \quad \forall q \ge 0, \tag{4.66}$$

$$|f^{(q)}(t)| \;\; \le \;\; C \tilde{H}^q (\lambda q)! \quad \forall q \ge 0, \; \forall t \in \mathbb{R}. \tag{4.67}$$

Pick $H := 1/(R'')^\lambda$ and $H_L := e^{e^{-1}}/(R_L)^\lambda$. Since $\hat{R} < R_L < R''$, we have $e^{e^{-1}}H < H_L < 1/(4N\lambda)^\lambda$. Then by Lemma 4.12, there exist M functions $h_0, h_1, \dots, h_{M-1} \in G^\lambda([t_1,t_2])$ such that for $k = 0, \dots, M-1$,

$$h_k^{(n)}(\tau) = d_n^k, \quad n \geq 0 \tag{4.68}$$

$$|h_k^{(n)}(t)| \leq C'PH_L^n(\lambda n)!, \quad n \geq 0,\ t \in [t_1,t_2]. \tag{4.69}$$

It follows at once from Stirling's formula that $(\lambda n)! \leq C_s \lambda^{\lambda n}(n!)^\lambda$ for some universal constant $C_s > 0$, so that for $k = 0, \dots, M-1$,

$$|h_k^{(n)}(t)| \leq C'PC_s(\lambda^\lambda H_L)^n (n!)^\lambda, \quad n \geq 0,\ t \in [t_1,t_2], \tag{4.70}$$

Note that $\lambda^\lambda H_L < 1/(4N)^\lambda$. So, if C is sufficiently small, then C' is as small as desired, and it follows then from Theorem 3.1 that we can pick a function $y \in G^{1,\lambda}([-1,1]\times[t_1,t_2])$ satisfying (3.1) with $k_i := h_i$ for $0 \leq i \leq M-1$. In particular, for all $n \in \mathbb{N}$ and $k = 0, \dots, M-1$, we have $\partial_t^n \partial_x^k y(0,\tau) = h_k^{(n)}(\tau) = d_n^k$. Using the third Item of Proposition 4.7, we infer that $\partial_t^n \partial_x^k y(0,\tau) = d_n^k$ for $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Moreover, we can check in the proof of Proposition 4.7 (case $0 \leq n \leq N-1$) that if $(d_n^k)_{(n,k)\in\mathbb{N}^2} = \Lambda^\infty(A_k)_{k\in\mathbb{N}}$, then $d_n^k = A_k^n$ for $k \in \mathbb{N}$ and $0 \leq n \leq N-1$. In particular, $\partial_t^n\partial_x^k y(0,\tau) = A_k^n = \partial_x^k y_0^n(0)$ for $k \in \mathbb{N}$ and $0 \leq n \leq N-1$, where $Y_0 = (y_0^0, y_0^1, \dots, y_0^{N-1})$, and hence (4.64) holds. Since $Y^x(0,t) = (h_0(t), \dots, h_{M-1}(t))$ by construction, the estimate (4.65) follow from (4.70) with $C'' := C'PC_s$ and $\lambda^\lambda H_L = \left(\lambda e^{(\lambda e)^{-1}}/R_L\right)^\lambda$. The proof of Proposition 4.11 is complete for the case $|\zeta_M| = 1$.

In the general case, assuming $\tau = 0$ without loss of generality, we proceed as in Remark 4 and define $\widetilde{P} = |\zeta_M|^{-1}P$ and $\widetilde{f} = |\zeta_M|^{-1}f$ for which the result is proved for any interval in time. We have therefore a solution $\widetilde{y}$ of $\partial_t^N \widetilde{y} = |\zeta_M|^{-1}P\widetilde{y} + |\zeta_M|^{-1}f(x,\widetilde{Y^x})$ on $[-1,1]\times[|\zeta_M|^{-1}t_1, |\zeta_M|^{-1}t_2]$ with $\widetilde{Y}^t(x,0) = Y_0(x)$. Moreover, $\widetilde{y}$ satisfies (4.65) with $|\zeta_M| = 1$. Now, we define $y(x,t) := \widetilde{y}(x, |\zeta_M|^{1/N}t)$ which is a solution of $\partial_t^N y = Py + f(x,Y^x)$ on $[-1,1]\times[t_1,t_2]$ with $Y^t(x,0) = Y_0(x)$. By scaling, Y^x satisfies

$$|\partial_t^n Y^x(0,t)| \leq |\zeta_M|^{n/N}\left|(\partial_t^n \widetilde{Y^x})(0, |\zeta_M|^N t)\right| \leq |\zeta_M|^{n\lambda/M} C''(n!)^\lambda \left(\frac{D}{R_L}\right)^{n\lambda}. \tag{4.71}$$

□

Proof of Lemma 4.12. We want to apply [25, Proposition 3.6] (stated below in Proposition 4.13) with the choice $a_0 = 1$ and k becoming $k-1$ so that $M_q = (\lambda q)!$ $a_k = \frac{(\lambda(k-1))!}{(\lambda k))!} = \Gamma(\lambda)^{-1}B(\lambda(k-1)+1,\lambda) = \Gamma(\lambda)^{-1}\int_0^1 t^{\lambda(k-1)}(1-t)^{\lambda-1}dt$ for $k \geq 1$. All the terms being positive, we obtain for

$p \geq 1$

$$\begin{aligned}
\sum_{k>p} a_k &= \Gamma(\lambda)^{-1} \int_0^1 (1-t)^{\lambda-1} (\sum_{k>p} t^{\lambda(k-1)}) dt \\
&= \Gamma(\lambda)^{-1} \int_0^1 (1-t)^{\lambda-1} \frac{t^{\lambda p}}{1-t^\lambda} dt \leq \Gamma(\lambda)^{-1} \int_0^1 (1-t)^{\lambda-2} t^{\lambda p} dt \\
&= \frac{\lambda p}{\Gamma(\lambda)(\lambda-1)} \int_0^1 (1-t)^{\lambda-1} t^{\lambda p-1} dt \leq \frac{\lambda}{(\lambda-1)} p a_p < +\infty,
\end{aligned}$$

where we have used twice $t^\lambda \leq t$ for $t \in [0,1]$ and $\lambda > 1$, and performed an integration by parts. In particular, the three conditions of Proposition 4.13 are fulfilled with $A := \frac{\lambda}{\lambda-1} + 1$ and $M_q = (\lambda q)!$. This completes the proof of Lemma 4.12. □

For the convenience of the reader, we state the following proposition that we used before to construct the suitable Gevrey functions.

Proposition 4.13 (Proposition 3.6 of [25]). *Pick any sequence $(a_q)_{q\in\mathbb{N}}$ satisfying*

- $1 = a_0 \geq a_1 \geq a_2 \geq \cdots > 0$
- $\sum_{k\geq 1} a_k < +\infty$
- $p a_p + \sum_{k>p} a_k \leq A p a_p, \quad \forall p \geq 1,$

for some constant $A \in (0, +\infty)$. Let $M_q := (a_0 \cdots a_q)^{-1}$ for $q \geq 0$. Then for any sequence of real numbers $(d_q)_{q\geq 0}$ such that

$$|d_q| \leq C H^q M_q, \quad \forall q \geq 0$$

for some $H > 0$ and $C > 0$, and for any $\widetilde{H} > e^{e^{-1}} H$, there exists a function $f \in C^\infty(\mathbb{R})$ such that

$$\begin{aligned}
f^{(q)}(0) = d_q \quad \forall q \geq 0, \\
|f^{(q)}(x)| \leq C \widetilde{H}^q M_q \quad \forall q \geq 0, \forall x \in \mathbb{R}.
\end{aligned}$$

Corollary 4.14. *Let Y_0 satisfying the assumptions of Proposition 4.7 and let $(d_n^k)_{(n,k)\in\mathbb{N}^2}$ be the sequence introduced in Proposition 4.7 (2). Then we have the relationship*

$$d_n^k = J_n^k(0, A_0, A_1, ..., A_{M[\frac{n}{N}]+k}) \tag{4.72}$$

where the J_n^k are the functions defined in (4.28).
Moreover, if $Y_0 \in \mathcal{C}$ and if we set $D_n := (d_n^0, d_n^1, \ldots, d_n^{M-1})$, then $BD_n = 0$ for all $n \in \mathbb{N}$.

Proof. Let y be a solution given by Proposition 4.7. Then (4.72) holds since both sides of the equality agree with $\partial_t^n \partial_x^k y(0, \tau)$, according to (4.28) for the right-hand side and to Proposition 4.7 (2) for the left-hand side. Moreover if $Y_0 \in \mathcal{C}$, then $BD_n = BJ_n = 0$ by (1.17). □

Proposition 4.15 (Existence of solution with boundary condition). *Consider the same assumptions and constants as in Proposition 4.11. Then, for any $Y^0 \in \left(\mathcal{R}_{\tilde{R},C}\right)^N \cap \mathcal{C}$, we can find a solution of (1.1)-(1.2) for $(x,t) \in [-1,1] \times [t_1, t_2]$ satisfying (4.64) and (4.65).*

Proof. The proof is similar to those of Proposition 4.11. The modifications to ensure the boundary conditions are the following.
The sequence $(D_n)_{n\geq 0}$ defined in Corollary 4.14 satisfies $BD_n = 0$ for all $n \in \mathbb{N}$. We can then proceed as in the proof of Proposition 4.11, replacing Lemma 4.12 by Lemma 4.16 (see below). The advantage of using Lemma 4.16 is that the condition $BH_0(t) = 0$ is satisfied by the function $H_0 = (h_0, ..., h_{M-1}) \in G^\lambda([t_1, t_2])^M$ it provides. Then, using Theorem 3.1 again with that boundary condition H_0, the equation (3.1) gives $Y^x(0, t) = H_0(t)$, so that the boundary condition $BY^x(0, t) = 0$ is satisfied, as expected. This gives a solution of the system (1.1)-(1.2). The conditions (4.64) and (4.65) are fulfilled for the same reasons as in Proposition 4.11. □

Lemma 4.16. *Let $(D_q)_{q\geq 0}$ be a sequence in $\mathbb{C}^M$ such that*

$$\begin{aligned} \|D_q\|_\infty &\leq CH^q(\lambda q)! \quad \forall q \geq 0, \\ BD_q &= 0 \quad \forall q \geq 0 \end{aligned}$$

for some $H > 0$ and $C > 0$. Then for all $\tilde{H} > e^{e^{-1}} H$, there exists a function $F \in C^\infty(\mathbb{R})^M$ such that

$$F^{(q)}(0) = D_q \quad \forall q \geq 0, \tag{4.73}$$

$$\|F^{(q)}(t)\|_\infty \leq C\tilde{H}^q(\lambda q)! \quad \forall q \geq 0, \ \forall t \in \mathbb{R}, \tag{4.74}$$

$$BF^{(q)}(t) = 0 \quad \forall q \geq 0, \ \forall t \in \mathbb{R}. \tag{4.75}$$

Proof. Let $e_i \in \mathbb{C}^M$, $i = 1, \cdots, dim(\ker(B))$, be the vectors of a basis of $\ker(B)$. In particular, we can write $D_q = \sum_i D_{q,i} e_i$. By assumption, the real sequence $(D_{q,i})_{q\in\mathbb{N}}$ satisfies the assumptions of Lemma 4.12, so that there are some functions $f_i \in C^\infty(\mathbb{R})$ satisfying $f_i^{(q)}(0) = D_{q,i}$ and $|f_i^{(q)}(t)| \leq C\tilde{H}^q(\lambda q)!$ for all $q \geq 0$, $t \in \mathbb{R}$. The function $F = \sum_i f_i e_i$ satisfies the requested properties. □

We also infer from the existence of solutions given by Proposition 4.11 the following uniqueness result for the functions J_l^k.

Lemma 4.17. *Let $l \in \mathbb{N}$. Then there exists some number $\varepsilon > 0$ such that if two applications $J_l, \widetilde{J}_l : [-1, 1] \times (\mathbb{R}^N)^{m(l)+1} \to \mathbb{R}^M$ satisfy (1.16) for any smooth solution y of (4.1), then they coincide on $[-1, 1] \times B(0, \varepsilon)$. In particular, if both functions are analytic, then they are equal.*

Proof. Since (1.16) is assumed to be satisfied, it is sufficient to prove that for any $(x_0, Y_0, Y_1, \cdots, Y_{m(l)}) \in [-1, 1] \times B(0, \varepsilon)$, there exists one solution of $y \in C^\infty([-1, 1] \times [t_1, t_2])$ solution of (4.1) with $(Y_0, Y_1, \cdots, Y_{m(l)}) = (Y^t, \partial_x Y^t(x_0, \tau), ..., \partial_x^{m(l)} Y^t(x_0, \tau)))$. Thanks to Proposition 4.11, it suffices to find $Y^0 \in (\mathcal{R}_{\tilde{R},C})^N$ so that $(Y^0(x_0), \cdots, \partial_x^{m(l)} Y^0(x_0)) = (Y_0, Y_1, \cdots, Y_{m(l)})$. This is simple analytic interpolation if ε is chosen small enough with respect to $\tilde{R}, C$. □

5. Proofs of Theorem 1.2 and Proposition 1.3

Proof of Theorem 1.2. Let $R > \hat{R} := 4N\lambda e^{(\lambda e)^{-1}}$ and let $\widetilde{C}$ be the constant given by Proposition 4.11. Let $Y^0, Y^1 \in (\mathcal{R}_{R,\widetilde{C}})^N \cap \mathcal{C}$. We infer from Proposition 4.15 applied with $[t_1, t_2] = [0, T]$

and $\tau = 0$ (resp. $\tau = T$) the existence of two functions $\hat{y}, \tilde{y} \in G^{1,\lambda}([-1,1]\times[0,T])$ satisfying (1.1)-(1.2) and such that

$$\hat{Y}^t(x,0) = Y^0(x) \ \text{ and } \ \tilde{Y}^t(x,T) = Y^1(x), \quad \forall x \in [-1,1].$$

Let $\rho \in C^\infty(\mathbb{R})$ be such that

$$\rho(t) = \begin{cases} 1 & \text{if } t \le \frac{T}{4}, \\ 0 & \text{if } t \ge \frac{3T}{4}, \end{cases}$$

and $\rho_{|[0,T]} \in G^{\frac{\lambda+1}{2}}([0,T])$. (Note that $(\lambda+1)/2 > 1$.) Let

$$K_0(t) \ = \ \rho(t)\hat{Y}^x(0,t) + (1-\rho(t))\tilde{Y}^x(0,t), \quad t \in [0,T].$$

Then $K_0 \in G^\lambda([0,T])^M$ by [25, Lemma 3.7], and assuming $Y^0, Y^1 \in (\mathcal{R}_{R,\widehat{C}})^N \cap \mathcal{C}$ with $0 < \widehat{C} < \widetilde{C}$, $\widehat{C}$ small enough, we can assume that (3.2) is satisfied. It follows then from Theorem 3.1 that there exists a solution $y \in G^{1,\lambda}([-1,1]\times[0,T])$ of (3.1). Then y satisfies (1.1)-(1.2) together with $Y^t(x,T) = Y^1(x)$ for $x \in [-1,1]$.
Indeed, since $\rho(t) = 0$ for $t > 3T/4$, we have

$$\partial_t^n Y^x(0,T) = K_0^{(n)}(T) \ = \ \partial_t^n \tilde{Y}^x(0,T), \quad \forall n \in \mathbb{N},$$

It follows then from Proposition 4.1 that $\partial_x^n Y^t(0,T) = \partial_x^n \tilde{Y}^t(0,T) = \partial_x^n Y^1(0)$ for all $n \in \mathbb{N}$, and hence $Y^t(.,T) = Y^1$. We can prove in the same way that $Y^t(.,0) = Y^0$. The proof of Theorem 1.2 is achieved. □

Let us now proceed to the proof of Proposition 1.3 describing the compatibility set in cases where parity arguments can be used.

Proof of Proposition 1.3. We first consider the Dirichlet case. We will give the modifications of the proof for the Neumann case after.
Consider first the Dirichlet case when $BY^x(0,t) = 0$ reduces to $\partial_x^{2j} y(0,t) = 0$ for $2j \le M-1$. It means that, following the definition (1.17) and denoting J_l^i the ith component of the vector $J_l \in \mathbb{R}^M$, we have

$$\mathcal{C} = \left\{ Y_0 \in C^\infty([0,1])^N; \quad J_l^{2j}(0, Y_0, \partial_x Y_0, ..., \partial_x^{m(l)} Y_0)_{x=0} = 0, \quad \forall 0 \le 2j \le M-1, \forall l \in \mathbb{N} \right\}$$

So, we need to show $\mathcal{C} = \widetilde{\mathcal{C}}$, where

$$\begin{aligned} \widetilde{\mathcal{C}} \ &:= \ \left\{ Y_0 = (y_0, y_1, ..., y_{N-1}) \in C^\infty([0,1])^N; \quad \partial_x^{2j} y_l(0) = 0, \quad \forall j \in \mathbb{N}, \ \forall l = 0, ..., N-1 \right\} \\ &= \ \left\{ Y_0 \in C^\infty([0,1])^N; \quad \partial_x^{2j} Y_0(0) = 0, \forall j \in \mathbb{N} \right\}. \end{aligned}$$

We first prove that $\widetilde{\mathcal{C}} \subset \mathcal{C}$.
The set $\widetilde{\mathcal{C}}$ is the set of smooth functions that admit a smooth odd extension to $[-1,1]$. We still denote $Y_0 \in C^\infty([-1,1])^N$ this extension. We use the notation $(Y_0)_k^x$ for the vector $(Y_0)_k^x = (Y_0, \partial_x Y_0, ..., \partial_x^k Y_0) \in E_{k,N-1}$. A vectorial variant of property (4.8) is then

$$I((Y_0)_k^x)(x) = (Y_{0,-})_k^x(-x) \tag{5.1}$$

where $Y_{0,-}$ is the reflected application $Y_{0,-}(x) = Y_0(-x)$.

The derivatives at zero are not modified, so we need to prove that $J_l^{2j}(0,(Y_0)^x_{m(l)})_{x=0} = 0$ for this extension. Using Lemma 4.6 and property (5.1)

$$J_l^k(-x,-(Y_{0,-})^x_{m(l)}(-x)) = J_l^k(-x,-I((Y_0)^x_{m(l)})(x)) = (-1)^{k+1}J_l^k(x,(Y_0)^x_{m(l)}(x)). \tag{5.2}$$

But since Y_0 is odd, $Y_{0,-} = -Y_0$ and $(Y_{0,-})^x_{m(l)} = -(Y_0)^x_{m(l)}$, which gives

$$J_l^k(-x,-(Y_{0,-})^x_{m(l)})(-x))) = J_l^k(-x,(Y_0)^x_{m(l)}(-x)).$$

In particular, thanks to (5.2), the function $x \mapsto J_l^{2j}(x,(Y_0)^x_{m(l)}(x))$ is odd and $J_l^{2j}(0,(Y_0)^x_{m(l)}(0)) = 0$.

Next we prove that $\mathcal{C} \subset \widetilde{\mathcal{C}}$. Let $Y_0 \in \mathcal{C}$. We prove by induction on k the following equivalent fact: $I((Y_0)^x_k) = -(Y_0)^x_k$ at $x = 0$.

For $k \leq M-1$, we notice from the proof of Lemma 4.2 that for $0 \leq l < N$, we have $H_l^k = 0$ so that for $Y_0 = (y_0, y_1, ..., y_{N-1})$, we have $J_l^k(x, Y_0, \ldots, \partial_x^k Y_0) = \partial_x^k y_l$. So the assumption $J_l^k(x, Y_0, \ldots, \partial_x^k Y_0)_{x=0} = 0$ for k even, $k \leq M-1$ implies $\partial_x^k y_l = 0$ for k even, $k \leq M-1$.

Now, assume that $I((Y_0)^x_{2k-1}) = -(Y_0)^x_{2k-1}$ at $x = 0$ for some $k \in \mathbb{N}$ with $2k-1 \geq M-1$. Write $2k = Mn + i$ with $0 \leq i < M$ (necessarily even) and pick any $l = Nn + j$, where j is arbitrary with $0 \leq j < N$.

By (4.31), since i is even, we have $H_l^i(0,-I(Y)) = -H_l^i(0,Y)$ for all Y. We have by the inductive hypothesis $I((Y_0)^x_{Mn+i-1}) = -(Y_0)^x_{Mn+i-1}$ at $x = 0$, so that $H_l^i(0,(Y_0)^x_{Mn+i-1}(0)) = -H_l^i(0,(Y_0)^x_{Mn+i-1}(0))$, and hence $H_l^i(0,(Y_0)^x_{Mn+i-1}(0)) = 0$. Now, using the definition (4.28) of J_l^i and the assumption $Y_0 \in \mathcal{C}$ which gives $J_l^i(x,(Y_0)^x_{Mn+i})_{x=0} = 0$ (since i is even), we obtain $P^n \partial_x^i y_j = 0$ if we denote $Y_0 = (y_0, \ldots, y_{N-1})$. By the structure of P, this gives the result at step $2k = Mn + i$ since $0 \leq j < N$ is arbitrary. This implies that the result is also true at step $2k+1$.

For the Neumann case, we modify the proof as follows.
This time, we are in the case when $BY^x(0,t) = 0$ reduces to $\partial_x^{2j+1} y(0,t) = 0$ for $2j+1 \leq M-1$, and using (1.17), we have

$$\mathcal{C} = \left\{ Y_0 \in C^\infty([0,1])^N; \quad J_l^{2j+1}(0, Y_0, \partial_x Y_0, ..., \partial_x^{m(l)} Y_0)_{x=0} = 0, \quad \forall 0 \leq 2j+1 \leq M-1, \forall l \in \mathbb{N} \right\}.$$

So, we have to show that $\mathcal{C} = \widetilde{\mathcal{C}}$ with

$$\begin{aligned} \widetilde{\mathcal{C}} &:= \left\{ Y_0 = (y_0, y_1, ..., y_{N-1}) \in C^\infty([0,1])^N; \quad \partial_x^{2j+1} y_l(0) = 0, \quad \forall j \in \mathbb{N}, \ \forall l = 0, ..., N-1 \right\} \\ &= \left\{ Y_0 \in C^\infty([0,1])^N; \quad \partial_x^{2j+1} Y_0(0) = 0, \forall j \in \mathbb{N} \right\}. \end{aligned}$$

We first prove that $\widetilde{\mathcal{C}} \subset \mathcal{C}$. In this case, the set $\widetilde{\mathcal{C}}$ is the set of smooth functions that admit a smooth even extension to $[-1,1]$. So we need to prove that $J_l^{2j+1}(0, Y_0, \partial_x Y_0, ..., \partial_x^{m(l)} Y_0)_{x=0} = 0$ for this extension. Using the second part of Lemma 4.6 and property (5.1)

$$J_l^k(-x,(Y_{0,-})^x_{m(l)}(-x)) = J_l^k(-x, I((Y_0)^x_{m(l)})(x)) = (-1)^k J_l^k(x,(Y_0)^x_{m(l)}(x)). \tag{5.3}$$

But since Y_0 is even, $Y_{0,-} = Y_0$ and $(Y_{0,-})^x_{m(l)} = (Y_0)^x_{m(l)}$, which gives this time

$$J^k_l(-x, (Y_{0,-})^x_{m(l)}(-x)) = J^k_l(-x, (Y_0)^x_{m(l)}(-x)).$$

In particular, thanks to (5.3), the function $x \to J^{2j+1}_l(x, (Y_0)^x_{m(l)}(x))$ is odd and

$$J^{2j+1}_l(0, (Y_0)^x_{m(l)}(0)) = 0.$$

In order to prove that $\mathcal{C} \subset \widetilde{\mathcal{C}}$, we prove by induction on k that for all $k \in \mathbb{N}$, $I((Y_0)^x_k) = (Y_0)^x_k$ at $x = 0$.
For $k \leq M - 1$, we still have $H^k_l = 0$ and the same arguments as in the Dirichlet case gives $\partial^k_x y_l = 0$ for k odd in the range we consider.

Assume that $I((Y_0)^x_{2k}) = (Y_0)^x_{2k}$ at $x = 0$ for some $k \in \mathbb{N}$ with $2k \geq M-1$. Write $2k+1 = Mn+i$ with $0 \leq i < M$ (necessarily odd), and pick $l = Nn + j$ where j is arbitrary with $0 \leq j < N$.

By (4.33), since i is odd, we have $H^i_l(0, I(Y)) = -H^i_l(0, Y)$ for all Y. But we have from the inductive hypothesis $I((Y_0)^x_{Mn+i-1}) = (Y_0)^x_{Mn+i-1}$ at $x = 0$, so that $H^i_l(0, (Y_0)^x_{Mn+i-1}) = -H^i_l(0, (Y_0)^x_{Mn+i-1})$, and hence $H^i_l(0, (Y_0)^x_{Mn+i-1}) = 0$. Now, using the definition (4.28) of J^i_l and the assumption $J^i_l(x, (Y_0)^x_{Mn+i})_{x=0} = 0$ (since i is odd), we obtain $P^n \partial^i_x y_j = 0$ if $Y_0 = (y_0, \ldots, y_{N-1})$. Since $0 \leq j < N$ is arbitrary, this gives the result at step $2k+1$ and also at step $2k + 2$. □

Appendix

6. A Lemma of complex analysis

Lemma 6.1. *Consider*

$$\mathcal{B}_{R,C} := \left\{ z : [-1,1] \to \mathbb{C};\ \exists f \in H^\infty_R,\ \|f\|_{L^\infty(B(0,R))} \leq C,\ f\Big|_{[-1,1]} = z \right\}.$$

Then, for any $1 < r < R$ *and* $C > 0$

$$\mathcal{B}_{R,C} \subset \mathcal{R}_{R,C} \subset \mathcal{B}_{r, C(1-\frac{r}{R})^{-1}}.$$

Proof. For given $z \in \mathcal{B}_{R,C}$, if f denotes its analytic extension to $B(0,R)$, writing $f(\xi) = \sum_{n=0}^\infty \alpha_n \frac{\xi^n}{n!}$ for $|\xi| < R$, we have by Cauchy's formula that for any $n \in \mathbb{N}$ and any $r < R$:

$$|\alpha_n| = |f^{(n)}(0)| = \left| \frac{n!}{2\pi i} \int_{|\xi|=r} \frac{f(\xi)}{\xi^{n+1}} d\xi \right| \leq \frac{n!}{r^n} \|f\|_{L^\infty(B(0,R))},$$

and hence $|\alpha_n| \leq \|f\|_{L^\infty(B(0,R))} \frac{n!}{R^n}$ by letting $r \to R^-$. On the other hand, if $z \in \mathcal{R}_{R,C}$ is given by $z(\xi) = f(\xi) := \sum_{n=0}^\infty \alpha_n \frac{\xi^n}{n!}$ for $\xi \in [-1,1]$ and $1 < r < R$, then for $|\xi| < r$ we have that $|f(\xi)| \leq C \sum_{n=0}^\infty (\frac{r}{R})^n = C(1 - \frac{r}{R})^{-1} < \infty$.

□

6.1. **Gevrey regularity of the solution of** (3.1) **provided in Theorem 3.1.** Assume that f satisfies (1.7)-(1.10), still under the assumption $|\zeta_M| = 1$. Let us show that $y \in G^{1,\lambda}([-1,1] \times [t_1, t_2])$. Let $L_0 = L(s_0) = e^{\frac{1-s_0}{N}} L_1 < \frac{e^{\frac{1}{N}}}{(4N)^\lambda}$ where $s_0 \in [0,1]$ and L_1 are given in the proof of Theorem 3.1. Then, we can pick some numbers R_1, R_2 such that $\frac{4M}{e^{\frac{1}{M}}} < R_1 < R_2 < \frac{\lambda}{L_0^{\frac{1}{\lambda}}}$. Let us prove that there exists some constant $Q > 0$ such that (3.3) holds. To this end, picking any $\mu > M+1$, we prove by induction on $k \in \mathbb{N}$ that

$$|\partial_x^k \partial_t^n y(x,t)| \leq C_k \frac{(\lambda n + k)!}{R_1^k R_2^{\lambda n} (\lambda n + k + 1)^\mu} \quad \forall (x,t) \in [-1,1] \times [t_1, t_2], \ \forall n \in \mathbb{N}, \tag{6.1}$$

with $\sup_{k \in \mathbb{N}} C_k < \infty$, the sequence C_k being nondecreasing. Let us start with $k \in \{0, ..., M-1\}$. We already know that $y \in C^\infty([-1,1] \times [t_1, t_2])$ and that $U = (y, \partial_x y, ..., \partial_x^{M-1}) \in C([-1,1], X_{s_0})$ for some $s_0 \in (0,1)$, the space X_{s_0} being defined in (3.29); that is, $U \in C([-1,1], \mathcal{X}_{L_0})$ with $L_0 = L(s_0) = e^{r(1-s_0)} L_1 = e^{\frac{1-s_0}{N}} L_1 \leq e^{\frac{1}{N}} L_1$. Thus, we have for some constant $C > 0$ and for all $n \in \mathbb{N}$ and all $(x,t) \in [-1,1] \times [t_1, t_2]$

$$\begin{aligned}
|\partial_x^k \partial_t^{n+1} y(x,t)| &\leq C\, L_0^{|n - \frac{M-1-k}{\lambda}|} \Gamma(n + 1 - \frac{M-1-k}{\lambda})^\lambda (1+n)^{-2}, \\
|\partial_x^k \partial_t^{n+1} y(x,t)| &\leq C, \\
|\partial_x^{M-1} \partial_t^{n+1} y(x,t)| &\leq C\, L_0^n (n!)^\lambda (1+n)^{-2},
\end{aligned}$$

for $0 \leq k < M-1, n \leq \left\lfloor \frac{M-k-1}{\lambda} \right\rfloor + 1$.

We readily infer from Stirling's formula $\Gamma(x+1) \sim (\frac{x}{e})^x \sqrt{2\pi x}$ that $\Gamma(x+a) \sim \Gamma(x) x^a$ as $x \to \infty$, for any $a \in \mathbb{R}$, and that $(n!)^\lambda \sim (2\pi n)^{\frac{\lambda-1}{2}} \lambda^{-\frac{1}{2}} (\lambda n)! / \lambda^{\lambda n}$. It follows that for some constant $C > 0$

$$|\partial_x^k \partial_t^{n+1} y(x,t)| \leq C L_0^n L_0^{-\frac{M-1-k}{\lambda}} [n!(n+1)^{-\frac{M-1-k}{\lambda}}]^\lambda (n+1)^{-2} \leq C L_0^n \frac{(\lambda n)!}{\lambda^{\lambda n}} (n+1)^{-(M-1-k)} n^{\frac{\lambda-1}{2}}$$

for $0 \leq k \leq M-1$. Thus there are some positive constants C_k, $0 \leq k \leq M-1$, such that (6.1) holds, provided that $R_2 < \lambda / L_0^{\frac{1}{\lambda}}$.

Assume now that (6.1) is true for $k \in \{0, ..., l+M-1\}$ for some $l \in \mathbb{N}$. Let us show that (6.1) is true for $k = l+M$; that is, for all $n \geq 0$ and all $(x,t) \in [-1,1] \times [t_1, t_2]$

$$\left|\partial_x^{l+M} \partial_t^n y(x,t)\right| \leq C_{l+M} \frac{(\lambda n + l + M)!}{R_1^{l+M} R_2^{\lambda n} (\lambda n + l + M + 1)^\mu},$$

for some constant $C_{l+M} > 0$. Since $|\zeta_M| = 1$, using (3.1), we have that

$$\begin{aligned}
|\partial_x^{l+M}\partial_t^n y| &= |\partial_x^l\partial_t^n(\zeta_M\partial_x^M y)| \\
&= |\partial_x^l\partial_t^n(\partial_t^N y - \sum_{j=0}^{M-1}\zeta_j\partial_x^j y - f(x,y,\partial_x y,...,\partial_x^{M-1}y)| \\
&\leq |\partial_x^l\partial_t^{n+N}y| + |\partial_x^l\partial_t^n(\sum_{j=0}^{M-1}\zeta_j\partial_x^j y)| + |\partial_x^l\partial_t^n f(x,y,\partial_x y,...,\partial_x^{M-1}y)| \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Then using directly the iteration assumption and $\lambda N = M$, we have

$$I_1 \leq C_l\frac{(\lambda n+\lambda N+l)!}{R_1^l R_2^{\lambda n+\lambda N}(\lambda n+\lambda N+l+1)^\mu} = C_l\left(\frac{R_1}{R_2}\right)^M\frac{(\lambda n+l+M)!}{R_1^{l+M}R_2^{\lambda n}(\lambda n+l+M+1)^\mu}.$$

On the other hand, we have that

$$\begin{aligned}
I_2 &\leq \sum_{j=0}^{M-1}|\zeta_j|\,|\partial_x^{l+j}\partial_t^n y| \\
&\leq \sum_{j=0}^{M-1}|\zeta_j|C_{l+j}\,\frac{(\lambda n+l+j)!}{R_1^{l+j}R_2^{\lambda n}(\lambda n+l+j+1)^\mu} \\
&\leq \left(\sum_{j=0}^{M-1}|\zeta_j|C_{l+j}\frac{R_1^{M-j}}{(\lambda n+l+j+1)\cdots(\lambda n+l+M)}\left(\frac{\lambda n+l+M+1}{\lambda n+l+j+1}\right)^\mu\right) \\
&\quad\times\frac{(\lambda n+l+M)!}{R_1^{l+M}R_2^{\lambda n}(\lambda n+l+M+1)^\mu}.
\end{aligned}$$

Finally, as in the proof of Proposition 4.7 (see estimate (4.55) iterating Lemma 4.10), we have that for some positive constant $\overline{C}$

$$\begin{aligned}
I_3 &\leq \left|\partial_x^l\partial_t^n\sum_{\vec{p}\neq 0}A_{\vec{p}}y^{p_0}(\partial_x y)^{p_1}\cdots(\partial_x^{M-1}y)^{p_{M-1}}\right| \\
&\leq \sum_{\vec{p}\neq 0}K^{|\vec{p}|}\frac{\overline{C}}{b^{|\vec{p}|}}\frac{(\lambda n+l+M-1)!}{R_1^l R_2^{\lambda n}(\lambda n+l+1)^\mu}C_{l+M-1}^{|\vec{p}|}\prod_{j=0}^{M-1}\frac{1}{(R_1^j)^{p_j}}
\end{aligned}$$

Note that $R_1 > 1$. If, for some constant $\delta \in (0,1)$, we have

$$\frac{C_{l+M-1}K}{b} \leq \delta, \tag{6.2}$$

this yields

$$\begin{aligned} I_3 &\leq \delta\overline{C}\frac{(\lambda n+l+M-1)!}{R_1^l R_2^{\lambda n}(\lambda n+l+1)^\mu}\left(\frac{1}{1-\delta}\right)^M \\ &\leq \frac{\delta\overline{C}R_1^M}{(\lambda n+l+M)(1-\delta)^M}\left(\frac{\lambda n+l+M+1}{\lambda n+l+1}\right)^\mu \frac{(\lambda n+l+M)!}{R_1^{l+M}R_2^{\lambda n}(\lambda n+l+M+1)^\mu} \end{aligned}$$

It follows that

$$|\partial_x^{l+M}\partial_t^n y(x,t)| \leq C_{l+M}\frac{(\lambda n+l+M)!}{R_1^{l+M}R_2^{\lambda n}(\lambda n+l+M+1)^\mu}, \tag{6.3}$$

with

$$\begin{aligned} C_{l+M} := \max\Bigg(C_{l+M-1}, C_l\left(\frac{R_1}{R_2}\right)^M \\ + \Bigg(\sum_{j=0}^{M-1}|\zeta_j|C_{l+j}\frac{R_1^{M-j}}{(\lambda n+l+j+1)\cdots(\lambda n+l+M)}\left(\frac{\lambda n+l+M+1}{\lambda n+l+j+1}\right)^\mu \\ + \frac{\delta\overline{C}R_1^M}{(\lambda n+l+M)(1-\delta)^M}\left(\frac{\lambda n+l+M+1}{\lambda n+l+1}\right)^\mu\Bigg)\Bigg). \end{aligned}$$

Then, using the fact that $R_1 < R_2$, if $\frac{C_jK}{b} \leq \delta$ for $j = 0,1,...,l+M-1$, then $\frac{C_{l+M}K}{b} \leq \delta$ provided that l is large enough, say $l \geq l_0$. It is then sufficient to impose that

$$\max(C_0, ..., C_{l_0+M-1}) \leq \frac{\delta b}{K},$$

and this is the case provided that the constant C in (3.2) is small enough.

7. On the complex Ginzburg-Landau equation

Theorem 7.1. *Theorem 3.1 holds true for the complex Ginzburg-Landau equation.*

Proof. The fact that the equation is complex-valued does not change the proof. The only slight difference is for Lemma 3.8 where the nonlinearity contains some conjugate. The proof is even simpler since the sum is finite. We give a simpler proof for the convenience of the reader. In that case, $M = 2$, $N = 1$ and $\lambda = 2$. If $U = (u_0, u_1) \in L^\infty(K)^2$, and $F(x,U) = \begin{pmatrix} 0 \\ -e^{i\varphi}|u_0|^2u_0 \end{pmatrix}$ then

$$\begin{aligned} \|F(x,U)-F(x,V)\|_{X_{s'}} &= \left\|\begin{pmatrix} 0 \\ |u_0|^2u_0 - |v_0|^2v_0 \end{pmatrix}\right\|_{X_{s'}} \\ &= e^{-\tau(1-s')}\||u_0|^2u_0 - |v_0|^2v_0\|_{L(s')} \\ &= e^{-\tau(1-s')}\|(u_0-v_0)(u_0+v_0)\overline{u_0} + v_0^2(\overline{u_0}-\overline{v_0})\|_{L(s')} \\ &\leq \frac{3}{2}e^{-\tau(1-s')}\|u_0-v_0\|_{L(s')}\left(\|u_0\|_{L(s')}^2 + \|v_0\|_{L(s')}^2\right). \end{aligned}$$

We used the algebra property of Lemma 3.3 and the fact that the norm is invariant by conjugation. Using (3.18) and (3.31), we get, for a constant C depending on L_1, M and N,

$$\|u_0\|_{L(s')} \leq C\|u_0\|_{L(s'),\frac{M-1}{\lambda}} \leq C\left\|U\right\|_{\mathcal{X}_{L(s')}} = Ce^{\tau(1-s')}\left\|U\right\|_{X_{s'}} \leq Ce^{\tau}e^{-\tau(s-s')}\|U\|_{X_s}.$$

The same estimate is true for $u_0 - v_0$, and therefore we obtain

$$\|F(x,U) - F(x,V)\|_{X_{s'}} \leq$$
$$C^3e^{-\tau(s-s')}e^{3\tau}\left\|U-V\right\|_{X_s}\left(\|U\|^2_{X_s} + \|V\|^2_{X_s}\right) \leq \frac{C^3e^{-1}e^{3\tau}}{\tau(s-s')}D^2\left\|U-V\right\|_{X_s},$$

where we have used (3.35). For fixed τ, it can be made arbitrarily small when D is chosen small enough. The proof finishes the same way for the existence of the solution. Concerning the estimates given in Section 6.1, the only difference concerns the term I_3 that becomes $I_3 = |e^{i\varphi}\partial_x^l\partial_t^n(y^2\overline{y})|$. In this part of the proof the induction argument (6.1) is valid for $k \in \{0, ..., l+M-1\}$ for some $l \in \mathbb{N}$. The derivatives of $\overline{y}$ have the same bounds as those of y in (6.1), namely

$$|\partial_x^k\partial_t^n\overline{y}(x,t)| \quad\leq\quad C_k\frac{(2n+k)!}{R_1^kR_2^{2n}(2n+k+1)^{\mu}} \quad \forall(x,t) \in [-1,1]\times[t_1,t_2],\ \forall n \in \mathbb{N}. \tag{7.1}$$

We can apply similarly Lemma 4.10 twice to get

$$I_3 \quad=\quad \left|\partial_x^l\partial_t^n(y^2\overline{y})\right| \leq K^2C_l^3\frac{(2n+l)!}{R_1^lR_2^{2n}(2n+l+1)^{\mu}} \leq \widetilde{\beta}_{l+2}C_l\frac{(2n+l+2)!}{R_1^{l+2}R_2^{2n}(2n+l+2+1)^{\mu}}$$

with $\widetilde{\beta}_{l+2} = \sup_{n\in\mathbb{N}}\frac{K^2C_l^2R_1^2}{(2n+l+1)(2n+l+2)}\frac{(2n+l+2+1)^{\mu}}{(2n+l+1)^{\mu}} \leq \frac{K^2C_l^2R_1^2}{(l+1)(l+2)}3^{\mu}$. The rest of the estimate being the same, we can make the $\widetilde{\beta}_{l+2}$ arbitrarily small in a similar way. This completes the inductive step. □

Proposition 7.2. *Proposition 4.7 holds true for the complex Ginzburg-Landau equation.*

Proof. The reconstruction is exactly the same working in $\mathbb{C}$ instead of $\mathbb{R}$. The modifications of the estimates of the nonlinear term are done in the same way as in Theorem 7.1, noticing that $\overline{y}$ satisfies the same estimates as y. □

Proof of Theorem 2.5 and 2.6. This is the same as before with $\lambda = 2/1 = 2$. It only remains to check the condition about the non-linearity. We have $f(x,y_0,y_1) = e^{i\varphi}|y_0|^2y_0$. It satisfies $f(-x,-y_0,y_1) = -e^{i\varphi}|y_0|^2y_0 = -f(x,y_0,y_1)$ which is condition (1.18) for system (2.16), and $f(-x,y_0,-y_1) = e^{i\varphi}|y_0|^2y_0 = f(x,y_0,y_1)$ which is condition (1.19) for system (2.17). □

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