On stabilization and control for the critical Klein-Gordon equation on a 3-D compact manifold

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Abstract

[Résumé] On étudie la stabilisation et le contrôle interne de l'équation de Klein-Gordon critique sur des variétés de dimension 3. Sous des conditions géométriques légèrement plus fortes que la condition de contrôle géométrique classique, on prouve la décroissance exponentielle de solutions bornées dans l'espace d'énergie mais petites dans des normes plus faibles. La preuve combine la décomposition en profils et des arguments microlocaux. Cette décomposition, analogue à celle de Bahouri-Gérard [2] sur $\mathbb{R}^3$, nécessite l'analyse de certains effets dus à la géométrie. Elle utilise des résultats de S. Ibrahim [16] sur le comportement d'ondes de concentration sur les variétés.

Abstract

We study the internal stabilization and control of the critical nonlinear Klein-Gordon equation on 3-D compact manifolds. Under a geometric assumption slightly stronger than the classical geometric control condition, we prove exponential decay for some solutions bounded in the energy space but small in a lower norm. The proof combines profile decomposition and microlocal arguments. This profile decomposition, analogous to the one of Bahouri-Gérard [2] on $\mathbb{R}^3$, is performed by taking care of possible geometric effects. It uses some results of S. Ibrahim [16] on the behavior of concentrating waves on manifolds.

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1 Introduction

The controllability property is the possibility of modifying one system to drive the solution from an initial state to a final state fixed in advance. Here we will be interested in the internal control, that is by adding a forcing term g which will be asked to be supported in a certain open set ω.

For the stabilization, the equation is modified by a feedback term only depending on the solution itself. The question here is whether the solution will converge to a certain state and eventually at which rate. Here, we will study the exponential decay to zero of the Klein-Gordon equation modified by an internal damping of the form $\chi_\omega^2 \partial_t u$ where χ_ω is a function supported in an open set ω.

In these both situations, one of the main question is to find the optimal geometric conditions on ω that ensures the controllability or the stabilization.

In this note, we study the internal stabilization and exact controllability for the defocusing critical nonlinear Klein-Gordon equation on some 3D compact manifold M. The free equation reads as

$$\begin{cases} \Box u = \partial_t^2 u - \Delta u & = & -u - u^5 \quad \text{on} \quad [0, +\infty[\times M \\ (u(0), \partial_t u(0)) & = & (u_0, u_1) \in \mathcal{E}. \end{cases} \tag{1}$$

where Δ is the Laplace-Beltrami operator on M and $\mathcal{E}$ is the energy space $H^1(M) \times L^2(M)$ of real valued functions. The solution displays a conserved energy

$$E(t) = \frac{1}{2} \left(\int_M |\partial_t u|^2 + \int_M |u|^2 + \int_M |\nabla u|^2 \right) + \frac{1}{6} \int_M |u|^6. \tag{2}$$

Since this equation is defocusing (sign $+$ in front of the nonlinear term in the energy) the energy is nonnegative. Note that due to the local nature of the blowup process, the controllability for the focusing equation is certainly false for large data. The exponent u^5 is critical with respect to the energy space. Indeed, the L^6 norm in the nonlinear energy corresponds to the greater exponent for the Sobolev embedding $H^1 \hookrightarrow L^6$. In some sense, it is the exponent for which the linear part of the energy and the nonlinear part have the same weight. Moreover, the closely related equation $\Box u + u^5 = 0$ considered on $\mathbb{R}^{1+3}$ admits the following family of solution parametrized by $\lambda > 0$

$$u_\lambda = \frac{1}{\sqrt{\lambda}} u \left(\frac{t}{\lambda}, \frac{x}{\lambda} \right) \tag{3}$$

which all have the same energy norm. We will see that this family of solutions will be of great importance since they represent the main phenomenon for which the solutions differ from the linear solutions at high frequency. It is also worth noticing that the critical exponent is the greatest one for which the flow map is uniformly continuous (see [24][6]). The local existence theory is proved by Strichartz estimates for variable coefficients of Kapitanski [19]. Since the exponent is critical, the global existence requires adapted Morawetz estimates which were proved by Ibrahim and Majdoub [17] (some additional arguments are also necessary for the damped equation).

We will be interested in both the controllability and stabilization for this equation.

Let us begin by the existing linear results. The situation is quite well understood thanks to the works of Rauch and Taylor [27] and Bardos-Lebeau-Rauch [3]. The necessary and sufficient condition to get controllability and uniform stabilization in the energy space is the very natural Geometric Control Condition.

Assumption 1.1 (Geometric Control Condition). *There exists $T_0 > 0$ such that every geodesic travelling at speed 1 meets ω in a time $t < T_0$.*

In the presence of boundaries, we have to replace the geodesics by the rays of the geometric optics bouncing on the boundaries (see also the role of diffractive points for boundary control [5]).

Yet, some results remain if Assumption 1.1 is not fulfilled, but necessarily with some loss. For instance, if ω is arbitrary non empty, there is a logarithmic type decay of the energy for data in some smoother space (see [23] [25]). Some recent progress have also been made to understand the intermediate decay rate when the missed trajectories are hyperbolics (see for instance [7] [29]).

For the nonlinear equation, since the nonlinear energy plays a crucial role, it seems reasonable to consider first some situation where geometric control condition is fulfilled. There are many results in that direction. The more advanced in the subcritical case is certainly the work of Dehman-Lebeau-Zuazua [11]. We will describe its proof in section 2 below. The main idea is to try as much as possible to compare the nonlinear equation to its linear counterpart, with for instance the concept of linearizability of P. Gérard [15] and with some propagation results. Yet, as we will describe below, some important arguments of the proof do not extend to the critical case, as the lack of linearizability and the propagation of regularity.

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In this paper, we employ a strategy to avoid the lack of linearizability at the cost of an additional condition for the subset ω. It was already performed by B. Dehman and P. Gérard [9] in the case of $\mathbb{R}^3$ with a flat metric. In fact, in that case, this defect of linearizability is described by the profile decomposition of H. Bahouri and P. Gérard [2]. The purpose of this paper is to extend a part of this proof to the case of a manifold. This more complicated geometry leads to extra difficulties, in the profile decomposition and the stabilization argument.

We also mention the recent result of Aloui, Ibrahim and Nakanishi [1] for $\mathbb{R}^d$. Their method of proof is very different and uses Morawetz-type estimates. They obtain uniform exponential decay for a damping outside of a ball, for any nonlinearity, provided the solution exists globally. This result is stronger than ours, but their method does not seem to apply to the more complicated geometries we deal with. We can also notice that if the nonlinearity is of the type $f(u_t)$ with some appropriate f, it is possible to link the decay of the linear equation to the nonlinear one by energy estimates for the difference of the two solutions, see [8].

1.1   Main results

We will need some additional geometrical condition to prove controllability. It requires first the notion of focus, which also appeared in [16].

Definition 1.1. *We say that* $(x_1, x_2, t) \in M \times M \times \mathbb{R}_+^*$ *is a couple of focus at distance* t *if the set*

$$F_{x_1,x_2,t} := \left\{ \xi \in S_{x_1}^* M \,\middle|\, exp_{x_1} t\xi = x_2 \right\}$$

of directions of geodesics stemming from x_1 *and reaching* x_2 *in a time* t*, has a positive surface measure.*

We denote T_{focus} *the infimum of the* $t \in \mathbb{R}_+^*$ *such that there exists a couple of focus at distance* t*.*

If M is compact, we have necessarily $T_{focus} > 0$.
Our main assumption will be the following.

Assumption 1.2 (Geometric control before refocusing)**.** *The open set* ω *satisfies the Geometric Control Condition in a time* $T_0 < T_{focus}$*.*

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For example, for $\mathbb{T}^3$, there is no refocusing and the assumption is the classical Geometric Control Condition. Yet, for the sphere S^3, our assumption is stronger. For example, it is fulfilled if ω is a neighborhood of $\{x_4 = 0\}$. However, there are some geometric situations where the Geometric Control Condition is fulfilled while our condition is not, for example if we take only a a neighborhood of $\{x_4 = 0, x_3 \geq 0\}$. The exponential decay might be true in this specific case but requires additional arguments (see subsection 4.1 for more comments).

The main result of this article is the following theorem.

Theorem 1.1. *Let $R_0 > 0$ and ω satisfying Assumption 1.2. Then, there exist $T > 0$ and $\delta > 0$ such that for any (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1)$ in $H^1 \times L^2$, with*

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leq R_0; \qquad \|(\tilde{u}_0, \tilde{u}_1)\|_{H^1 \times L^2} \leq R_0$$
$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq \delta; \qquad \|(\tilde{u}_0, \tilde{u}_1)\|_{L^2 \times H^{-1}} \leq \delta$$

there exists $g \in L^\infty([0,T], L^2)$ supported in $[0,T] \times \omega$ such that the unique strong solution of

$$\begin{cases} \Box u + u + u^5 &= g \quad \text{on} \quad [0,T] \times M \\ (u(0), \partial_t u(0)) &= (u_0, u_1). \end{cases}$$

satisfies $(u(T), \partial_t u(T)) = (\tilde{u}_0, \tilde{u}_1)$.

By reversibility of the equation, the result is equivalent to the control to zero. We obtain it in two steps. The first one is obtained by taking as control the damping term of a stabilized equation. This allows to bring the solution close to zero in some large time. Once it is done, a theorem of local control for small data allows to bring the solution to zero by a perturbation argument of the linear controllability.

Let us discuss the assumptions on the size in Theorem 1.1. In some sense, it is a high frequency controllability result and expresses in a rough physical way that we can control some "small noisy data". In the subcritical case, two similar kind of results were proved : in Dehman-Lebeau-Zuazua [11] similar results were proved for the nonlinear wave equation but without the smallness assumption in $L^2 \times H^{-1}$ while in Dehman-Lebeau [10], they obtain similar high frequency controllability results for the subcritical equation but in a uniform time which is actually the time of linear controllability (see also

the work of the author [21] for the Schrödinger equation). Actually, this smallness assumption is made necessary in our proof because we are not able to prove in general the following unique continuation result.

Unique continuation property $u \equiv 0$ *is the unique strong solution in the energy space of*

$$\begin{cases} \Box u + u + u^5 &=& 0 & \text{on} & [0,T] \times M \\ \partial_t u &=& 0 & \text{on} & [0,T] \times \omega. \end{cases}$$

This kind of theorem can be proved with Carleman estimates under some additional geometrical conditions and once the solution is known to be smooth. We refer to subsection 2.2 for further discussion about this. In the case of $\mathbb{R}^3$ with flat metric and ω the complementary of a ball, B. Dehman and P. Gérard [9] prove this theorem using the existence of the scattering operator, which is not available on a manifold.

Moreover, as in the subcritical case, we do not know if the time of controllability does depend on the size of the data. This is actually still an open problem for several nonlinear evolution equations such as nonlinear wave or Schrödinger equation (even in the subcritical case). Note that for certain nonlinear parabolic equations, it has been proved that we can not have controllability in arbitrary short time while it is the case for the linear equation, see [12] for instance.

The most difficult part of the proof of Theorem 1.1 is in the first part where we have to prove exponential decay for a damped equation.

Theorem 1.2. *Let $R_0 > 0$, ω satisfying Assumption 1.2 and $\chi_\omega \in C^\infty(M)$ satisfying $\chi_\omega(x) > \eta > 0$ for all $x \in \omega$. Then, there exist $C, \gamma > 0$ and $\delta > 0$ such that for any (u_0, u_1) in $H^1 \times L^2$, with*

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leq R_0; \qquad \|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq \delta;$$

the unique strong solution of

$$\begin{cases} \Box u + u + u^5 + \chi_\omega^2 \partial_t u &=& 0 & \text{on} & [0,T] \times M \\ (u(0), \partial_t u(0)) &=& (u_0, u_1). \end{cases} \tag{4}$$

satisfies $E(u)(t) \leq Ce^{-\gamma t} E(u)(0)$.

This theorem has to be combined with the following local controllability theorem whose proof is done by a perturbation argument.

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Theorem 1.3. *Let ω satisfying Assumption 1.1 and $T > T_0$. Then, there exists $\delta > 0$ such that for any (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1)$ in $H^1 \times L^2$, with*

$$\|(u_0, u_1)\|_{\mathcal{E}} \leq \delta; \quad \|(\tilde{u}_0, \tilde{u}_1)\|_{\mathcal{E}} \leq \delta$$

there exists $g \in L^\infty([0, 2T], L^2)$ supported in $[0, 2T] \times \omega$ such that the unique strong solution of

$$\begin{cases} \Box u + u + u^5 &= g \quad \text{on} \quad [0, 2T] \times M \\ (u(0), \partial_t u(0)) &= (u_0, u_1). \end{cases}$$

satisfies $(u(2T), \partial_t u(2T)) = (\tilde{u}_0, \tilde{u}_1)$

Section 2 is devoted to a description of the proof of Dehman-Lebeau-Zuazua [11] of the exponential decay in the subcritical case, stressing the new difficulties that will appear in the critical case. Section 3 is devoted to a sketch of the proof of Theorem 1.2 and of the profile decomposition that it requires. We refer to the article [22] for more details. Section 4 discusses some possible extensions of the presented results.

Remark 1.1. *Some other type of control could be considered like boundary control. It has appeared that the results about linear boundary control were often close, in the spirit, to the one of internal control but with additional (difficult) technicalities. Therefore, it seems likely that the result we present could be extended to control or stabilization from the boundary. Yet, it would require some more precise understanding of the link between the nonlinearity and some nonhomogeneous boundary conditions.*

2 The compactness-uniqueness argument in the subcritical case (after Dehman-Lebeau-Zuazua)

2.1 Arguments of the proof

In this section, we describe the general method employed to prove the effective decay in nonlinear wave-type equation. It mainly describes the proof contained in Dehman-Lebeau-Zuazua [11]. We also emphasize which points fail in the critical case, that we will describe in the next section.

The aim is to prove exponential decay of the energy for the equation

$$\Box u + \chi_\omega^2 \partial_t u + f(u) = 0$$

where $f(u)$ is a regular subcritical defocusing nonlinearity. Their result is stated for $M = \mathbb{R}^3$ and ω the exterior of a ball, but their method of proof could easily be extended to a compact manifold and ω satisfying

- Geometric control condition

- Unique Continuation property (see subsection 2.2 for a discussion of situations where this condition is known to be fulfilled).

First, we notice that the energy is well decreasing and we have:

$$E(u)(t) = E(u)(0) - \int_0^t \int_M |\chi_\omega(x)\partial_t u|^2.$$

Then, to get an exponential decay, it is sufficient to prove an observability estimate for a certain time T.

$$E(u)(0) \leq C \int_0^T \int_M |\chi_\omega(x)\partial_t u|^2. \tag{5}$$

This gives $E(u)(T) \leq (1-C)E(u)(0)$ and it means that at each step $[nT, (n+1)T]$, a certain amount of the energy is dissipated, which easily yields exponential decay. But regarding the localisation of the support of $\chi_\omega(x)$, this is a way of measuring the amount of energy that passes through ω during a time T. The proof of such estimate will therefore involve some results of propagation of the information (compactness, regularity, nullity) from the open set ω to the whole manifold M.

The classical method to prove some observability estimates like (5) is the so called compactness-uniqueness method.

The argument is by contradiction. Let u_n be a bounded sequence of solutions contradicting (5) :

$$\int_0^T \int_M |\chi_\omega(x)\partial_t u_n|^2 \leq \frac{1}{n}E(u_n)(0). \tag{6}$$

Up to extraction, we can assume $E(u_n)(0) \to \alpha \geq 0$. Then, we have two cases: $\alpha > 0$ (large solutions) and $\alpha = 0$ (small solutions). We will only sketch the proof for the first case because the second one is similar but easier because the solution is small and therefore closer to a linear solution.

So assume $\alpha > 0$. We prove $u_n(0) \to 0$ in energy which is a contradiction.

Up to extraction we can assume $u_n \rightharpoonup u$ for the weak-$*$ convergence in $L^\infty([0,T], \mathcal{E})$ where u is solution of the same equation. So, to finish the proof, it is enough to prove that $u \equiv 0$ (uniqueness) and that the convergence is actually for the strong topology (compactness).

- By passing to the limit in (6), we easily get that $\partial_t u = 0$ on ω. So on ω, u is solution of a nonlinear subcritical elliptic equation and is indeed smooth, if for instance f is smooth. Then, since the nonlinearity is subcritical, they also prove that $f(u)$ is actually in a space $L^1([0,T], H^\varepsilon)$ with $\varepsilon > 0$. This allows to prove that the microlocal regularity at the level $H^{1+\varepsilon}$ propagates along the bicaracteristics as in the linear case. This gives $(u, \partial_t u)_{t=0} \in H^{1+\varepsilon} \times H^\varepsilon$ since u is regular on ω that satisfies geometric control condition. By iteration, u is C^∞.

  Some unique continuation arguments can then be applied to get $u \equiv 0$ (see subsection 2.2 below for more details). That is $u_n \rightharpoonup 0$.

- By linearizability property, $\|u_n - v_n\|_{L^\infty([0,T],\mathcal{E})} \underset{n\to\infty}{\longrightarrow} 0$ where v_n is solution of $\Box v_n = 0$ with same initial data as u_n. This property introduced by P. Gérard [15] essentially says that in a subcritical situation and at high frequency, the nonlinearity can be seen as a compact, and indeed is a small term.

- $\partial_t u_n \underset{L^2([0,T]\times\omega)}{\rightarrow} 0$ by (6) and so the same holds for v_n. This gives $\mu \equiv 0$ on $]0,T[\times\omega$ where μ is the microlocal defect measure associated to v_n (see [14][4]). Since, μ propagates along the Hamiltonian flow for a solution of the linear equation, this allows to propagate the compactness from the open set ω to M thanks to the geometric control condition. This gives $v_n \to 0$ in energy and the same holds for u_n.

This gives a contradiction to $\alpha > 0$.

In that proof, several argument fail when the nonlinearity becomes critical. Mainly two can be noticed

- the propagation of regularity does not work any more (at least with the same techniques). One way to avoid the proof of unique continuation is to prove a weaker observability estimate of the form

$$E(u)(0) \leq C \left(\iint_{[0,T]\times M} |\chi_\omega(x)\partial_t u|^2 \ dtdx + \|(u_0, u_1)\|_{L^2 \times H^{-1}} E(u)(0) \right).$$

When $\|(u_0, u_1)\|_{L^2 \times H^{-1}}$ is small, this gives the classical observability estimates. So, it allows to avoid the unique continuation property at the cost of a high frequency assumption.

Another solution would have been to use directly a unique continuation with rough potential as in Koch and Tataru [20]. The price to pay would have been some more restrictive geometrical assumption, since they are proved by Carleman estimates (see subsection 2.2 below).

- the linearizability is false. For $\mathbb{R}^3$ for instance, a sequence as (3) with $\lambda \to 0$ is a dilation of a nonlinear solution and is indeed not linearizable since the corresponding linear solution is a dilation of the corresponding linear solution. Therefore, this defect is local and is also an obstacle on a manifold. Then, the microlocal defect measure does not propagate as in the linear case. The remedy for this problem will be to describe very precisely the defect of linearizability. This is the aim of the profile decomposition described in subsection 3.1.

2.2 A few comments about unique continuation

The first part of the compactness-uniqueness argument is based on a unique continuation property as stated in the Introduction.

Our assumption of high frequency allows us to avoid the proof of such theorem. Yet, in this subsection, we discuss some known techniques to get similar results in linear or nonlinear situations.

There are two types of results in that direction. The first one is the Holmgren theorem and requires some analyticity of the coefficients of a linear equation. This result is very powerful in the sense that the geometrical assumption on the domain ω are very weak: you can propagate nullity along any noncaracteristic hypersurface. For the wave equation you essentially only have to wait for a time large enough not to contradict the finite speed of propagation. Yet, to apply such theorem in a nonlinear setting, we have to write the nonlinearity as $u^5 = u^4 u = Vu$ where V will be considered as a potential term. Therefore, the assumption of analyticity is very demanding concerning solutions of nonlinear equations.

Without that assumption, the available results to prove some unique continuation properties are based on Carleman estimates. Yet, the geometrical assumptions on the domain are very strong and not so natural.

First take a function $\varphi(t,x)$ strongly pseudoconvex for the wave operator. Then, you can propagate the nullity across the hypersurface $S = \{\varphi = 0\}$. That is, if a solution of $\Box u + Vu = 0$ satisfies $u \equiv 0$ on $\Omega_+ = \{\varphi > 0\}$, then it is also zero in a small neighborhood of each point of S. Yet, this requires some strong geometric conditions. Typically, for a function $\varphi(t,x) = \lambda t^2 - |x|^2$ on the euclidian space, this allows to extend the nullity from the exterior of a sphere to the interior for a time large enough. For a general manifold, we have to replace $|x|^2$ by a function $d(x)$ which must have a positive Hessian (with respect to the metric). We are often led to consider specific examples where we have to build locally the function φ "by hands" (see for some examples [21] in the similar context of nonlinear Schrödinger equation).

There are also some intermediate results (as [28]) which assume some analyticity only in certain variables (for instance the variable t). This also allows to obtain the uniqueness with some very weak geometrical assumptions. Yet, to use it for our purpose it would require to prove that the solution of the nonlinear equation is analytic in time. This seems hard to prove for finite times, since the propagation of analytic microlocal regularity is often more complicated to obtain for nonlinear equations. To overcome this, in a forthcoming article with R. Joly [18], we prove such analyticity in time for a solution satisfying $\partial_t u = 0$ on $\mathbb{R} \times \omega$ with a subcritical analytic nonlinearity. This would then give some unique continuation result in infinite time, which is sufficient for stabilization. The extension of such result to a critical exponent is not clear for the moment.

Note also that the idea of proving unique continuation in infinite time also appeared in the work [9] of Dehman-Gérard in $\mathbb{R}^3$. In this case, they use the existence of the scattering operator which allows to approximate the solution by a linear one for some large times. Then, the unique continuation for the nonlinear solution is obtained thanks to an observability estimate for the linear equation.

3 Sketch of proof in the critical case

3.1 The profile decomposition

The proof of the observability estimate in the subcritical case has used very strongly the linearizability property of the subcritical equation. As, we have noticed, some sequences of the type of (3) with $\lambda \to 0$ are not linearizable.

The interesting fact stressed by Bahouri-Gérard [2] is that this is almost the only obstacle. This is the object of the profile decomposition. The aim of this subsection is to precise this in the case of a manifold. Note that some similar result were also obained by Gallagher-Gérard [13] outside of a convex obstacle on $\mathbb{R}^3$.

Since we are on a compact manifold where there is no scaling, we have to define what will be a profile. This was also introduced in [16].

Definition 3.1. *Let $x_\infty \in M$ and $(f, g) \in \mathcal{E}_{x_\infty} = (\dot{H}^1 \times L^2)(T_{x_\infty} M)$. Given $[(f, g), \underline{h}, \underline{x}] \in \mathcal{E}_{x_\infty} \times (\mathbb{R}_+^* \times M)^{\mathbb{N}}$ such that $\lim_n (h_n, x_n) = (0, x_\infty)$ We call the associated concentrating data the class of equivalence, modulo sequences convergent to 0 in $\mathcal{E}$, of sequence in $\mathcal{E}$ that take the form*

$$h_n^{-\frac{1}{2}} \Psi_U(x) \left(f, \frac{1}{h_n} g \right) \left(\frac{x - x_n}{h_n} \right) + o(1)_\mathcal{E} \tag{7}$$

in some coordinate patch $U_M \approx U \subset \mathbb{R}^d$ containing x_∞ and for some $\Psi_U \in C_0^\infty(U)$ such that $\Psi_U(x) = 1$ in a neighborhood of x_∞. (Here, we have identified x_n, x_∞ with its image in U).

We can prove that this definition does not depend on the coordinate charts and on Ψ_U: two sequences defined by (7) in different coordinate charts are in the same class. In what follows, we will often call concentrating data associated to $[(f, g), \underline{h}, \underline{x}]$ an arbitrary sequence in this class.

Definition 3.2. *Let $\underline{t} = (t_n)$ a sequence in $\mathbb{R}$ converging to t_∞ and (f_n, g_n) a concentrating data associated to $[(f, g), \underline{h}, \underline{x}]$. A damped linear concentrating wave is a sequence v_n solution of*

$$\begin{cases} \Box v_n + v_n + a(x)\partial_t v_n &= 0 \quad \text{on} \quad \mathbb{R} \times M \\ (v_n(t_n), \partial_t v_n(t_n)) &= (f_n, g_n). \end{cases} \tag{8}$$

The associated damped nonlinear concentrating wave is the sequence u_n solution of

$$\begin{cases} \Box u_n + u_n + a(x)\partial_t u_n + u_n^5 &= 0 \quad \text{on} \quad \mathbb{R} \times M \\ (u_n(0), \partial_t u_n(0)) &= (v_n(0), \partial_t v_n(0)). \end{cases} \tag{9}$$

If $a \equiv 0$, we will only write linear or nonlinear concentrating wave.

Energy estimates yields that two representants of the same concentrating data have the same associated concentrating wave modulo strong convergence in $L^\infty_{loc}(\mathbb{R}, \mathcal{E})$. This is not obvious for the nonlinear evolution but will be a consequence of the study of nonlinear concentrating waves.

It can be easily seen that this kind of nonlinear solutions are not linearizable. Actually, it can be shown that this concentration phenomenon is the only obstacle to linearizability. We begin with the linear decomposition.

Theorem 3.1. *Let (v_n) be a sequence of solutions to the damped Klein-Gordon equation (8) with initial data at time $t = 0$, (φ_n, ψ_n) bounded in $\mathcal{E}$. Then, up to extraction, there exist a sequence of damped linear concentrating waves $(p^{(j)})$, as defined in Definition 3.2, associated to concentrating data $[(\varphi^{(j)}, \overline{\psi^{(j)}}), \underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)}]$, such that for any $l \in \mathbb{N}^*$,*

$$v_n(t,x) = v(t,x) + \sum_{j=1}^{l} p_n^{(j)}(t,x) + w_n^{(l)}(t,x),$$

$$\forall T > 0, \quad \varlimsup_{n \to \infty} \left\| w_n^{(l)} \right\|_{L^\infty([-T,T],L^6(M)) \cap L^5([-T,T],L^{10})} \xrightarrow[l \to \infty]{} 0$$

$$\|(v_n, \partial_t v_n)\|_{\mathcal{E}}^2 = \sum_{j=1}^{l} \left\| (p_n^{(j)}, \partial_t p_n^{(j)}) \right\|_{\mathcal{E}}^2 + \left\| (w_n^{(l)}), \partial_t w_n^{(l)}) \right\|_{\mathcal{E}}^2 + o(1), \text{ as } n \to \infty,$$

where $o(1)$ is uniform for $t \in [-T, T]$.

The nonlinear flow map follows this decomposition up to an error term in the strong following norm

$$|||u|||_I = \|u\|_{L^\infty(I, H^1(M))} + \|\partial_t u\|_{L^\infty(I, L^2(M))} + \|u\|_{L^5(I, L^{10}(M))}.$$

Theorem 3.2. *Let $T < T_{focus}/2$. Let u_n be the sequence of solutions to damped non linear Klein-Gordon equation (9) with initial data, at time 0, (φ_n, ψ_n) bounded in $\mathcal{E}$. Denote $p_n^{(j)}$ (resp v the weak limit) the linear damped concentrating waves given by Theorem 3.1 and $q_n^{(j)}$ the associated nonlinear damped concentrating wave (resp u the associated solution of the nonlinear equation with $(u, \partial_t u)_{t=0} = (v, \partial_t v)_{t=0}$). Then, up to extraction, we have*

$$u_n(t,x) = u + \sum_{j=1}^{l} q_n^{(j)}(t,x) + w_n^{(l)}(t,x) + r_n^{(l)}$$

$$\varlimsup_{n \to \infty} \left\| |r_n^{(l)}| \right\|_{[-T,T]} \xrightarrow[l \to \infty]{} 0$$

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where $w_n^{(l)}$ is given by Theorem 3.1.

The same theorem remains true if M is the sphere S^3 and $a(x) \equiv 0$ (undamped equation) without any assumption on the time T.

3.2 Description of a profile (after S. Ibrahim)

An important fact for completing the decomposition is the knowledge of the behavior of a single nonlinear profile. This has been described by S. Ibrahim [16].

To begin with, let us consider a profile on the euclidian space $\mathbb{R}^3$ (with $t_n = x_n = 0$ to simplify), that is a sequence of solutions $w_n = \frac{1}{\sqrt{h_n}} w \left(\frac{t}{h_n}, \frac{x}{h_n} \right)$ where $\Box_{\mathbb{R}^3} w + w^5 = 0$. According to the theory of scattering, there are two linear solutions v^+ and v^- such that $\|(v^\pm - w)(t)\|_{\mathcal{E}} \xrightarrow[t \to \pm\infty]{} 0$. The scattering operator S sends $v^-(0)$ into $v^+(0)$. Denote $v_n^\pm = \frac{1}{\sqrt{h_n}} v^\pm \left(\frac{t}{h_n}, \frac{x}{h_n} \right)$.

In that case, by invariance of the energy space by the scaling, it is worth noticing that if we take for instance $t = -\varepsilon$, $w_n(-\varepsilon)$ is asymptotically close to $v_n(-\varepsilon)$, as n tends to infinity. The same reasoning for $t = \varepsilon$ yields that, before and after concentration, the nonlinear solution w_n is close to a linear solution (it is linearizable during that time) but this linear solution is not the same and is modified according to the scattering operator S (since S is not the identity, this is a proof of non linearizability).

On the manifold, we have to be a little more precautious because the comparison with a concentrating function is only true for small times $[t_n - \Lambda h_n, t_n - \Lambda h_n]$ but with Λ arbitrary large. This is enough to conclude that during the time of concentration, the profile is changed by the euclidian scattering. But, after that time, the lower order terms are no more negligeable and the geometry of the manifold has to be taken into account.

Roughly speaking, for a nonlinear concentrating wave, there are mainly two different times that need to be considered

- For times close to the concentration, that is in $[t_n - \Lambda h_n, t_n - \Lambda h_n]$, the solution is close to a concentrating profile on the tangent plane $w_n = \frac{1}{\sqrt{h_n}} w \left(\frac{t-t_n}{h_n}, \frac{x-x_n}{h_n} \right)$ where $\Box_{\mathbb{R}^3} w + w^5 = 0$ as described above. So, the profile is changed according to a euclidian scattering operator.

- Away from a concentrating point, the behavior is linear. Therefore, the propagation of the energy (seen through the microlocal defect mea-

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sure) is well described by the geometric optic. It means that the energy propagates at high frequency along the bicaracteristic flow of the Hamiltonian.

Actually, the concentration-compactness principle of P.-L. Lions [26] combined with Strichartz estimates allows to say that, as long as the defect measure associated to the sequence does not charge any point, the nonlinearity is compact and the behavior remains linear (these ideas comes from the paper of P. Gérard [15] about linearizability). So, since we know that when the behavior is linear, the microlocal defect measure propagates along the Hamiltonian flow, it is possible to get that the nonlinear solution remains close to the linear one as long as there is no refocusing.

The exact description of this phenomenon of refocusing is quite hard to obtain. Yet, in the specific case of the sphere S^3 and without damping, the explicit knowledge of the linear solution allows to obtain a more precise description. Indeed, the linear concentrating wave concentrating on a North pole refocus on the South pole with a new profile applied with a reflection.

This allows to describe the nonlinear concentrating wave as follows. After a concentration on the North pole, the solution propagates linearly on the sphere, that means remains concentrated on a circle propagating to the South. When it reaches the South, it reconcentrates with a profile which is the reflection of the one on the North pole. Yet, during the concentration, the nonlinearity transforms the profile into a new one obtained by the euclidian scattering operator. Then, the solution propagates linearly to the North pole until it reconcentrates with a reflection of the one obtained on the South after scattering. And so on... So, there is a kind of ping-pong between the North and the South pole with at each time a modification of the profile according to the Euclidian scattering operator.

3.3   Sketch of proof of the profile decomposition

We follow the proof of Bahouri-Gérard [2] and Gallagher-Gérard [13] and adapt it to the case of a manifold and the presence of damping.

For the linear decomposition, the idea is to extract one after the other the different elements of the decomposition and to prove that, in some sense, the Strichartz norm of the new remainder term is decreasing.

The main steps are the following:

- decomposition in scales of oscillation: the point is to extract the scale of oscillation $h_n^{(j)}$. In our case, it was enough to prove that if a sequence is oscillating at scale $h_n^{(j)}$ on M (that means roughly that it is almost spectrally localized on some interval $[Ch_n^{-1}, Dh_n^{-1}]$) it is the same for any local coordinate chart of the manifold (when the oscillation is considered with respect to Euclidian Fourier transform). The proof that the remaining term is small in $L^\infty([0,T], L^6)$ is done by using the similar result in each coordinate chart.

- extraction of times and cores of concentration by exhaustion. We "track" the possible points of concentration of the energy according to a specific scale. We go on the process as long as there are some points where the energy concentrates.

- prove that the rest is small in Strichartz norm by some refined estimates (that is involving a compact term).

We also have to prove that the scales, times and cores of concentration are "orthogonal" in some sense that will allow to get that the non linear interactions are small. More precisely, two profiles are said orthogonal if either

- $\log \left| \dfrac{h_n^{(1)}}{h_n^{(2)}} \right| \underset{n \to \infty}{\longrightarrow} +\infty$

- $x_\infty^{(1)} \neq x_\infty^{(2)}$

- $h_n^{(1)} = h_n^{(1)} = h$ and $x_\infty^{(1)} = x_\infty^{(2)} = x_\infty$ and in some coordinate chart around x_∞, we have

$$\frac{\left| t_h^{(1)} - t_h^{(2)} \right|}{h} + \frac{\left| x_h^{(1)} - x_h^{(2)} \right|}{h} \underset{h \to 0}{\longrightarrow} +\infty.$$

For the nonlinear decomposition, we mainly have to establish that each element of the linear decomposition "behaves independently" in the nonlinear equation.

To illustrate the problem, let us take the examples of a sequence with two linear profiles $p_n^{(1)}$ and $p_n^{(2)}$, and denote $q_n^{(1)}$, $q_n^{(2)}$ the associated nonlinear profiles, solutions of the nonlinear equation with same initial data. We want to prove that the solution of $\Box u_n + u_n = u_n^5$ with initial data $p_n^{(1)} + p_n^{(2)}$ at time 0 can be approximately written $u_n \approx q_n^{(1)} + q_n^{(2)}$ in energy. Because both

of them have the same initial data, a boot strap argument allows to conclude if we prove that $\left(q_n^{(1)} + q_n^{(2)} \right)^5 \approx (q_n^{(1)})^5 + (q_n^{(2)})^5$ in $L^1([0,T], L^2)$. Such a result can be obtained because these profiles are orthogonal according to the previous definition. That means that they have different scales, points and times of concentration and indeed, the locus where the Strichartz norm is large are distincts.

The general result with an infinite number of profiles uses similar ideas. This proof is quite technical and we refer to [22] for more details. The main difficulties that appear are:

- to find equivalent on manifolds of the tools developed on $\mathbb{R}^3$ by Bahouri-Gérard: to extract scales of oscillation and to track the points of concentration

- the equation is damped : we lose the conservation of energy, of orthogonality and of spectral localization. Then, we only have almost conservation of orthogonality by propagation of joint microlocal defect measures and almost conservation of spectral localization.

- to take care of geometric effects: we have to prove that each non linear profile do not interact with the others. We use the description of the non linear profiles by S. Ibrahim described in subsection 3.2.

Remark 3.1. *The restriction on the time $T < T_{focus}/2$ in Theorem 3.2 comes from our lack of precise understanding of the phenomenon of refocusing for a linear concentrating wave. More precisely, let (x_1, x_2, t) be a couple of focus at distance t. Take two orthogonal concentrating waves concentrating at x_1 with the same frequency h_n but with orthogonal sequence x_n^1 tending to x_1 with different rates: $\frac{x_{n,1} - \widetilde{x}_{n,1}}{h_n} \to +\infty$. By using microlocal defect measure, we are able to prove that the associated linear concentrating waves v_n and $\widetilde{v}_n$ will reconcentrate at x_2 after a time t still at frequence h_n. Yet, we are not able to get that the new way of local concentration at x_2 will still be with orthogonal $x_{n,2}$ and $\widetilde{x}_{n,2}$ converging to x_2. It could happen (even if very unlikely) that the new way of convergence is the same. Then, the nonlinear analysis would be different because the resulting nonlinear solutions would not keep an independant behavior. When describing the solution after concentration, it would become necessary to study the scattering operator applied to the sum of the concentrating profiles at x_2.*

Yet, this scenario is certainly not possible, but to prove it, we would need to get good informations on the new way of refocusing for a concentrating wave associated to a couple of focus. This would require some more precise techniques as 2-microlocal defect measures or Fourier Integral Operator.

3.4   End of the argument for stabilization

To obtain the exponential decay at high frequency, we need to prove the following weak observability estimate:

$$E(u)(0) \leq C \left(\iint_{[0,T] \times M} |\chi_\omega(x) \partial_t u|^2 \ dt dx + \|(u_0, u_1)\|_{L^2 \times H^{-1}} E(u)(0) \right).$$

The proof by contradiction is similar to the one described in the subcritical case except that the linearizability is false in general. The idea is to compute the profile decomposition to describe the defect. The additional geometrical assumption will then help us to kill the profiles.

More precisely, by contradiction, suppose

$$\iint_{[0,T] \times M} |\chi_\omega(x) \partial_t u_n|^2 + \|(u_{0,n}, u_{1,n})\|_{L^2 \times H^{-1}} E(u_n)(0) \leq \frac{1}{n} E(u_n)(0) \quad (10)$$

and $E(u_n)(0) \to \alpha > 0$ (otherwise, as before, we have a linear behavior which is easier).

In that case, the second term of (10) allows to skip the unique continuation and we obtain directly, $u_n \rightharpoonup 0$.

We want to prove the linearizability property. After completing the profile decomposition according to Theorem 3.2, it is equivalent to proving $q_n^{(j)} \equiv 0$ for all $j \in \mathbb{N}^*$.

- Estimate (10) gives a strong convergence of $\partial_t u$ on ω. This gives $\mu = 0$ on $S^*(]0, T[\times \omega)$ where μ is the microlocal defect measure associated to u_n. This implies $e \equiv 0$ on $]0, T[\times \omega$ where e is the energy density limit of u_n (that is the measure weak-$*$ limit of the local energy of u)

- Using the fact that each nonlinear profile "behaves independently", e can be decomposed as

$$e(t, x) = \sum_{j=1}^{+\infty} e^{(j)}(t, x) + e_f(t, x)$$

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where $e^{(j)}$ is associated to each profile and e_f is the energy density limit of a sequence of linear solutions. This is actually the profile decomposition seen from the point of view of the density of energy. The positivity of all the measures in the decomposition and the fact that $e \equiv 0$ on $]0, T[\times\omega$ allow to conclude that $e^{(j)} \equiv 0$ on $]0, T[\times\omega$. This also gives $\mu^{(j)} = 0$ on $S^*(]0, T[\times\omega)$ where $\mu^{(j)}$ is the microlocal defect measure associated to $q_n^{(j)}$.

- Let T_0 be the time in the assumption Geometric Control Condition before refocusing. Since $T_0 < T_{focus}$, for each $j \in \mathbb{N}^*$, we can find a subinterval I_j of lenght greater T_0 on which $q_n^{(j)}$ does not focus. Therefore, on that interval I_j, $q_n^{(j)}$ has a linear behavior and $\mu^{(j)}$ propagates along the Hamiltonian flow. Since the geometric control condition is satisfied on (ω, I_j) and $\mu^{(j)} = 0$ on $S^*(]0, T[\times\omega)$, we obtain $\mu^{(j)} = 0$ on $S^*(I_j \times M)$ and so $q_n^{(j)} \equiv 0$.

This proves the linearizability and allows to finish the proof as in the sub-critical case.

Remark 3.2. *We have a little simplified the proof since, in fact, we are only able to prove the "independant behavior" of the profiles for some time less than T_{focus} while we need a time larger than $2T_0$ to kill all the profiles. In practice, we apply a process that kills the profiles according to their first time of concentration.*

4 Further problems

4.1 Suppressing the assumption about focusing?

In order to go beyond the Geometric Control before refocusing, it is necessary to have a better understanding of the behavior at the focus and use some specific properties of the euclidian scattering operator. One additional argument that could be used is the conservation of the momentum of the nonlinear scattering operator.

For instance, in some specific case for S^3, it may be possible to avoid the strong assumption of Geometric Control before refocusing with ω a neighborhood of half the equator:

ω is a neighborhood of $\left\{ x = (x_1, x_2, x_3, x_4) \in S^3 \,|\, x_4 = 0 \text{ and } x_1 > 0 \right\}$

We briefly explain how to kill a profile concentrating on the North pole at time 0. Shortly after the time 0, the profile is linear. Then, the energy microlocal defect measure at time t propagates along some transport equation related to the Hamiltonian flow. Yet, since there is strong convergence to zero of the energy on $]0, T[\times\omega$, the specific geometry allows to obtain that the momentum of the initial concentrating profile has a negative component on the axis x_1. We easily obtain that the linear profile refocuses at time π in the South Pole with a new reflected profile, therefore with a momentum concentrated in the direction $x_1 \geq 0$. So after time π, the nonlinear solution is close to a linear solution obtained by a reflection and the scattering operator applied to the previous one. By conservation of the momentum by the scattering operator, its momentum projected on the axis x_1 is also positive. The same reasonning as before on a time $]\pi, 2\pi[$ and the specific geometry gives that this profile is an integral in Fourier with negative projection on the axis x_1. This combined with the sign of the momentum gives that the profile is zero.

As explained before, if we want to avoid our additional assumption of geometric control before refocusing, it becomes necessary to have a better understanding of the behavior of the scattering operator S on $\mathbb{R}^3$ with respect to Fourier direction. If a set of initial data (u_0, u_1) is supported in Fourier in a certain cone V (or more precisely, $u_1 \pm i\sqrt{-\Delta}u_0$ in direction $\pm V$), is that still true for $S(u_0, u_1)$? We have seen that it can remain true in some convex cone by using the conservation of momentum, but the general case is open (and might be false?). It is likely that this could be useful in other situations.

4.2   Suppressing the high frequency assumption?

As we have seen in section 2.2, the unique continuation is very problematic and imposes some additionnal conditions.

Concerning Carleman estimates, the results of H. Koch and D. Tataru [20] prove some unique continuation result in the critical case. This should allow to suppress the high frequency assumption for some specific geometries.

It would be also interesting to obtain the unique continuation, with some more general geometric assumption on ω as the geometric control condition.

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