Unique continuation and applications

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Chapter 1

Introduction and generalities

The focus is put here on second order operators and applications arising from physics, namely through the operators Δ (Laplace operator), $\partial_t^2 - \Delta$ (wave operator, or d'Alembert operator), $i\partial_t - \Delta$ (Schrödinger operator), $\partial_t - \Delta$ (heat operator)...

There are many references on Carleman estimates and unique continuation. We mention here a nonexhaustive list:

- this course was influenced by the course of Nicolas Lerner, that can be found on his website http://webusers.imj-prg.fr/~ nicolas.lerner/m2carl.pdf,
- as well as the survey article by Jérôme Le Rousseau and Gilles Lebeau [LRL12];
- The most classical reference on unique continuation for partial differential operators is the Chapter XXVIII of Lars Hörmander's treatise [Hör94]. The latter gives a more general framework for what is described in Chapter 2.
- the book of Claude Zuily [Zui83] is another classical reference;
- We also refer to the complete notes of Daniel Tataru available at https://math.berkeley.edu/~ tataru/papers/ucpnotes.ps.
- finally, the presentation of Chapter 3, concerning the wave operator, is inspired by the article [Hör97].

1.1 Motivation and applications

We start with presenting different applications to motivate the more technical parts of these notes. All these applications are discussed in detail later on in the notes.

1.1.1 Tunneling estimates for eigenfunctions

Given a compact Riemannian manifold $\mathcal{M}$ with or without boundary $\partial\mathcal{M}$, we consider the eigenfunction problem

$$-\Delta_g \psi_\lambda = \lambda \psi_\lambda, \quad \psi_\lambda|_{\partial\mathcal{M}} = 0. \tag{1.1}$$

It is known that the equation $(-\Delta_g + 1)u = f, u|_{\partial\mathcal{M}} = 0$ has for any fixed $f \in L^2(\mathcal{M})$ a unique solution $u \in H_0^1(\mathcal{M})$ (consequence of the Riesz representation theorem in H_0^1). The map $(-\Delta_g + 1)^{-1} : L^2(\mathcal{M}) \to H_0^1(\mathcal{M})$ is hence compact $L^2 \to L^2$, and this implies that the eigenfunction equation (1.1) has solutions for only a discrete number of values of λ. The latter are real, nonnegative, since $-\int_{\mathcal{M}}(\Delta_g u)v dx = \int_{\mathcal{M}} \nabla_g u \cdot \nabla_g v dx$ for $u, v \in H^2(\mathcal{M})$ with $u|_{\partial\mathcal{M}} = v|_{\partial\mathcal{M}} = 0$.

This allows to introduce the eigenvalues $\lambda_j \in \mathbb{R}^+$, for $j \in \mathbb{N}$. Compactness of $(-\Delta_g + 1)^{-1}$ also implies that $\lambda_j \to +\infty$ as $j \to +\infty$. These eigenvalues are delivered with associated eigenfunctions, i.e. $\psi_j \in H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})$ such that $-\Delta_g \psi_j = \lambda_j \psi_j$, which, once L^2-normalized, form a Hilbert basis of $L^2(\mathcal{M})$, and in particular satisfy $(\psi_i, \psi_j)_{L^2(\mathcal{M})} = \int_{\mathcal{M}} \psi_i(x)\overline{\psi_j}(x)dx = \delta_{ij}$. Note that in dimension 1, these eigenvalues/eigenfunctions are particularly simple. For instance, on the interval $\mathcal{M} = [0, L]$, for $L > 0$, we have for $j \in \mathbb{N}^*$

$$\lambda_j = \left(\frac{j\pi}{L}\right)^2, \quad \psi_j(x) = \sqrt{\frac{2}{L}} \sin\left((\frac{j\pi}{L})x\right), \qquad \int_0^L |\psi_j(x)|^2 dx = 1.$$

(one could also consider the even simpler boundaryless situation $\mathcal{M} = \mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$, for which we have $\psi_k^\pm(x) = (2\pi)^{-1/2} e^{\pm ikx}$, and associated eigenvalues $\lambda_k^\pm = k^2$).

These eigenfunctions describe the resonant states of the domain (drum) $\mathcal{M}$. They are also particularly useful to describe solutions to evolution equations involving Δ_g. We have solutions of the heat equation:

$$\begin{cases} (\partial_t - \Delta_g)u = 0, & \text{on } \mathbb{R}_*^+ \times \mathrm{Int}(\mathcal{M}) \\ u = 0, & \text{on } \mathbb{R}_*^+ \times \partial\mathcal{M} \end{cases} \iff u(t, x) = \sum_{j \in \mathbb{N}} u_j e^{-\lambda_j t} \psi_j(x).$$

Similarly, solutions to the wave equation write

$$\begin{cases} (\partial_t^2 - \Delta_g)u = 0, & \text{on } \mathbb{R} \times \mathrm{Int}(\mathcal{M}) \\ u = 0, & \text{on } \mathbb{R} \times \partial\mathcal{M} \end{cases} \iff u(t, x) = \sum_{j \in \mathbb{N}} u_j^+ e^{i\sqrt{\lambda_j} t} \psi_j(x) + u_j^- e^{-i\sqrt{\lambda_j} t} \psi_j(x).$$

From these above considerations, it appears that an important question concerning eigenfunctions is the following: where are the eigenfunctions ψ_j localized? A preliminary question can be formulated as follows:

Can eigenfunctions ψ_j identically vanish on a nonempty open set $\omega \subset \mathcal{M}$?

As a consequence of a unique continuation result for the Laplace operator, we shall see that this does never happen. The next natural question is then

Can eigenfunctions ψ_j *asymptotically* vanish on the (nonempty) open set ω as $j \to +\infty$?

and if so, at which rate? In the abovementioned one dimensional situation $\mathcal{M} = [0, L]$, with $\omega = (a, b) \subset [0, L]$, we have

$$\int_\omega |\psi_j|^2(x) dx \to \frac{|\omega|}{L}, \qquad \text{as } j \to +\infty.$$

Indeed, we have

$$\int_a^b |\psi_j|^2(x) dx = \frac{2}{L} \int_a^b \left|\sin\left((\frac{j\pi}{L})x\right)\right|^2 dx = \frac{2}{L} \int_a^b \left[\frac{1}{2} - \frac{1}{2}\cos\left((\frac{2j\pi}{L})x\right)\right] dx$$
$$= \frac{b-a}{L} - \frac{1}{2j\pi}\left[\sin\left((\frac{2j\pi}{L})x\right)\right]_a^b = \frac{b-a}{L} + O\left(\frac{1}{j}\right), \qquad \text{as } j \to +\infty.$$

That is to say that, in this particular situation, the eigenfunctions ψ_j equidistribute in $[0, L]$ asymptotically. Of course, this very strong property does not hold in general; one may however want to quantify the property $\|\psi_j\|_{L^2(\omega)} > 0$. We shall actually prove that eigenfunctions never decay exponentially: namely, for all nonempty open set $\omega \subset \mathcal{M}$, there is $C, \kappa > 0$ such that

$$\|\psi_\lambda\|_{L^2(\omega)} \geq C e^{-\kappa\sqrt{\lambda}},$$

for all (λ, ψ_λ) solutions to (1.1) with $\|\psi_\lambda\|_{L^2(\mathcal{M})} = 1$. This is a manifestation of what is called "tunneling effect" in quantum mechanics. One can prove that this is optimal in general (in the sense that there exist $(\mathcal{M}, g, \omega)$ for which there are indeed eigenfunctions with $\|\psi_j\|_{L^2(\omega)} \leq C e^{-\kappa\sqrt{\lambda_j}}$, see Section 2.4.4). Of course, this can be much improved in different situations, as in the 1D case discussed above, for which we have a uniform lower bound $\|\psi_\lambda\|_{L^2(\omega)} \geq C$.

1.1.2 Penetration of waves into the shadow region

In this section, we consider the wave equation outside a convex obstacle in $\mathbb{R}^n$. Namely, let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded smooth open subset, and consider $\mathcal{M} = \mathbb{R}^n \setminus \mathcal{O}$. We consider the Laplace operator Δ and $u(t,x)$ the solution to the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & \text{on } \mathbb{R} \times \mathrm{Int}(\mathcal{M}), \\ u = 0, & \text{on } \mathbb{R} \times \partial\mathcal{M}, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), & \text{on } \mathcal{M}. \end{cases} \tag{1.2}$$

The quantity $u(t,x)$ might for instance model

- the displacement of a membrane;
- the intensity of a light;
- the pressure of a sound...

measured at point $x \in \mathcal{M}$ and at time t.

Now, we consider a compact set $K \subset \mathcal{M}$, and assume that the initial data (u_0, u_1) are supported in K. If the set K is not too large, there is a whole region of $\mathcal{M}$ which does not intersect any ray of geometric optics in $\mathcal{M}$ (i.e. straight line in $\mathrm{Int}(\mathcal{M})$, which reflects according to Snell-Descartes laws at the boundary $\partial\mathcal{M}$) passing through K. Taking an open set ω in this shadow region, the question under consideration is the following:

Can one recover (u_0, u_1) from the observation of u on the set $(-T, T) \times \omega$?

And if so, what is the time T required? By linearity of (1.2), this can be reformulated under the following unique continuation question:

$$\big(u \text{ solution to (1.2)}, \quad u|_{(-T,T)\times\omega} = 0\big) \implies (u_0, u_1) = 0?$$

(and hence $u \equiv 0$).

We shall prove that this is false if T is too small, but that this is actually the case if T is large enough. The limit time will be expressed as a natural geometric quantity.

1.1.3 Approximate controllability for the wave equation

In this section, we consider a wave equation in a compact manifold $\mathcal{M}$ (or the closure of a bounded open set $\mathcal{M} \subset \mathbb{R}^n$), controlled from a subdomain. Namely, given $\omega \subset \mathcal{M}$ a nonempty open set, the equation

$$\begin{cases} \partial_t^2 u - \Delta_g u = \mathbb{1}_\omega f, & \text{on } (0,T) \times \mathrm{Int}(\mathcal{M}), \\ u = 0, & \text{on } (0,T) \times \partial\mathcal{M}, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), & \text{on } \mathcal{M}. \end{cases} \tag{1.3}$$

The term f in this equation plays the role of a forcing term. Controllability problems concern the ability of driving the solution u to (1.3) from the initial state (u_0, u_1) to a final target state (v_0, v_1) at time T, using only the action of f on ω. This property depends a priori on the data/target, and is too complicated. More tractable questions, arising from applications in engineering are the following

Definition 1.1.1. We say that (1.3) is *exactly controllable* from (ω, T) if for all data $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ and all target state $(v_0, v_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$, there is a function $f \in L^2((0,T)\times\omega)$ such that the solution to (1.3) satisfies $(u, \partial_t u)|_{t=T} = (v_0, v_1)$.

We say that (1.3) is *approximately controllable* from (ω, T) if for all data $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$, all target state $(v_0, v_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$, and all precision $\varepsilon > 0$, there is a function $f = f_\varepsilon \in L^2((0,T)\times\omega)$ such that the solution to (1.3) satisfies $\|(u, \partial_t u)|_{t=T} - (v_0, v_1)\|_{H_0^1(\mathcal{M})\times L^2(\mathcal{M})} \leq \varepsilon$.

Due to finite speed of propagation for waves, if $\overline{\omega} \neq \mathcal{M}$, a minimal time will be required for controllability to hold. Here, we will mostly be interested in the (weaker) approximate controllability question.

Linearity of the equation shows it is enough to consider zero initial conditions $(u_0, u_1) = (0, 0)$. Introducing the linear map

$$\begin{array}{rcl} F : L^2((0,T) \times \omega) & \to & H_0^1(\mathcal{M}) \times L^2(\mathcal{M}) \\ f & \mapsto & (u, \partial_t u)|_{t=T}, \end{array}$$

where u denotes the solution of (1.3) associated to $(u_0, u_1) = (0, 0)$, approximate controllability is equivalent to range(F) being dense in $H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$. This can be reformulated as $\ker(F^*) = \{0\}$, where F^* is an appropriate adjoint (or dual) to F. After some work, one can identify F^* to be the map

$$\begin{array}{rcl} F^* : L^2(\mathcal{M}) \times H^{-1}(\mathcal{M}) & \to & L^2((0,T) \times \omega) \\ (w_0, w_1) & \mapsto & w|_{(0,T)\times\omega}, \end{array}$$

where w is the unique solution to

$$\left\{ \begin{array}{rl} \partial_t^2 w - \Delta_g w = 0, & \text{on } (0,T) \times \text{Int}(\mathcal{M}), \\ w = 0, & \text{on } (0,T) \times \partial\mathcal{M}, \\ (w, \partial_t w)|_{t=0} = (w_0, w_1), & \text{on } \mathcal{M}. \end{array} \right. \tag{1.4}$$

Again, $\ker(F^*) = \{0\}$ is the unique-continuation property

$$\big(w \text{ solution to (1.4)}, \quad w|_{(0,T)\times\omega} = 0\big) \implies (w_0, w_1) = 0,$$

which now appears to characterize the approximate controllability of (1.3).

1.1.4 Trend to equilibrium for the damped wave equation

♣ `Prove energy goes to zero for all solutions?`

1.1.5 Controllability of the heat equation

In this section, we discuss similar controllability issues as in Section 1.1.3, but for the heat equation

$$\left\{ \begin{array}{rl} \partial_t u - \Delta_g u = \mathbb{1}_\omega f, & \text{on } (0,T) \times \text{Int}(\mathcal{M}), \\ u = 0, & \text{on } (0,T) \times \partial\mathcal{M}, \\ u|_{t=0} = u_0, & \text{on } \mathcal{M}. \end{array} \right. \tag{1.5}$$

The term f again acts as a localized control on the state u (temperature). Because of the smoothing properties of the heat equation (e.g., if $u_0 \in L^2(\mathcal{M})$ and $f \in L^2((0,T) \times \omega)$ the solution of (1.5) will satisfy for all $t \in (0,T)$ $u(t,\cdot) \in C^\infty(\text{Int}(\mathcal{M}) \setminus \bar{\omega})$), it is hopeless to control exactly to *any* target u_T in the state space $L^2(\mathcal{M})$. Two alternatives are in order: approximate controllability and controllability to trajectories (i.e. to target states v that are solutions at time T to the free heat equation (1.5) with $f = 0$).

Definition 1.1.2. We say that (1.5) is *controllable to trajectories* from (ω, T) if for all $u_0, v_0 \in L^2(\mathcal{M})$, there is a function $f \in L^2((0,T) \times \omega)$ such that the solution to (1.5) satisfies $u|_{t=T} = e^{T\Delta_g} v_0$ (where $e^{T\Delta_g} v_0$ denotes the solution at time T to the free heat equation (1.5) with $f = 0$ and initial datum v_0).

We say that (1.5) is *null-controllable* from (ω, T) if for all data $u_0 \in L^2(\mathcal{M})$, there is a function $f \in L^2((0,T) \times \omega)$ such that the solution to (1.5) satisfies $u|_{t=T} = 0$.

We say that (1.5) is *approximately controllable* from (ω, T) if for all data $u_0 \in L^2(\mathcal{M})$, all target state $v_1 \in L^2(\mathcal{M})$, and all precision $\varepsilon > 0$, there is a function $f = f_\varepsilon \in L^2((0,T) \times \omega)$ such that the solution to (1.5) satisfies $\|u|_{t=T} - v_1\|_{L^2(\mathcal{M})} \leq \varepsilon$.

Linearity of the equation (1.5) implies that controllability to trajectories is equivalent to null-controllability. Similar arguments as for the wave equation reduce these controllability questions to unique continuation/observability issues for the free heat equation

$$\left\{ \begin{array}{rl} \partial_t w - \Delta_g w = 0, & \text{on } (0,T) \times \text{Int}(\mathcal{M}), \\ w = 0, & \text{on } (0,T) \times \partial\mathcal{M}, \\ w|_{t=0} = w_0, & \text{on } \mathcal{M}, \end{array} \right. \tag{1.6}$$

observed from $(0,T) \times \omega$.

More precisely, the observation map is given by

$$\begin{array}{rcl} F^* : L^2(\mathcal{M}) & \to & L^2((0,T)\times\omega) \\ w_0 & \mapsto & w|_{(0,T)\times\omega}, \end{array}$$

where w is the solution to (1.4). As for the wave equation, a duality argument (functional anaysis) proves that

- approximate controllability from (ω, T) is equivalent to F^* being injective, i.e. to the unique continuation property
$$\big(w \text{ solution to (1.6)}, \quad w|_{(0,T)\times\omega} = 0\big) \implies w_0 = 0;$$
- null-controllability from (ω, T) is equivalent to the observability estimate: there is $C > 0$ such that
$$\|w(T)\|^2_{L^2(\mathcal{M})} \leq C \int_0^T \|w(t,\cdot)\|^2_{L^2(\omega)}\, dt, \qquad \text{for all } w_0 \in L^2(\mathcal{M}) \text{ and associated } w \text{ solution to (1.6)}.$$

Note that this last inequality observes the norm at time T, which is much weaker than observing the solution at time 0 (which, in turn, would be equivalent to exact controllability, and thus never holds if $\bar{\omega} \neq \mathcal{M}$). This is in strong contrast with the wave equation, for which energy is conserved. These two properties again take the form of (global) qualitative/quantitative unique continuation properties for the heat operator.

As for the wave equation, the above two properties will reflect the way of propagation of the energy for solutions to the heat equation, namely instantaneously (with infinite propagation speed), and in all directions.

1.2 Generalities about unique continuation

1.2.1 The unique continuation problem

All above described problems amount to a *unique continuation* property (or a quantitative unique continuation property) for a differential operator $P = -\Delta_g - \lambda$ (eigenfunctions), $P = \partial_t - \Delta_g$ (heat), $P = \partial_t^2 - \Delta$ (waves).

The general problem of *unique continuation* can be set into the following form: given a differential operator $P = \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha$ on an open set $\Omega \subset \mathbb{R}^n$, and given a small subset U of Ω, do we have (for u regular enough):

$$\left\{ \begin{array}{rcl} Pu & = & 0 \text{ in } \Omega, \\ u & = & 0 \text{ in } U \end{array} \right. \implies u = 0 \text{ on } \Omega. \tag{1.7}$$

In cases where (1.7) is known to hold, it is often interesting to prove a quantitative version of

$$\left\{ \begin{array}{lll} Pu & \text{small in} & \Omega, \\ u & \text{small in} & U \end{array} \right. \implies u \text{ small in } \Omega.$$

A more tractable problem than (1.7) is the so called *local unique continuation across an hypersurface* problem: given an oriented local hypersurface $S = \{\Psi = 0\}$ at a point x_0 (that is $\Psi(x_0) = 0$ and $d\Psi(x_0) \neq 0$), do we have the following implication:

There is a neighborhood Ω of x_0 so that

$$\left\{ \begin{array}{rcl} Pu & = & 0 \text{ in } \Omega, \\ u & = & 0 \text{ in } \Omega \cap S^+ \end{array} \right. \implies u = 0 \text{ in a neighborhood of } x_0. \tag{1.8}$$

where $S^+ = \{\Psi > 0\}$ is one side of S.

It turns out that proving (1.8) for a suitable class of hypersurfaces S (with regards to the operator P) is in general a key step in the proof of properties of the type (1.7).

Note that under a certain geometric condition on the surface (namely, assuming S is *non-characteristic* for P at x_0, see ♣ below), the local unique continuation question (1.8) is equivalent to the so called *uniqueness in the Cauchy problem*, namely the question whether there is a neighborhood Ω of x_0 such that

$$\left\{\begin{array}{l} Pu = 0 \text{ in } \Omega, \\ u|_S = \partial_\nu u|_S = \cdots = \partial_\nu^{m-1} u|_S = 0 \text{ on } S \end{array}\right. \implies u = 0 \text{ in a neighborhood of } x_0, \tag{1.9}$$

where ∂_ν denotes a normal vector field to S, and m is the order of the differential operator P.

Here, we collect some simple and very informative situations in which the local unique continuation property (1.8) is well-understoood: ♣ `draw pictures for vector-fields`

1. (dimension one) If $n = 1$, a differential operator writes $P = \sum_{k=0}^m a_k(x) \frac{d^k}{dx^k}$ where we assume a_k are defined in a neighborhood Ω of 0 and are smooth enough. The problem $Pu = 0$ in Ω is only an ordinary differential equation. Here the surface $S = \{0\}$ is a point. If u is a regular enough solution to the equation $Pu = 0$ in Ω such that $u = 0$ in $S^+ = \{x > 0\}$, then we have $\frac{d^k}{dx^k} u(0) = 0$ for all $k \in \mathbb{N}$ and, assuming P is non-degenerate at zero ($a_m(0) \neq 0$) the Cauchy-Lipschitz theorem gives directly the (local) uniqueness since $u = 0$ in a neighborhood of 0.

2. ($\overline{\partial}$ operator) If $n = 2$, $P = \overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ is the so-called Cauchy-Riemann operator. That $Pu = 0$ in an open set $\Omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ implies that u is an analytic function in Ω. In particular, if $u = 0$ on any nonempty open subset of Ω (or more generally on any set containing an accumulation point) then $u = 0$ identically on Ω.

3. (Linear vector fields and flat hypersurface) Consider the operator $P = \frac{\partial}{\partial x_1}$ in $\mathbb{R}^n$ in a neighborhood of 0 and a hyperplane $S = \{x \in \mathbb{R}^n, \langle b, x\rangle = 0\}$, where $b = (b_1, \cdots, b_n) \in \mathbb{R}^n \setminus \{0\}$. Then $Pu = 0$ if and only if u does not depend on x_1, i.e. $u(x_1, x_2, \cdots, x_n) = u^0(x_2, \cdots, x_n)$ for all $x_1 \in \mathbb{R}$. Assume further $b_1 \neq 0$ and that $u = 0$ in $S^+ = \{x \in \mathbb{R}^n, \langle b, x\rangle > 0\}$ in a neighborhood of zero. Then, taking $(x_2, \cdots, x_n)$ in a neighborhood of 0, and x_1 with $b_1 x_1$ large enough, we have $(x_1, x_2, \cdots, x_n) \in S^+$ so that $u(x_1, \cdots, x_n) = 0 = u^0(x_2, \cdots, x_n) = 0$. Hence $u^0 = 0$ and $u = 0$. Local unique continuation across S thus holds if $b_1 \neq 0$. The converse is also true. Indeed, if $b_1 = 0$, choose $u(x) = \chi(\langle b, x\rangle)$ where $s \mapsto \chi(s)$ is any function $\neq 0$ on $s > 0$ and $= 0$ on $s \leq 0$. Then, u is x_1-invariant hence $Pu = 0$, and satisfies $\text{supp}(u) = S^+$. In conclusion, in this simple setting, a necessary and sufficient condition for local unique continuation across S is that the vectorfield $\frac{\partial}{\partial x_1}$ is *not tangent* to the hyperplane S.

4. (Real non-degenerate vector-fields) We give here (without proof) a more general context for this last result. We assume that P is a general vectorfield (or, equivalently, first order ($m = 1$) homogeneous differential operator) near 0, that is $P = \sum_{k=0}^n a_k(x)\partial_{x_k}$. Assume further that it is nondegenerate at 0, that is $a(0) = (a_1(0), \cdots, a_n(0)) \neq 0$. Take $S = \{\Psi = 0\}$ where $\Psi(0) = 0$ and $d\Psi(0) \neq 0$. Then, a sufficient condition for having local unique continuation (1.8) is that $\langle d\Psi(0), a(0)\rangle \neq 0$, that is that the vector-field P is transversal to S at 0. This condition is a "non-characteristicity assumption", see Definition 1.2.3 below. The local straightened model in this case is that of the former example. Note that the condition $\langle d\Psi(0), a(0)\rangle \neq 0$ is not necessary for unique continuation to hold, see the discussion in Example 5 below.

5. (Linear vector fields and curved hypersurface) Here (as opposed to previous examples), we shall see that the orientation of the surface may play a role. Consider the operator $P = \frac{\partial}{\partial x_1}$ (as in Item 3) in $\mathbb{R}^2$ in a neighborhood of 0, but the curved hypersurface $S = \{x = (x_1, x_2) \in \mathbb{R}^2, \Psi(x) = 0\}$, where $\Psi(x_1, x_2) = x_2 - x_1^2$. Notice first that S is tangent to P at 0 since $\langle d\Psi(0), P\rangle = 0$. We shall see that unique continuation holds from $S^+ = \{\Psi > 0\}$ (outside the parabola) to $S^- = \{\Psi > 0\}$ (inside the parabola), but not from S^- to S^+.

   Indeed solutions u to $Pu = 0$ write $u(x_1, x_2) = u^0(x_2)$ for all $x_1 \in \mathbb{R}$. The first statement then follows from the fact that any line $x_2 = cst > 0$ intersects S^+ in a neighborhood of zero, thus showing that if $u^0(x_2) = 0$ for all x_1 in a neighborhood of zero, then $u^0 = 0$. Choosing $u^0 \in C_c^\infty(\mathbb{R})$ such that $u^0(x_2) \neq 0$ on $0 > x_2 > -1$ and $u^0(x_2) = 0$ on $x_2 \geq 0$ yields the second statement.

6. (One dimensional wave operator) Consider the wave operator $P = \partial_t^2 - \partial_x^2$ on $\mathbb{R}_t \times \mathbb{R}_x$. Then P factorizes as $P = (\partial_t + \partial_x)(\partial_t - \partial_x)$ and all solutions to $Pu = 0$ write $u(t,x) = f(x+t) + g(x-t) + C_0 t + C_1 x + C_2$, where f, g are functions and C_j constants. Take for instance $g = 0$, $C_j = 0$ and $f \in C^\infty(\mathbb{R})$ with $\operatorname{supp}(f) = [0,1]$. Then $u(t,x) = f(x+t)$ and the surface $S = \{x + t = 0\}$ thus does not satisfy the unique continuation property (at any point). More precisely, up to linear changes of variables, this problem reduces to that of linear vectorfields discussed above, and one sees that the only hyperplanes S not satisfying the unique continuation property (at any point) are $S_\pm^\alpha = \{x \pm t = \alpha\}$, for $\alpha \in \mathbb{R}$.

The above examples 3-4-5-6 concerning first order partial differential operators (namely, vector fields) and the wave operator show that geometrical conditions linking the operator P and the surface S are often needed for unique continuation to hold. Note that the example 2 also "suggests" that no geometric condition is needed for *elliptic* operators.

As stressed in Section 1.1, many important differential operators arising from physics are linked to the Laplace operator. In $\mathbb{R}^n$, it is simply defined by

$$\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}.$$

One of the main focuses of these notes is the wave operator, which has a strong geometric and physical content. Let us now discuss some features of this operator in more detail.

1.2.2 Remarks on the wave operator

In this section, we collect known facts for the wave equation in the flat space $\mathbb{R}^n$, that are related to unique continuation questions. We start with local energy estimates and a proof of finite speed of propagation in this context.

Theorem 1.2.1 (Finite speed of propagation for the wave equation). *Let u be a $C^2(\mathbb{R}^{1+n})$ (real-valued) solution of*

$$(\partial_t^2 - \Delta)u = 0 \qquad \text{on } \mathbb{R} \times \mathbb{R}^n, \tag{1.10}$$

and define the local energy in the ball of radius r at time t by:

$$E_r(t) = \frac{1}{2} \int_{|x| \le r} \left((\partial_t u(t,x))^2 + |\nabla_x u(t,x)|^2 \right) dx.$$

Then, for any $r_0 > 0$ and any $t \in [0, r_0]$, we have

$$E_{r_0 - t}(t) \le E_{r_0}(0). \tag{1.11}$$

In particular, if $u|_{t=0}(x) = \partial_t u|_{t=0}(x) = 0$ for $|x| \le r_0$, then $u = 0$ in the cone

$$C_{r_0} = \left\{ (t,x) \in \mathbb{R}^{1+n} \text{ s.t. } t \in [0, r_0] \text{ and } |x| \le r_0 - t \right\}.$$

Denoting by

$$e(t,x) = \frac{1}{2} \left((\partial_t u(t,x))^2 + |\nabla_x u(t,x)|^2 \right) \tag{1.12}$$

the density of energy, the proof also yields the estimate

$$\int_{C_{r_0}} e(t,x) dt dx \le r_0 \int_{|x| \le r_0} e(0,x) dx.$$

Proof. Multiply the equation by $\partial_t u$ to obtain $\partial_t^2 u \partial_t u - \Delta u \partial_t u = 0$. First, we notice $\partial_t^2 u \partial_t u = \frac{1}{2} \partial_t (\partial_t u(t,x))^2$. Moreover, by the Leibnitz formula $\operatorname{div}_x(fX) = f \operatorname{div}(X) + \nabla f \cdot X$ valid for f a C^1 function and X a C^1 vector field, we have

$$-\Delta u \partial_t u = -\operatorname{div}_x(\nabla u) \partial_t u = -\operatorname{div}_x(\nabla_x u \partial_t u) + \nabla_x u \cdot \nabla_x \partial_t u = -\operatorname{div}_x(\nabla u \partial_t u) + \partial_t \frac{|\nabla_x u|^2}{2}.$$

Therefore, recalling the definition of e in (1.12), we have obtained the local energy balance:

$$\partial_t e - \operatorname{div}(\nabla u \partial_t u) = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^n. \tag{1.13}$$

The main tool will be the Stokes theorem ♣ `reference ?`.

Lemma 1.2.2 (Stokes formula)**.** *Let $d \in \mathbb{N}^*$ and $X = \sum_{k=0}^{d} a_k(x)\partial_{x_k}$ be a C^1 vector field on a bounded domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$ being piecewise C^1 and graph-Lipschitz. Denote by $N(x)$ the outward pointing normal vector on $\partial\Omega$ (being piecewise C^1 as well). Then, we have the formula*

$$\int_{\Omega} \operatorname{div}(X)(x)dx = \int_{\partial\Omega} X(x) \cdot N(x) d\sigma(x),$$

where $d\sigma$ is the surface measure on $\partial\Omega$ and $\operatorname{div}(X)(x) = \sum_{k=0}^{d} \partial_{x_k} a_k(x)$.

We now integrate the energy balance (1.13) in time-space on the truncated cone, defined for $t_0 \leq r_0$, by

$$C_{r_0,t_0} = \left\{(t,x) \in \mathbb{R}^{1+n} \text{ s.t. } t \in [0,t_0] \text{ and } |x| \leq r_0 - t\right\}.$$

♣ `picture`

Note that it is piecewise C^1 and graph-Lipschitz, and that its boundary $\partial C_{r_0,t_0}$ is the union of three pieces with the following normal vector field:

- in the bottom part $S_0 = \left\{(0,x) \in \mathbb{R}^{1+n} \text{ s.t. } |x| \leq r_0\right\} = \{0\} \times \overline{B}_{r_0}$, the outward normal is $N(x) = (-1_t, 0_x)$;
- in the top part $S_{t_0} = \left\{(t_0,x) \in \mathbb{R}^{1+n} \text{ s.t. } |x| \leq r_0 - t_0\right\} = \{t_0\} \times \overline{B}_{r_0-t_0}$, the outward normal is $N(x) = (1_t, 0_x)$;
- in the lateral boundary $M_0^{t_0} = \left\{(t,x) \in \mathbb{R}^{1+n} \text{ s.t. } t \in [0,t_0] \text{ and } |x| = r_0 - t\right\}$, the outward normal is $N(t,x) = (1, x/|x|)/|(1,x/|x|)| = (1,x/|x|)/\sqrt{2}$.

We now apply the integration by part of Lemma 1.2.2 to the set $\Omega = C_{r_0,t_0}$ and the vector field $X = X_1 + X_2$ where:

- $X_1 = e(t,x)(1_t, 0_x) = e(t,x)\partial_t$ so that $\operatorname{div}_{t,x} X_1 = \partial_t e$;
- $X_2 = (0_t, -\partial_t u \nabla_x u)$ so that $\operatorname{div}_{t,x} X_2 = -\operatorname{div}_x(\partial_t u \nabla_x u)$.

Equation (1.13) expresses $\operatorname{div}_{t,x} X = 0$ and hence

$$\begin{aligned} 0 &= \int_{C_{r_0,t_0}} \operatorname{div}_{t,x} X_1 + \operatorname{div}_{t,x} X_2 \\ &= \int_{S_0} X \cdot (-1_t, 0_x) + \int_{S_{t_0}} X \cdot (1_t, 0_x) + \frac{1}{\sqrt{2}} \int_{M_0^{t_0}} X \cdot (1, x/|x|)\; d\sigma \\ &= -\int_{S_0} e(0,x)dx + \int_{S_{t_0}} e(t_0,x)dx + \frac{1}{\sqrt{2}} \int_{M_0^{t_0}} \left(e(t,x) - \partial_t u \nabla_x u \cdot \frac{x}{|x|} \right) d\sigma. \end{aligned}$$

Now, we remark that the integral on the lateral boundary is nonnegative since

$$\left| \partial_t u \nabla_x u \cdot \frac{x}{|x|} \right| \leq |\partial_t u||\nabla_x u| \leq \frac{1}{2}\left(|\partial_t u|^2 + |\nabla_x u|^2\right) = e. \tag{1.14}$$

We have thus obtained

$$-\int_{S_0} e(0,x)dx + \int_{S_{t_0}} e(t_0,x)dx \leq 0,$$

which is precisely the inequality (1.11) at time t_0.

Finally, let us prove the unique continuation property. The assumption implies that $E_{r_0}(0) = 0$. So, the inequality implies $E_{r_0-t}(t) = 0$ for $t \in [0,r_0]$ and in particular $\partial_t u = 0$ and $\nabla_x u = 0$ in the cone C_{r_0}. By connexity, this implies that $u = cste$ in C_{r_0}. This constant needs to be zero since $u(0,x) = 0$ for $|x| \leq r_0$, which concludes the proof of the theorem. □

Note that in the proof, the angle of the cone is the limiting one so that the Cauchy Schwarz in (1.14) holds.

Theorem 1.2.1 leads to the following definitions:

- The cone of dependence of a point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ is the cone

$$D_{(t_0,x_0)} = \left\{(t,x) \in \mathbb{R}^{1+n} \text{ s.t. } t \in [0,t_0] \text{ and } |x - x_0| \leq t_0 - t\right\}.$$

  The value $u(t_0, x_0)$ of the solution u to the wave equation (1.10) at the point (t_0, x_0) only depends on the values of u in $D_{(t_0,x_0)}$.

- The cone of influence of a point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ is the cone

$$I_{(t_0,x_0)} = \left\{(t,x) \in \mathbb{R}^{1+n} \text{ s.t. } t \geq t_0 \text{ and } |x - x_0| \leq t - t_0\right\}.$$

  The value $u(t,x)$ of the solution u to the wave equation (1.10) at a point (t,x) depends on the value of u at the point (t_0, x_0) if and only if $(t,x) \in I_{(t_0,x_0)}$.

We can infer an interesting consequence of Theorem 1.2.1 concerning the unique continuation property for the wave operator: unique continuation holds across the hypersurface $\{t = 0\}$ and actually, we have some nice local linear quantification of the unique continuation. This situation is actually a particular case of a more general situation in which the differential operator P (here $\partial_t^2 - \Delta$) is said to be hyperbolic with respect to the surface S (here e.g. $\{t = 0\}$). We refer to ♣ for more precisions.

As we have seen in above Example 6, the one dimensional wave equation is considerably simpler to analyse, since the d'Alembert operator factorizes as :

$$\partial_t^2 - \partial_x^2 = (\partial_t - \partial_x)(\partial_t + \partial_x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{1.15}$$

Hence, solutions to the wave equation reduce to solutions to two transport equations. The situation is radically different in higher dimensions. This is linked with the fact that the polynomial $\tau^2 - \sum_{j=1}^n \xi_j^2$ does not factorize in a product of polynomials of degree 1. This translates the fact that the values of solutions to (1.10) are not "transported". To see this, we can actually solve the wave equation (1.10). For instance, in $\mathbb{R}^3$, the Kirchhoff formula

$$u(t,x) = \frac{1}{4\pi t}\int_{|y-x|=t} u_1(y) dS_t(y) = \frac{t}{4\pi}\int_{\mathbb{S}^2} u_1^h(x - t\sigma) dS_1(\sigma), \quad u(-t) = -u(t), \quad t > 0 \tag{1.16}$$

gives the unique solution to (1.10) with $(u, \partial_t u)|_{t=0} = (0, u_1)$, $u_1 \in C^0(\mathbb{R}^3)$. In the first formula, the integration set is the (2 dimensional) sphere centered at x and of radius t; in the second it is the unit sphere. The integration measure dS is the surface measure on the sphere of radius t (induced by the Euclidean measure dx on $\mathbb{R}^3$). ♣ `See Exercice... for a proof of this formula, together with a similar formula in dimension 2.`

As a consequence of this explicit solution, we see that if we choose $u_1(x) = \chi(x)$ with $\chi \in C_c^\infty(\mathbb{R}^3)$, $\chi \geq 0$ and $\chi > 0$ on $B(0,r)$, $r > 0$ the associated solution u is smooth and satisfies $u \geq 0$ on $\mathbb{R}^{1+3}$. Moreover, notice that $u_1(x - t\sigma) = 0$ iff $x - t\sigma \notin B(0,r)$, we have $u(t,x) = 0$ as soon as $t\mathbb{S}^2 \cap B(x,r) = \emptyset$. As a consequence, we have

$$\operatorname{supp}(u) \cap \{t \geq 0\} = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, t - r \leq |x| \leq t + r\}. \tag{1.17}$$

Several remarks are in order. The fact that the solution u at time t vanishes in the ball $|x| \leq t - r$ corresponds to the strong Huygens principle; this is strongly related to the fact that the dimension 3 of $\mathbb{R}^3$ is odd, the metric is flat, and the wave operator has no lower order term. A contrario, the fact that the support of the solution at time t is contained in the ball $|x| \leq t + r$ translates the finite speed of propagation. This piece of information is already contained in Theorem 1.2.1. Finally, (1.17) also tells us that any point in the annulus $t - r \leq |x| \leq t + r$ is actually in the support of $u(t, \cdot)$. This new piece of information is very important for what follows. It implies that unique continuation cannot hold across an hypersurface tangent to the cone $|x| = t + r$.

♣ `Discuss finite propagation speed in a general riemannian setting, with lower order terms`

1.2.3 A general local unique continuation result in the analytic Category

The first general unique continuation result of the form (1.8) is the Holmgren Theorem, stating that, for operators with analytic coefficients, unique continuation holds across any noncharacteristic hypersurface S. Proper definitions of a differential operator of order m and its principal symbol $p_m(x,\xi)$ are given in Definition 1.2.6 below.

Definition 1.2.3. Let P be a differential operator of order m on Ω, $x_0 \in \Omega$ and S a local hypersurface passing through x_0, that is $S = \{\Psi = 0\}$, $\Psi(x_0) = 0$ and $d\Psi(x_0) \neq 0$ with $\Psi \in C^1(\Omega)$. We say that S is characteristic (resp. non-characteristic) for P at x_0 if $p_m(x_0, d\Psi(x_0)) = 0$ (resp. $p_m(x_0, d\Psi(x_0)) \neq 0$).

Also, given a local hypersurface $S = \{\Psi = 0\}$, it has locally two sides which we write

$$S^{\pm} = \{x \in \Omega; \pm\Psi(x) > 0\}.$$

Theorem 1.2.4 (Holmgren Theorem). *Let P be a differential operator of order m on Ω, having all coefficients* real analytic *in a neighborhood of $x_0 \in \Omega$ and $S \ni x_0$ being a local hypersurface. Assume that S is non characteristic for P at x_0. Then, there exists a neighborhood V of x_0 so that every $u \in \mathcal{D}'(\Omega)$ satisfying $Pu = 0$ on Ω and $u = 0$ in the set S^+ vanishes identically in V.*

Another way of writing the conclusion is to say that $x_0 \notin \operatorname{supp}(u)$. We refer e.g. to [Hör63, Theorem 5.3.1] for a proof of Theorem 1.2.4. Note that this unique continuation property does not take into account the orientation of the surface S, i.e. it holds from S^+ to S^- as well as from S^- to S^+.

The non-characteristicity condition is very weak, and in some sense optimal. Indeed, we saw in Examples 3 and 4 in Section 1.2.1 for linear vector-fields that unique continuation holds for non-characteristic surfaces, and does not hold for some characteristic surfaces. We also saw in Section 1.2.2 for the wave operator that local uniqueness does not hold across hypersurfaces that are tangent to the cone $|x| = t + r$. These are precisely characteristic surfaces: the principal symbol of the wave operator $\partial_t^2 - \Delta$ is given by $p_2(t, x, \xi_t, \xi_x) = -\xi_t^2 + |\xi_x|^2$, and a surface $\{\Psi(t,x) = 0\}$ tangent to $\{|x| = t + r\}$ at the point (t_0, x_0) has $|\partial_t \Psi(t_0, x_0)| = |d_x \Psi(t_0, x_0)|$. Remark however that the non-characteristicity condition is a "first order condition": it only cares about the tangent space of the surface. We saw in Example 5 in Section 1.2.1 in the case of first order differential operators a more subtle "second order condition" (curvature condition) on the surface may yield unique continuation across a characteristic surface. This is linked to the so-called pseudoconvexity condition (see e.g. Definition 2.3.1 below).

We recall that a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{C}$ is *real analytic* if for every $y \in \Omega$, there is a convergence radius $R > 0$ and coefficients $a_\alpha \in \mathbb{C}^n$, $\alpha \in \mathbb{N}^n$ such that

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (x - y)^\alpha = \sum_{\alpha_1, \cdots, \alpha_n \in \mathbb{N}} a_\alpha (x_1 - y_1)^{\alpha_1} \cdots (x_n - y_n)^{\alpha_n}, \quad \text{for all } x \in B(y, R) \subset \Omega,$$

where the series is convergent. For every compact set $K \subset \Omega \subset \mathbb{R}^n$, such a function f can be extended to a complex neighborhood of K in $\mathbb{C}^n$ as a complex analytic function. Analyticity is a very demanding regularity assumption. In Theorem 1.2.4, we stress that *all* the coefficients of P should have this regularity. In most situations, however, this requirement is much too strong. As an example, even for the wave equation on a flat (and hence analytic) metric, this theorem does not allow for the addition of a C^∞ time independent potential $V(x)$. This is a very strong drawback to the result. Therefore, we would like to avoid the analyticity assumption on the coefficients. This will require sometimes some stronger assumption say of pseudoconvexity condition (see e.g. Definition 2.3.1 below) and will be the object of Chapter 2. The following chapter 3 will deal with some intermediate case where the analyticity is with respect to only one variable (we will actually treat the simpler case where it is independent on one variable).

1.2.4 Notation

We consider complex valued functions defined on $\mathbb{R}^n$.

We will denote the duality in $L^2(\mathbb{R}^n)$, denoted L^2 when there is not ambiguity, by

$$(f, g)_{L^2} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx.$$

For any multiindex $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, we define its length $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbb{C}^n$, ζ^α is defined by $\zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}$.

For $1 \leq j \leq n$, write $\partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}$ and we denote $D_j = \frac{\partial_j}{i}$.

For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, we denote

$$\begin{aligned} \partial^\alpha &= \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \\ D^\alpha &= \frac{\partial^\alpha}{i^{|\alpha|}} = D_1^{\alpha_1} \cdots D_n^{\alpha_n}. \end{aligned}$$

With this multiindex notation, the Leibnitz formula (derivatives of a product) writes

$$\partial^\alpha (fg) = \sum_{\beta + \gamma = \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^\gamma g),$$

where $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d}$ with $\binom{n}{k} = C_n^k = \frac{n!}{k!(n-k)!}$.

We recall The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is defined as the vector space of $C^\infty(\mathbb{R}^n; \mathbb{C})$ functions u such that

$$p_{\alpha,\beta}(u) := \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta u(x) \right| < +\infty, \quad \text{for all multiindices } \alpha, \beta \in \mathbb{N}^n.$$

The quantity $(p_{\alpha,\beta})_{\alpha,\beta \in \mathbb{N}^n}$ define a countable family of seminorms, which equip $\mathcal{S}(\mathbb{R}^n)$ with a Fréchet space structure. We shall also sometimes use the dual space $\mathcal{S}'(\mathbb{R}^n)$ of temperate distributions, that is, linear functionals $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ with the continuity property

$$\begin{aligned} &\text{for all } \alpha, \beta \in \mathbb{N}^n, \text{ there is } C_{\alpha,\beta} > 0 \text{ such that} \\ &\left| \langle T, \varphi \rangle_{\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)} \right| \leq C_{\alpha,\beta} p_{\alpha,\beta}(\varphi), \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}^n). \end{aligned}$$

This defines a proper subset of the set of distributions $\mathcal{D}'(\mathbb{R}^n)$. For $T \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier transform is well-defined by the formula

$$\langle \hat{T}, \varphi \rangle_{\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)} = \langle T, \hat{\varphi} \rangle_{\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)}, \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The interest of using D instead of ∂ comes from the Fourier transform. Namely, taking the following normalization for the Fourier transform

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) dx, \quad u \in \mathcal{S}(\mathbb{R}^n),$$

we have

$$\widehat{D^\alpha u}(\xi) = \xi^\alpha \widehat{u}(\xi). \tag{1.18}$$

With this convention, the Fourier inversion formula is

$$u(x) = \mathcal{F}^{-1} \widehat{u} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{u}(\xi) d\xi,$$

while Plancherel formula reads

$$(u, v)_{L^2} = \frac{1}{(2\pi)^n} (\widehat{u}, \widehat{v})_{L^2}, \tag{1.19}$$

$$\|u\|_{L^2} = \frac{1}{(2\pi)^{n/2}} \|\widehat{u}\|_{L^2}. \tag{1.20}$$

Another interest of using D instead of ∂ is that the former is (formally) selfadjoint whereas the latter is skewadjoint: on the Fourier side, using the Plancherel formula (1.19), we have

$$(D^\alpha u, v)_{L^2} = \int_{\mathbb{R}^n} D^\alpha u \overline{v} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi^\alpha \widehat{u} \overline{\widehat{v}} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{u} \overline{\xi^\alpha \widehat{v}} = \int_{\mathbb{R}^n} u \overline{D^\alpha v} = (u, D^\alpha v)_{L^2}. \tag{1.21}$$

We also have for $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\widehat{f * g} = \widehat{f}\widehat{g}, \tag{1.22}$$
$$\widehat{fg} = \frac{1}{(2\pi)^n}\widehat{f} * \widehat{g}, \tag{1.23}$$

where

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

When applied to $a = \widehat{f}$ and $b = \widehat{g}$ and taking the inverse Fourier transform $\mathcal{F}^{-1}$, this also yields:

$$\mathcal{F}^{-1}(ab) = \mathcal{F}^{-1}(a) * \mathcal{F}^{-1}(b), \tag{1.24}$$
$$\mathcal{F}^{-1}(a * b) = (2\pi)^n \mathcal{F}^{-1}(a)\mathcal{F}^{-1}(b). \tag{1.25}$$

Definition 1.2.5. For $s \in \mathbb{R}$ and $u \in \mathcal{S}(\mathbb{R}^n)$, we set

$$\|u\|_{H^s(\mathbb{R}^n)} = \left\|(|D|^2+1)^{\frac{s}{2}}u\right\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2}\left\|(|\xi|^2+1)^{\frac{s}{2}}\widehat{u}\right\|_{L^2(\mathbb{R}^n)}.$$

We define the space $H^s(\mathbb{R}^n)$ as the completion of $\mathcal{S}(\mathbb{R}^n)$ for this norm.

Note that for $s_1 \geq s_0$, we have $\|u\|_{H^{s_0}(\mathbb{R}^n)} \leq \|u\|_{H^{s_1}(\mathbb{R}^n)}$ and thus $H^{s_1}(\mathbb{R}^n) \hookrightarrow H^{s_0}(\mathbb{R}^n)$. In particular, for $s \geq 0$, we have $H^s(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n) \hookrightarrow H^{-s}(\mathbb{R}^n)$.

For $s = k \in \mathbb{N}$, the $H^s(\mathbb{R}^n)$ norm is equivalent to the norm

$$\|u\|_{H^k(\mathbb{R}^n)}^2 = \sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{L^2(\mathbb{R}^n)}^2.$$

and we do not introduce another notation (the constants involved in the equivalence only depend on k and the dimension n).

Given an open set $\Omega \subset \mathbb{R}^n$, we will sometimes use the notation $\|\cdot\|_{H^1(\Omega)}$ for

$$\|f\|_{H^1(\Omega)}^2 = \int_\Omega |\nabla f(x)|^2 dx + \int_\Omega |f(x)|^2 dx, \quad f \in C_c^\infty(\mathbb{R}^n).$$

The completion of $C_c^\infty(\mathbb{R}^n)$ for this norm leads to a Hilbert space $H^1(\Omega) \subset L^2(\Omega)$ (which we shall not use in the following).

We finally define properly differential operators. Recall first that a function f on $\mathbb{R}^n$ is said homogeneous of degree $m > 0$ if

$$f(\lambda\xi) = \lambda^m f(\xi), \quad \text{for all } \lambda > 0 \text{ and } \xi \in \mathbb{R}^n.$$

Definition 1.2.6 (Classical differential operators)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set and $m \in \mathbb{N}$.

- We say that P is a (linear) differential operator of order m on Ω if there are coefficients $a_\alpha \in C^\infty(\Omega)$ having all derivatives bounded uniformly on Ω, such that $P = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$ with $m = \max\{|\alpha|, a_\alpha \neq 0\}$.
- We denote $\mathrm{Diff}^m(\Omega)$ the set of differential operators of order m on Ω
- We say that the function $p(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$, $(x,\xi) \in T^*\Omega = \Omega \times \mathbb{R}^n$ is the full symbol of P. It is a polynomial of degree m in the variable ξ.
- We say that the function $p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ is the principal symbol of P. It is a *homogeneous* polynomial of degree m in the variable ξ.
- We denote by $\Sigma^m(\Omega \times \mathbb{R}^n)$ the set of functions $p(x,\xi)$ on $\Omega \times \mathbb{R}^n$ that are polynomials of degree m in the variable ξ with coefficients being smooth functions of $x \in \Omega$.

- If $P \in \text{Diff}^m(\Omega)$, then p and p_m belong to $\Sigma^m(\Omega \times \mathbb{R}^n)$.

  Its full symbol will be denoted $p(x,\xi) = \sum_{|\alpha|\leq m} p_\alpha(x)\xi^\alpha$. It belongs to the set of polynomials of degree m of the variable ξ. Respectively, if $p \in \Sigma^m$, we will denote by $p(x, D)$ the operator with symbol p.

  We denote $p_m(x,\xi) = \sum_{|\alpha|=m} p_\alpha(x)\xi^\alpha$ its principal symbol. It is homogeneous of degree m in ξ.

Example 1.2.7. We have $a(x)D_j \in \text{Diff}^1(\mathbb{R}^n)$ with (full) symbol $a(x)\xi_j$, and $-\Delta \in \text{Diff}^2(\mathbb{R}^n)$ with (full) symbol $|\xi|^2$.

Note that from (1.18), we have that

$$(a_\alpha(x)D^\alpha)u(x) = a_\alpha(x)(D^\alpha u)(x) = a_\alpha(x)\mathcal{F}^{-1}_{\xi\to x}\left(\xi^\alpha \mathcal{F}(u)(\xi)\right) = \mathcal{F}^{-1}_{\xi\to x}\left(a_\alpha(x)\xi^\alpha \mathcal{F}(u)(\xi)\right),$$

so that by linearity

$$p(x,D)u(x) = \mathcal{F}^{-1}_{\xi\to x}\left(p(x,\xi)\mathcal{F}(u)(\xi)\right) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}p(x,\xi)\widehat{u}(\xi)d\xi. \tag{1.26}$$

The augmented set $\Omega \times \mathbb{R}^n$, in which the symbols p, p_m live, may be seen as a "phase space" containing both the position variable x and the Fourier/frequency/momentum variable $\xi \in \mathbb{R}^n$. The latter is to be understood as a cotangent variable $\xi \in T^*_x\Omega$, as we shall see below.

Finally, another class of interesting operators is the class of Fourier multiplier.

1.2.5 The general strategy of Carleman

We consider here Ω a bounded open subset of $\mathbb{R}^n$, P a differential operator on Ω, $x_0 \in \Omega$ a point, and a surface $S = \{\Psi = 0\}$ containing x_0. We aim at proving local unique continuation for an operator P across the surface $S = \{\Psi = 0\}$ (say, a statement like (1.8)). In particular, we want to prevent the situation in which a smooth function w both solves $Pw = 0$ and vanishes (possibly "flately", in the sense that all its derivatives vanish) on S. We thus need to "emphasize" the local behavior of functions close to the hypersurface S.

The general idea of Carleman to do so, and thus prove unique continuation, is to consider weighted estimates of the form

$$\left\|e^{\tau\Phi}w\right\|_{L^2(\Omega)} \leq C\left\|e^{\tau\Phi}Pw\right\|_{L^2(\Omega)}, \tag{1.27}$$

which hold:

- for some well-chosen weight function $\Phi : \overline{\Omega} \to \mathbb{R}$ (related to Ψ as discussed below);
- for all $w \in C^\infty_c(\Omega)$ (related to u as discussed below);
- and *uniformly* for τ sufficiently large, i.e. $\tau \geq \tau_0$.

To prove the relevance/efficiency of this approach, two different things need to be explained:

1. what is the link between Carleman estimates like (1.27) and unique continuation properties like (1.7)?
2. how to prove such Carleman estimates?

Let us first discuss point 1. Note first that (1.27) says directly that if $w \in C^\infty_c(\Omega)$ is solution of $Pw = 0$ on $\{\Phi \geq 0\}$, then the right hand side will tend to zero as τ tends to infinity. Therefore, the left hand side will converge to zero, which implies that w is supported in $\{\Phi \leq 0\}$.

However, statements like (1.8) that are useful in applications are not concerned with functions w having compact support. Moreover, in general, as we shall see, usual differential operators P do not admit solutions w to $Pw = 0$ having compact support!

The heart of the Carleman method to pass from the estimate (1.27) to the unique continuation statement (1.8) resides in applying (1.27) to $w = \chi u$, where u is the function for which unique continuation has to be proved (hence solving $Pu = 0$ in Ω and $u = 0$ on $\Psi \geq 0$), and $\chi \in C_c^\infty(\Omega)$ is a cut-off function (to be chosen) allowing to apply (1.27).

Using that $P\chi u = \chi P u + [P, \chi]u = [P, \chi]u$ (where $[P, \chi]$ denotes the commutator of P and the multiplication operator by χ), this then yields

$$\left\| e^{\tau\Phi}\chi u \right\|_{L^2(\Omega)} \leq C \left\| e^{\tau\Phi}[P, \chi]u \right\|_{L^2(\Omega)}.$$

We then notice that $\operatorname{supp}[P, \chi] \subset \operatorname{supp} \nabla\chi$. If we now *assume* (this can be achieved if Φ is a slight convexification of Ψ, see Figure ♣) that the functions Ψ, Φ, χ are chosen such that $\operatorname{supp}(\nabla\chi) \cap \{\Psi \leq 0\} \subset \{\Phi \leq -\eta\}$, for some $\eta > 0$ (small!), then the support property of u (namely $u = 0$ on $\Psi \geq 0$) implies that $\operatorname{supp}([P, \chi]u) \subset \{\Phi \leq -\eta\}$, and we thus obtain

$$\left\| e^{\tau\Phi}\chi u \right\|_{L^2(\Omega)} \leq C_u e^{-\eta\tau}, \quad \text{for all } \tau \geq \tau_0.$$

The following lemma then implies that χu vanishes identically in $\{\Phi \geq -\eta\}$ which contains a neighborhood of the point x_0.

Lemma 1.2.8. *Assume $w \in L^2(\Omega)$ satisfies $\left\| e^{\tau\Phi} w \right\|_{L^2(\Omega)} \leq C e^{-\eta\tau}$ for all $\tau \geq \tau_0$. Then we have $w = 0$ on $\{\Phi \geq -\eta\}$.*

The proof of the lemma reduces first to the case $\eta = 0$ by changing Φ in $\Phi + \eta$. Then, it suffices to notice that if w does not vanish a.e. on $\{\Phi > 0\}$, there are $\varepsilon > 0$ and a compact set $E \subset \{\Phi > 0\}$ of positive measure such that $|w| \geq \varepsilon > 0$ a.e. on E. This yields

$$C^2 \geq \left\| e^{\tau\Phi} w \right\|^2_{L^2(\Omega)} \geq \int_E e^{2\tau\Phi}|w|^2 \geq \varepsilon^2 \int_E e^{2\tau \min_E \Phi} = \varepsilon^2 |E| e^{2\tau \min_E \Phi} \to_{\tau \to +\infty} +\infty,$$

and hence a contradiction.

To conclude, this brief discussion of point 1 suggests that unique continuation (1.8) will hold (across $\{\Psi = 0\}$) provided the Carleman estimate (1.27) is true for some weight function Φ satisfying an appropriate *geometric convexity condition* as in Figure ♣ .

As stated in point 2, the other issue is how to prove Carleman estimates, and, in particular, understand the conditions on Φ for which 2 can hold. As far as this analysis is concerned, the exponential weight is not convenient to work with. One might thus want to eliminate it by setting $v = e^{\tau\Phi} w$. Then (1.27) is equivalent to $\|v\|_{L^2(\Omega)} \leq C \|P_\Phi v\|_{L^2(\Omega)}$, with $P_\Phi = e^{\tau\Phi} P e^{-\tau\Phi}$ is the so-called conjugated operator. Note that again here, we slightly abuse notation and make the confusion between the function $e^{\tau\Phi}$ and the operator of multiplication by $e^{\tau\Phi}$. We are thus left to prove a lower bound for the operator P_Φ.

Writing $\partial_j(e^{-\tau\Phi}u) = e^{-\tau\Phi}(\partial_j u - \tau u \partial_j \Phi)$ implies that

$$e^{\tau\Phi} D_j e^{-\tau\Phi} = D_j + i\tau\partial_j\Phi. \tag{1.28}$$

The first effect of conjugation is that there is no exponential factor in the right-handside, which is much more convenient. Second, the conjugation changes D_j into an operator having one derivative and one exponent of τ. We thus expect (and we will check) that for general differential operators $P = \sum_\alpha a_\alpha(x) D^\alpha$, the associated conjugated operator P_Φ will have as many derivatives as exponents of τ. Since we want to obtain estimates that are uniform for large τ, we have to think of τ as having the same weight as a derivative. We describe this calculus in the next section.

1.3 Operators depending on a large parameter τ

In this section, we describe the setting in which Carleman estimates like (1.27) shall be proved (see Chapters 2 and 3 below). The main new feature is the presence of a large parameter $\tau > 0$, and the calculus makes things uniform for τ large. One may think to τ as having the same weight as a derivative, i.e. as the Fourier variable ξ. Since τ is aimed at being large, we will always assume $\tau \geq 1$ when dealing with estimates uniform in τ.

1.3.1 Sobolev spaces

Definition 1.3.1. We define the H^s_τ norm of a function $u \in \mathcal{S}(\mathbb{R}^n)$ as

$$\|u\|_{H^s_\tau} = \left\|(|D|^2+\tau^2)^{\frac{s}{2}}u\right\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2}\left\|(|\xi|^2+\tau^2)^{\frac{s}{2}}\widehat{u}\right\|_{L^2(\mathbb{R}^n)}.$$

Note that for fixed τ, this norm is equivalent to the usual H^s norm (Definition 1.2.5), since, for $\tau \geq 1$, we have

$$|\xi|^2+1 \leq |\xi|^2+\tau^2 \leq \tau^2(|\xi|^2+1).$$

That is to say that $\|u\|_{H^s(\mathbb{R}^n)} \leq \|u\|_{H^s_\tau} \leq \tau^s \|u\|_{H^s(\mathbb{R}^n)}$ for all $\tau \geq 1$.

Note also that, as for usual Sobolev spaces, the definition of the H^s_τ norm has a uniformly equivalent definition in case $s = k \in \mathbb{N}$.

Lemma 1.3.2. *Let $k \in \mathbb{N}$. Then, there is $C > 1$ such that for all $\tau \geq 1$ and all $u \in H^k(\mathbb{R}^n)$, we have*

$$C^{-1}\|u\|^2_{H^k_\tau(\mathbb{R}^n)} \leq \sum_{|\alpha|+\beta\leq k} \tau^{2\beta}\|\partial^\alpha u\|^2_{L^2(\mathbb{R}^n)} \leq C\|u\|^2_{H^k_\tau(\mathbb{R}^n)}.$$

In particular, we often use the case $k = 1$,

$$\|u\|_{H^1_\tau} \approx \|u\|_{H^1} + \tau\|u\|_{L^2},$$

uniformly for $\tau \geq 1$.

Proof of Lemma 1.3.2. The Plancherel formula yields

$$\|u\|^2_{H^k_\tau} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(|\xi|^2+\tau^2)^k|\hat{u}(\xi)|^2 d\xi,$$

together with

$$\sum_{|\alpha|+\beta\leq k}\tau^{2\beta}\|\partial^\alpha u\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\sum_{|\alpha|+\beta\leq k}\tau^{2\beta}\|\xi^\alpha\hat{u}\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\sum_{|\alpha|+\beta\leq k}\tau^{2\beta}|\xi^\alpha|^2|\hat{u}(\xi)|^2 d\xi.$$

Now, for $|\alpha|+\beta \leq k$, each term in the sum is bounded as

$$\tau^{2\beta}|\xi^\alpha|^2 = \tau^{2\beta}\xi^{2\alpha} \leq \tau^{2\beta}|\xi|^{2|\alpha|} \leq (\tau^2+|\xi|^2)^{\beta+|\alpha|} \leq (\tau^2+|\xi|^2)^k,$$

implying the second inequailty of the lemma.

Concerning the first inequality, notice that the sum contains in particular the instance $\alpha = 0, \beta = k$, the instance $\alpha_1 = k, \alpha_j = 0$ for $j = 2, \cdots, n$ and $\beta = 0$, etc..., yielding

$$\sum_{|\alpha|+\beta\leq k}\tau^{2\beta}|\xi^\alpha|^2 \geq \tau^{2k} + |\xi_1^{2k}| + \cdots + |\xi_n^{2k}| \geq c(|\xi|^2+\tau^2)^k$$

for some $c > 0$ uniformly for $(\tau, \xi) \in \mathbb{R}^n \times \mathbb{R}^+$ (these are two homogeneous functions of degree k which do not vanish on the sphere). This proves the first inequality, and concludes the proof of the lemma. □

We finally give a duality statement between the spaces/norms H^s_τ and H^{-s}_τ.

Lemma 1.3.3. *For all $s \in \mathbb{R}$ and all $u, v \in \mathcal{S}(\mathbb{R}^n)$, $\tau > 0$, we have*

$$|(u,v)_{L^2(\mathbb{R}^n)}| \leq \|u\|_{H^{-s}_\tau}\|v\|_{H^s_\tau}.$$

Moreover, for all $u \in \mathcal{S}(\mathbb{R}^n)$ and $\tau > 0$, we have

$$\|u\|_{H^{-s}_\tau} = \sup_{v\in\mathcal{S}(\mathbb{R}^n), \|v\|_{H^s_\tau}\leq 1}|(u,v)_{L^2(\mathbb{R}^n)}|.$$

Proof. The first statement comes from the Plancherel theorem and the Cauchy Schwarz inequality as follows:

$$\begin{aligned}|(u,v)_{L^2(\mathbb{R}^n)}| &= \frac{1}{(2\pi)^n}\left|\int_{\mathbb{R}^n}\hat{u}(\xi)\bar{\hat{v}}(\xi)d\xi\right| = \frac{1}{(2\pi)^n}\left|\int_{\mathbb{R}^n}(|\xi|^2+\tau^2)^{-\frac{s}{2}}\hat{u}(\xi)(|\xi|^2+\tau^2)^{\frac{s}{2}}\bar{\hat{v}}(\xi)d\xi\right| \\ &\leq \frac{1}{(2\pi)^n}\|(|\xi|^2+\tau^2)^{-\frac{s}{2}}\hat{u}(\xi)\|_{L^2}\|(|\xi|^2+\tau^2)^{\frac{s}{2}}\hat{v}(\xi)\|_{L^2} = \|u\|_{H_\tau^{-s}}\|v\|_{H_\tau^s}.\end{aligned}$$

The second statement comes from the first one together with the fact that for

$$v := \|u\|_{H_\tau^{-s}}^{-1}(|D|^2+\tau^2)^{-s}u \in \mathcal{S}(\mathbb{R}^n),$$

we have

$$\|v\|_{H_\tau^s} = \frac{\left\|(|D|^2+\tau^2)^{-s}u\right\|_{H_\tau^s}}{\|u\|_{H_\tau^{-s}}} = 1,$$

and

$$(u,v)_{L^2(\mathbb{R}^n)} = \frac{1}{\|u\|_{H_\tau^{-s}}}(u,(|D|^2+\tau^2)^{-s}u)_{L^2(\mathbb{R}^n)} = \frac{1}{\|u\|_{H_\tau^{-s}}}\|(|D|^2+\tau^2)^{-\frac{s}{2}}u\|^2_{L^2(\mathbb{R}^n)} = \|u\|_{H_\tau^{-s}}.$$

□

In the next section, we describe some important properties of operators depending on a large parameter τ.

1.3.2 Differential operators

Definition 1.3.4 (Differential operators depending on τ)**.** Let $m \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ an open set. We denote $\mathrm{Diff}_\tau^m(\Omega)$ the set of differential operators of the form $P = \sum_{|\alpha|+\beta\leq m} p_{\alpha,\beta}(x)\tau^\beta D^\alpha$ with $p_{\alpha,\beta} \in C^\infty(\Omega)$ such that all derivatives of $p_{\alpha,\beta}$ are bounded uniformly on Ω.

For $P \in \mathrm{Diff}_\tau^m(\Omega)$, we define its full symbol by $p(x,\xi,\tau) = \sum_{|\alpha|+\beta\leq m} p_{\alpha,\beta}(x)\tau^\beta\xi^\alpha$. It belongs to the set of polynomials of degree m of the variable (ξ,τ), with coefficients smooth functions of $x \in \Omega$, that we denote $\Sigma^m(\Omega \times \mathbb{R}^n \times \mathbb{R}^+)$.

Respectively, if $p \in \Sigma^m(\Omega \times \mathbb{R}^n \times \mathbb{R}^+)$, we will denote $p(x,D,\tau)$ the operator with symbol p.

We finally define $p_m(x,\xi,\tau) = \sum_{|\alpha|+\beta=m} p_{\alpha,\beta}(x)\tau^\beta\xi^\alpha$ its principal symbol. It is homogeneous of degree m in (ξ,τ), in the sense that

$$p_m(x,\lambda\xi,\lambda\tau) = \lambda^m p_m(x,\xi,\tau), \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^n, \tau \geq 0 \text{ and } \lambda > 0.$$

Recall that, the order m being fixed, the set of smooth homogeneous functions of degree m in this sense identify (through the restriction map) to smooth functions on the half-sphere bundle over Ω, namely

$$\{(x,\xi,\tau) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^+, |\xi|^2+\tau^2 = 1\}.$$

Remark that this Definition 1.3.4 is almost the same definition as Definition 1.2.6, except for the dependance on τ which changes the definition of the principal symbol. Note also that if $p \in \Sigma^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+)$, the inversion Fourier formula gives, as in (1.26), for $u \in \mathcal{S}(\mathbb{R}^n)$

$$p(x,D,\tau)u(x) = \mathcal{F}^{-1}_{\xi\to x}\left(p(x,\xi,\tau)\mathcal{F}(u)(\xi)\right) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}p(x,\xi,\tau)\widehat{u}(\xi)d\xi. \tag{1.29}$$

Let us finally remark that for $P = p(x,D,\tau) \in \mathrm{Diff}_\tau^m(\Omega)$, with principal symbol p_m, we have

$$P - p_m(x,D,\tau) \in \mathrm{Diff}_\tau^{m-1}(\Omega),$$

that is to say that these two differential operators of order m only differ by a differential operator of order $m-1$.

Now, we want to describe the calculus of differential operators with a large parameter. This consists in explaining the properties of such operators with respect to usual operations (composition, commutators, taking the adjoint), their mapping properties (in τ dependent Sobolev spaces) and positivity properties. Moreover, we want to link such properties with those of the symbol of the operators. The philosophy is that we want to recover properties of the operators only from their principal symbols (which are simpler objects to manipulate, namely functions on the augmented space $\Omega_x \times \mathbb{R}^n_\xi \times \mathbb{R}^+_\tau$). The general Heuristic is that a these differential operators act as if they were multiplication by $p_m(x,\xi,\tau)$, modulo lower order terms.

If P, A, B of respective order m, m_1 and m_2, with respective principal symbol p, a, b, a rough summary of the calculus properties proved below is the following:

1. (action on Sobolev spaces) P maps continuously H^s_τ into H^{s-m}_τ;
2. (composition) $AB = A \circ B$ is of order $m_1 + m_2$ with principal symbol ab;
3. (commutators) $[A,B] = AB - BA$ is of order $m_1 + m_2 - 1$ with principal symbol $\frac{1}{i}\{a,b\}$, where
$$\{a,b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b = \sum_{j=1}^n \left(\partial_{\xi_j} a \partial_{x_j} b - \partial_{x_j} a \partial_{\xi_j} b\right)$$
is the Poisson bracket.
4. (adjoint) P^*, the formal adjoint in L^2 is of order m with principal symbol $\overline{p}$;
5. (Gårding estimate) $p \geq C(\xi^2 + \tau^2)^{m/2}$ implies $\operatorname{Re}(Pu,u)_{L^2} \geq C' \|u\|^2_{H^{m/2}_\tau}$ for large τ (we consider only the case $m = 2$ here but need a slightly more general class of operators than Diff^2_τ).

1.3.3 The calculus of differential operators with a large parameter

Proposition 1.3.5 (Action on Sobolev spaces)**.** *Let $P \in \mathrm{Diff}^m_\tau(\mathbb{R}^n)$ and fix $s \in \mathbb{R}$. Then, there exists a constant $C > 0$ (depending on s and the coefficients of P) such that*
$$\|Pu\|_{H^{s-m}_\tau} \leq C \|u\|_{H^s_\tau}, \quad \textit{for all } u \in \mathcal{S}(\mathbb{R}^n) \textit{ and all } \tau \geq 1.$$
In particular, P extends uniquely as a bounded operator from H^s_τ to H^{s-m}_τ, uniformly for $\tau \geq 1$.

Although the result is stated for all $s \in \mathbb{R}$, we prove it only for $s = k \in \mathbb{Z}$, which simplifies considerably the proof. The general case can be deduced by interpolation ♣ `ref`. A direct proof is also possible (but requires more work).

Proof. By the triangle inequality, it is enough to prove that any term $p_{\alpha,\beta}(x)\tau^\beta D^\alpha$ with $|\alpha| + \beta \leq m$ is bounded from H^k_τ to H^{k-m}_τ. Then, we want to decompose
$$\left\|p_{\alpha,\beta}(x)\tau^\beta D^\alpha\right\|_{H^k_\tau \to H^{k-m}_\tau} \leq \|p_{\alpha,\beta}\|_{H^{k-m}_\tau \to H^{k-m}_\tau} \left\|\tau^\beta D^\alpha\right\|_{H^k_\tau \to H^{k-m}_\tau} \tag{1.30}$$
and it suffices to prove that each term on the right handside is finite. Firstly we have,
$$\begin{aligned}\left\|\tau^\beta D^\alpha u\right\|^2_{H^{k-m}_\tau} &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (\tau^2 + |\xi|^2)^{k-m} |\tau^\beta \xi^\alpha \hat{u}(\xi)|^2 d\xi \\ &\leq \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (|\xi|^2 + \tau^2)^{k-m} \left(|\xi|^2 + \tau^2\right)^{|\alpha|+\beta} |\widehat{u}(\xi)|^2 d\xi \\ &\leq \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (|\xi|^2 + \tau^2)^{k} |\widehat{u}(\xi)|^2 d\xi = \|u\|^2_{H^k_\tau} \end{aligned} \tag{1.31}$$
where we used $|\alpha| + \beta \leq m$ in the last inequality. This proves that $D^\alpha \tau^\beta$ applies H^k_τ into H^{k-m}_τ. It only remains to prove that the multiplication by a smooth function f bounded as well as its derivatives (here

$p_{\alpha,\beta}$) is bounded from H^k_τ to H^k_τ for any $k \in \mathbb{Z}$. For $k \in \mathbb{N}$, we use the characterization of $\|\cdot\|_{H^k_\tau}$ given by Lemma 1.3.2 together with the Leibnitz formula. This yields

$$\begin{aligned}
\|fu\|^2_{H^k_\tau} &\leq C \sum_{|\alpha|+\beta\leq k} \tau^{2\beta} \|\partial^\alpha(fu)\|^2_{L^2(\mathbb{R}^n)} \leq C \sum_{|\alpha|+\beta\leq k} \tau^{2\beta} \sum_{\delta+\gamma=\alpha} \left\|(\partial^\delta f)(\partial^\gamma u)\right\|^2_{L^2(\mathbb{R}^n)} \\
&\leq C \sum_{|\alpha|+\beta\leq k} \tau^{2\beta} \sum_{\delta+\gamma=\alpha} \left\|\partial^\delta f\right\|^2_{L^\infty(\mathbb{R}^n)} \|\partial^\gamma u\|^2_{L^2(\mathbb{R}^n)} \\
&\leq C \|f\|^2_{W^{k,\infty}(\mathbb{R}^n)} \sum_{|\gamma|+\beta\leq k} \tau^{2\beta} \|\partial^\gamma u\|^2_{L^2(\mathbb{R}^n)} \leq C(f)^2 \|u\|^2_{H^k_\tau}. \qquad (1.32)
\end{aligned}$$

Now, still for $k \in \mathbb{N}$, we prove that the multiplication by f is bounded on H^{-k}_τ. We proceed by duality and use the characterization of H^{-k}_τ in Lemma 1.3.3. Indeed, first remark that

$$\left|(fu, v)_{L^2(\mathbb{R}^n)}\right| = \left|(u, \overline{f}v)_{L^2(\mathbb{R}^n)}\right| \leq \|u\|_{H^{-k}_\tau} \left\|\overline{f}v\right\|_{H^k_\tau} \leq \|u\|_{H^{-k}_\tau} C(f) \|v\|_{H^k_\tau},$$

where we used Lemma 1.3.3 together with (1.32). The characterization of $\|fu\|_{H^{-k}_\tau} = \sup_{\|v\|_{H^k_\tau}=1} (fu, v)_{L^2(\mathbb{R}^n)}$ in Lemma 1.3.3 yields

$$\|fu\|_{H^{-k}_\tau} \leq C(f) \|u\|_{H^{-k}_\tau}.$$

This, together with (1.32) proves that multiplication by f is bounded on H^k_τ for all $k \in \mathbb{Z}$. Finally recalling (1.30) and (1.31) concludes the proof of the proposition. □

Proposition 1.3.6 (Composition)**.** *Let $A \in \mathrm{Diff}^{m_1}_\tau(\Omega)$ and $B \in \mathrm{Diff}^{m_2}_\tau(\Omega)$ with principal symbols $a_{m_1}(x, \xi, \tau)$ and $b_{m_2}(x, \xi, \tau)$. Then, the operator $AB = A \circ B$ is in $\mathrm{Diff}^{m_1+m_2}_\tau(\Omega)$ Moreover, it can be written as*

$$A \circ B = (a_{m_1} b_{m_2})(x, D, \tau) + r(x, D, \tau)$$

with $r(x, D, \tau) \in \mathrm{Diff}^{m_1+m_2-1}_\tau(\Omega)$. In particular, the principal symbol of $A \circ B$ is $a_{m_1} b_{m_2}$.

A direct proof with the Leibnitz formula is given in Appendix B.1.1.

Proof. We prove it by induction on $m = m_1 + m_2$.

- Case $m = 0$: $A = f(x)$ and $B = g(x)$ and the result is clear.
- Induction $m - 1 \to m$: By linearity with respect to A and B, it is enough to prove the result for $A = f(x)\tau^\beta D^\alpha$ and $B = g(x)\tau^{\beta'} D^{\alpha'}$. Since $m_1 + m_2 = m + 1$, at least one of the β, $|\alpha|$, β', $|\alpha'|$ is bigger than 1. If it is either $\beta \geq 1$, $\beta' \geq 1$ or $|\alpha'| \geq 1$, the result is a consequence of the induction assumption at rank $m - 1$. If $|\alpha| \geq 1$, take k such that $\alpha_k \geq 1$. Then $A = \widetilde{A} D_k$ with $\widetilde{A} \in \mathrm{Diff}^{m_1-1}_\tau$ and we have

  $$ABu = \widetilde{A} D_k[g(x)\tau^{\beta'} D^{\alpha'}]u = \widetilde{A}(D_k g(x))\tau^{\beta'} D^{\alpha'} + \widetilde{A} g(x)\tau^{\beta'} D^{\alpha'} D_k u.$$

  The induction assumption at rank $m-1$ implies that the first term is in Diff^{m-1}_τ and $\widetilde{A} g(x)\tau^{\beta'} D^{\alpha'} \in \mathrm{Diff}^{m-1}_\tau$ with principal symbol $\widetilde{a} g(x)\tau^{\beta'} \xi^{\alpha'}$. We deduce that $\widetilde{A} g(x)\tau^{\beta'} D^{\alpha'} D_k \in \mathrm{Diff}^m_\tau$ with principal symbol $\widetilde{a} g(x)\tau^{\beta'} \xi^{\alpha'} \xi_k = ab$.

□

Notice then that both AB and BA belong to $\mathrm{Diff}^{m_1+m_2}_\tau(\Omega)$ and have the same principal symbol $a_{m_1} b_{m_2}$. This in particular implies

$$[A, B] = AB - BA \in \mathrm{Diff}^{m_1+m_2-1}_\tau(\Omega),$$

for the commutator. It is natural when comparing these two operators to study the principal symbol of $[A, B]$. Notice that basic algebra shows

$$[A, B] = -[B, A], \qquad (1.33)$$
$$[A, BC] = [A, B]C + B[A, C]. \qquad (1.34)$$

We need the following notation and definition.

Definition 1.3.7 (Poisson bracket)**.** Given $a, b \in C^\infty(\Omega \times \mathbb{R}^n; \mathbb{C})$, we define the Poisson bracket of a and b by

$$\{a, b\} := \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b = \sum_{j=1}^{n} \left(\partial_{\xi_j} a \partial_{x_j} b - \partial_{x_j} a \partial_{\xi_j} b \right).$$

Notice that we have the properties:

$$\{a, b\} = -\{b, a\}, \tag{1.35}$$
$$\{a, bc\} = \{a, b\} c + b \{a, c\}. \tag{1.36}$$

The second formula is a Leibnitz formula, yielding that the map $b \mapsto \{a, b\}$ is a derivation on $C^\infty(\Omega \times \mathbb{R}^n; \mathbb{C})$. When comparing these to (1.33)-(1.34), it seems natural to obtain the following result.

Proposition 1.3.8 (Commutation)**.** *Let $A \in \mathrm{Diff}_\tau^{m_1}(\Omega)$ and $B \in \mathrm{Diff}_\tau^{m_2}(\Omega)$ with principal symbols $a_{m_1}(x, \xi, \tau)$ and $b_{m_2}(x, \xi, \tau)$. Then, the operator $[A, B]$ is in $\mathrm{Diff}_\tau^{m_1+m_2-1}(\Omega)$. Moreover, it can be written as*

$$[A, B] = \frac{1}{i} \{a_{m_1}, b_{m_2}\}(x, D, \tau) + r(x, D, \tau),$$

with $r(x, D, \tau) \in \mathrm{Diff}_\tau^{m_1+m_2-2}(\Omega)$.

Note that since a_{m_1} is homogeneous of degree m_1 in (ξ, τ), then $\partial_x a_{m_1}$ has the same homogeneity whereas $\partial_\xi a_{m_1}$ is homogeneous of degree $m_1 - 1$ in (ξ, τ) (and similarly for b_{m_2}). Therefore $\{a_{m_1}, b_{m_2}\}$ is homogeneous of degree $m_1 + m_2 - 1$, which is consistent with the formula.

We could give a direct proof with the Leibnitz formula (given in Appendix B.1.1). The latter would however be technical and not very informative, so we prefer to give a simple inductive proof based on (1.33)-(1.34)-(1.35)-(1.36).

Proof. We first treat the case where one of the operators is of order 1. This amounts to prove, by induction on m, the following property: for any $A = f(x)D_k$ and $B \in \mathrm{Diff}_\tau^m$, $[A, B] \in \mathrm{Diff}_\tau^m$ with principal symbol $\frac{1}{i}\{f\xi_k, b_m\}$.

- Case $m = 0$: $B = g(x)$ and we have

  $$[A, B]u = [f(x)D_k, g(x)]u = f(x)D_k(g(x)u) - g(x)f(x)D_k u = f(x)D_k(g(x))u = \frac{1}{i} f(x)\partial_k(g(x))u.$$

  This operator is in Diff_τ^0 with principal (and full) symbol $\frac{1}{i} f(x)\partial_{x_k}(g(x))$. And for the principal symbols $a_1 = f(x)\xi_k$ and $b_0 = g(x)$, we have $\{a_1, b_0\} = \{a_1, b\} = f\partial_{x_k} g$.

- Case $m = 1$ (this is only needed as a partial result): By linearity (and the case $m = 0$), it is enough to have the result for $B = \tau g(x)$ or $B = g(x)D_l$. The first case reduces to the case $m = 0$ (since τ commutes), so we only need to treat the second. We have

  $$\begin{aligned} [A, B]u &= [f(x)D_k, g(x)D_l]u = f(x)D_k [g(x)D_l u] - g(x)D_l [f(x)D_k u] \\ &= f(x)D_k(g(x))D_l u - g(x)D_l(f(x))D_k u. \end{aligned}$$

  This operator belongs to Diff_τ^1 and has principal (and full) symbol $\frac{1}{i}\big(f(x)\partial_{x_k}(g(x))\xi_l - g(x)\partial_{x_l}(f(x))\xi_k\big)$ which turns out to be equal to $\frac{1}{i}\{f(x)\xi_k, g(x)\xi_l\}$.

- Induction $m \to m+1$: the main idea is to use (1.33)-(1.34). More precisely, by linearity ($B \in \mathrm{Diff}_\tau^{m+1}$ writes $B = B_0\tau + \sum_{j=1}^m B_j D_j + \check{B}$ where $B_j, \check{B} \in \mathrm{Diff}_\tau^m$), it is enough to consider $B = \tau\widetilde{B}$ or $B = \widetilde{B}D_l$ with $\widetilde{B} \in \mathrm{Diff}_\tau^m$, with principal symbol $\tilde{b}_m$. In the first case, we have $[A, B] = \tau[A, \widetilde{B}]$, and the induction assumption at step m gives the result. In the second case, we have

  $$[A, B] = [A, \widetilde{B}D_l] = [A, \widetilde{B}]D_l + \widetilde{B}[A, D_l].$$

The case $m = 1$ and the induction assumption then yield (after using Proposition 1.3.6) that $[A, B] \in \mathrm{Diff}^m_\tau$ with principal symbol

$$\frac{1}{i}\left(\left\{a, \tilde{b}_m\right\}\xi_l + \tilde{b}_m\left\{a, \xi_l\right\}\right) = \frac{1}{i}\left\{a, \tilde{b}_m\xi_l\right\} = \frac{1}{i}\left\{a, b\right\},$$

where we have used (1.36).

The result is now proved for any $A = f(x)D_k$ and $B \in \mathrm{Diff}^m_\tau$ (or $A \in \mathrm{Diff}^m_\tau$ and $B = g(x)D_l$ by antisymmetry). The final result can then be proved similarly by induction on $m = m_1 + m_2$ using the same strategy.

- Case $m = 0$: the result is clear since then $A = f(x)$ and $B = g(x)$.
- Induction $m \to m + 1$ By linearity with respect to both variable, it is enough to get the result for $A = f(x)\tau^\beta D^\alpha$ and $B = g(x)\tau^{\beta'} D^{\alpha'}$. Since $m_1 + m_2 = m + 1$, at least one of the β, $|\alpha|$, β', $|\alpha'|$ is bigger than 1. By symmetry, we can assume that either $\beta \geq 1$ or $|\alpha| \geq 1$. In the first case, we apply directly the induction assumption at rank m. In the second case, $|\alpha| \geq 1$, we take k such that $\alpha_k \geq 1$. Then $A = \widetilde{A}D_k$ where $\widetilde{A}$ is of order $m_1 - 1$. We have again

$$[A, B] = [\widetilde{A}D_k, B] = \widetilde{A}[D_k, B] + [\widetilde{A}, B]D_k,$$

and we conclude similarly by the induction assumption at rank m for the second term and for the first term by the previous result proved in the specific case $A = D_k$.

□

Given an operator P, we now would like to discuss its formal adjoint P^* (if it exists) in the sense that

$$(Pu, v)_{L^2} = (u, P^*v)_{L^2}, \qquad \text{for all } u, v \in C^\infty_c(\Omega). \tag{1.37}$$

We only talk about "formal adjoint" because the test functions in (1.37) are in $C^\infty_c(\Omega)$. This is linked with the fact that we did not define differential operators as closed operators on the Hilbert space $L^2(\Omega)$ (such a definition would require to define their domains, which we do not do/need), but rather as acting on $C^\infty_c(\Omega) \to C^\infty_c(\Omega)$.

Proposition 1.3.9 (Formal adjoint)**.** *Let $P \in \mathrm{Diff}^m_\tau(\Omega)$ with principal symbol p_m. There exists a unique operator $P^* \in \mathrm{Diff}^m_\tau(\Omega)$ satisfying (1.37). Moreover, the principal symbol of P^* is $\overline{p_m}$, that is $P^* - \overline{p_m}(x, D, \tau) \in \mathrm{Diff}^{m-1}_\tau(\Omega)$.*

Note in particular that an operator $P \in \mathrm{Diff}^m_\tau(\Omega)$ with *real-valued principal symbol* is formally selfadjoint modulo $\mathrm{Diff}^{m-1}_\tau(\Omega)$, in the sense that $P - P^* \in \mathrm{Diff}^{m-1}_\tau(\Omega)$. This fact will be used several times in the proof of Carleman estimates.

Proof. By linarity, it is enough to prove the result for $P = a(x)\tau^\beta D^\alpha$. We recall from (1.21) that D^α is formally selfadjoint, so that

$$\left(a(x)\tau^\beta D^\alpha u, v\right)_{L^2} = \left(\tau^\beta D^\alpha u, \overline{a(x)}v\right)_{L^2} = \left(u, \tau^\beta D^\alpha \overline{a(x)}v\right)_{L^2}.$$

As a consequence, $P^* = \tau^\beta D^\alpha \overline{a(x)}$ and we know from Proposition 1.3.6 that $P^* \in \mathrm{Diff}^m_\tau$ with principal symbol $\overline{a(x)}\tau^\beta\xi^\alpha = \overline{p_m}$. □

1.3.4 The conjugated operator

As described in Section 1.2.5, the introduction of the calculus with the large parameter τ is motivated by the conjugated operator $P_\Phi := e^{\tau\Phi}Pe^{-\tau\Phi}$. We here prove that it belongs to the class Diff^m_τ, and compute its principal symbol.

Lemma 1.3.10 (The conjugated operator)**.** *Let* $P = \sum_{|\alpha|\le m} p_\alpha(x) D^\alpha \in \mathrm{Diff}^m(\Omega)$ *be a (classical) differential operator with principal symbol* p_m *and let* $\Phi \in C^\infty(\Omega)$ *be real-valued and bounded as well as all its derivatives.*

Then, the operator P_Φ *defined by* $P_\Phi v = e^{\tau\Phi} P(e^{-\tau\Phi} v)$ *satisfies* $P_\Phi \in \mathrm{Diff}^m_\tau(\Omega)$*, and its principal symbol, denoted by* $p_\Phi = p_{\Phi,m}$ *(with a slight abuse of notation), is given by*

$$p_\Phi(x,\xi,\tau) = p_m(x, \xi + i\tau d\Phi(x)) = \sum_{|\alpha|=m} p_\alpha(x)(\xi + i\tau d\Phi(x))^\alpha.$$

Roughly speaking, the previous Lemma says that p_Φ is obtained by replacing ξ by $\xi + i\tau d\Phi(x)$ in p_m. Note that it implies in particular that p_Φ has a complex-valued symbol if p_m is real-valued: The conjugation turns selfadjoint operators into non-selfadjoint ones.

Proof. As already checked in (1.28), we have $e^{\tau\Phi} D_j(e^{-\tau\Phi} u) = D_j u + i\tau(\partial_j \Phi)u$. In particular, the conjugated operator $e^{\tau\Phi} D_j e^{-\tau\Phi}$ lies in the class Diff^1_τ with principal symbol $\xi_j + i\tau\partial_j\Phi$. We now write

$$\begin{aligned} e^{\tau\Phi} D_j^{\alpha_j} e^{-\tau\Phi} &= e^{\tau\Phi} D_j e^{-\tau\Phi} e^{\tau\Phi} D_j \cdots e^{\tau\Phi} D_j e^{-\tau\Phi} \qquad (\alpha_j \text{ times}) \\ &= (e^{\tau\Phi} D_j e^{-\tau\Phi})(e^{\tau\Phi} D_j e^{-\tau\Phi}) \cdots (e^{\tau\Phi} D_j e^{-\tau\Phi}). \end{aligned}$$

Therefore, using Proposition 1.3.6 $\alpha_j - 1$ times, we obtain that this is a differential operator depending on τ of order α_j with principal symbol $(\xi_j + i\tau\partial_j\Phi)^{\alpha_j}$ (note that the full symbol is more complicated).

So, since $D^\alpha = D_1^{\alpha_1} \cdots D_j^{\alpha_j} \cdots D_n^{\alpha_n}$, we obtain similarly that $e^{\tau\Phi} D^\alpha e^{-\tau\Phi} \in \mathrm{Diff}^{|\alpha|}_\tau$, with principal symbol

$$\prod_{j=1}^n (\xi_j + i\tau\partial_j\Phi)^{\alpha_j} = (\xi + i\tau d\Phi)^\alpha,$$

using the notation of Section 1.2.4. Since p_α commutes with $e^{\tau\Phi}$ and $P = \sum_\alpha p_\alpha(x) D^\alpha$, this provides the result by summing up. □

Example 1.3.11 (the Laplace operator)**.** Here, we take $P = -\Delta \in \mathrm{Diff}^2(\mathbb{R}^n)$, having (full and principal) symbol $|\xi|^2$, and make a direct computation of the full and the principal symbol of P_Φ. We have

$$\begin{aligned} e^{\tau\Phi}(-\Delta)e^{-\tau\Phi} u &= -e^{\tau\Phi}\left[\Delta(e^{-\tau\Phi})u + e^{-\tau\Phi}\Delta u + 2\nabla u \cdot \nabla(e^{-\tau\Phi})\right] \\ &= -e^{\tau\Phi}\left[-\tau(\Delta\Phi)e^{-\tau\Phi} + \tau^2|\nabla\Phi|^2 e^{-\tau\Phi} u + e^{-\tau\Phi}\Delta u - 2\tau\nabla u \cdot \nabla\Phi e^{-\tau\Phi}\right] \\ &= \tau(\Delta\Phi)u - \tau^2|\nabla\Phi|^2 u - \Delta u + 2\tau\nabla u \cdot \nabla\Phi, \end{aligned}$$

where we have used

$$\begin{aligned} \nabla(e^{-\tau\Phi}) &= -\tau\nabla\Phi e^{-\tau\Phi} \\ \Delta(e^{-\tau\Phi}) &= \mathrm{div}(\nabla(e^{-\tau\Phi})) = -\tau\,\mathrm{div}(\nabla\Phi e^{-\tau\Phi}) = -\tau(\Delta\Phi)e^{-\tau\Phi} - \tau\nabla\Phi\cdot\nabla(e^{-\tau\Phi}) \\ &= -\tau(\Delta\Phi)e^{-\tau\Phi} + \tau^2|\nabla\Phi|^2 e^{-\tau\Phi}. \end{aligned}$$

$\tau^2|\nabla\Phi|^2 u$, Δu and $\tau\nabla u \cdot \nabla\Phi$ are of order 2 with respective symbol $\tau^2|\nabla\Phi|^2$, $-|\xi|^2$ and $i\tau\xi\cdot\nabla\Phi$ (remember that $\nabla u = (\partial_1 u, \cdots, \partial_n u) = i(D_1 u, \cdots, D_n u)$ has complex symbol denoted for short iD).

So, denoting $p_{\Phi,full}$ the full symbol of P and p_Φ its principal symbol, we have

$$\begin{aligned} p_{\Phi,full}(x,\xi,\tau) &= |\xi|^2 - \tau^2|\nabla\Phi(x)|^2 + 2i\tau\xi\cdot\nabla\Phi(x) + \tau\Delta\Phi(x) \\ p_\Phi(x,\xi,\tau) &= |\xi|^2 - \tau^2|\nabla\Phi(x)|^2 + 2i\tau\xi\cdot\nabla\Phi(x) \end{aligned}$$

Note that we have $p_{\Phi,2}(x,\xi,\tau) = (\xi + i\tau\nabla\Phi(x))\cdot(\xi + i\tau\nabla\Phi(x)) = p_2(x, \xi + i\tau\nabla\Phi(x))$, (beware that here, $\cdot$ denotes the *real* inner product in $\mathbb{R}^n$) in accordance with Lemma 1.3.10.

♣ `Fix:` $d\Phi$ `or` $\nabla\Phi$

Example 1.3.12 (second order operators with real-valued principal symbol)**.** Below, we will be particularly interested in *second order differential operators with real-valued principal symbol*, namely $P \in \mathrm{Diff}^2(\Omega)$ with p_2 real-valued. The principal symbol of such operators write $p_2(x,\xi) = \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j$ with real coefficients a^{ij}. This encompasses of course the case of the Laplace operator discussed in Example 1.3.11.

Notice first that $\xi \mapsto p_2(x,\xi) = \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j$ is a real quadratic form for all $x \in \Omega$. In particular, we have the canonical polar form:

$$p_2(x,\xi) = \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j = \sum_{i,j=1}^n \frac{1}{2}\left(a^{ij}(x) + a^{ji}(x)\right)\xi_i\xi_j,$$

and we can thus assume that

$$\text{the matrix } (a^{ij}(x))_{i,j} \text{ is symmetric, i.e. } a^{ij}(x) = a^{ji}(x) \text{ for all } 1 \le i,j \le n. \tag{1.38}$$

Concerning the operator P, we then have

$$P - p_2(x,D) \in \mathrm{Diff}^1(\Omega), \quad p_2(x,D) = \sum_{i,j=1}^n a^{ij}(x)D_iD_j = \sum_{i,j=1}^n D_i a^{ij}(x)D_j + R_1,$$

where $R_1 = -\sum_{i,j=1}^n D_i(a^{ij})D_j \in \mathrm{Diff}^1(\Omega)$. The operator $\sum_{i,j=1}^n D_i a^{ij}(x)D_j$ is formally selfadjoint (equivalently, one can say that it is of divergence form with respect to the measure dx). This last form thus states in a clearer way that the operator is formally self-adjoint modulo $\mathrm{Diff}^1(\Omega)$. Also, $\xi \mapsto p_2(x,\xi) = \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j$ is a real quadratic form with (1.38), and thus Lemma 1.3.10 states that the principal symbol of the associated conjugated operator P_Φ is given by

$$p_\Phi(x,\xi,\tau) = p_2(x,\xi + i\tau d\Phi(x)) = p_2(x,\xi) - \tau^2 p_2(x, d\Phi(x)) + 2i\tau\widetilde{p_2}(x,\xi,d\Phi(x)),$$

where $\widetilde{p_2}(x,\xi,\eta) = \sum_{i,j=1}^n a^{ij}(x)\xi_i\eta_j$ is the polar bilinear form of the quadratic form $p_2(x,\xi)$.

1.3.5 A Gårding inequality for a class of operators with a large parameter

In this section, we prove that operators having a real positive principal symbol are positive (referred to as a Gårding inequality).

However, for the need of Carleman estimates, the class of differential operators is not quite sufficient. We need to consider a slightly larger class, that also includes the operator

$$(-\Delta + \tau^2)^{-1} = (|D|^2 + \tau^2)^{-1}, \quad \tau \ge 1,$$

defined as a Fourier multiplier:

$$\mathcal{F}((-\Delta + \tau^2)^{-1}u)(\xi) = (|\xi|^2 + \tau^2)^{-1}\hat{u}(\xi), \quad u \in \mathcal{S}(\mathbb{R}^n).$$

Note that, as opposed to differential operators, the operator $(-\Delta + \tau^2)^{-1}$ is non-local (in the sense that it does not satisfy $\mathrm{supp}(Pu) \subset \mathrm{supp}(u)$ for all $u \in C_c^\infty(\mathbb{R}^n)$).

We write in this section a weak form of Gårding estimates for (almost-)differential operators of order 2, which is at the core of the Carleman method. A general Gårding inequality (that we shall not need for Carleman estimates) will be stated in the next section (about pseudodifferential operators with large parameters).

Let us first state an elementary Gårding-type lemma for Fourier multipliers.

Lemma 1.3.13. *Let $q = q(\xi,\tau) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+) \cap \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^+)$ real-valued and satisfying*

$$q(\xi,\tau) \ge C_0(|\xi|^2 + \tau^2), \quad \textit{for all } (\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Then we have

$$\mathrm{Re}\,(q(D,\tau)u,u)_{L^2} = (q(D,\tau)u,u)_{L^2} \ge C_0 \|u\|^2_{H^1_\tau}, \quad \textit{for all } u \in \mathcal{S}(\mathbb{R}^n), \tau \ge 0.$$

Proof of Lemma 1.3.13. We have $\mathcal{F}\left(q(D,\tau)u\right)(\xi) = q(\xi,\tau)\widehat{u}(\xi)$, so that using the Plancherel formula, we obtain

$$(q(D,\tau)u,u)_{L^2} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} q(\xi,\tau)\widehat{u}(\xi)\overline{\widehat{u}(\xi)}d\xi \geq C_0\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(|\xi|^2+\tau^2)|\widehat{u}(\xi)|^2 d\xi = C_0\left\|u\right\|^2_{H^1_\tau},$$

which proves the lemma (in particular the first identity implies that $(q(D,\tau)u,u)_{L^2}$ is real). □

We now state a local Gårding inequality for a family of operators including Diff^2_τ. The main additional difficulty when compared to Lemma 1.3.13 is the dependence of these operators on the x-variable.

Proposition 1.3.14 (A local Gårding inequality for particular operators)**.** *Assume Ω is an open set with $0\in\Omega$ and let P be an operator of the form*

$$P = A + \sum_{i=1}^{k} B_i \circ (-\Delta+\tau^2)^{-1}\circ B_i \tag{1.39}$$

with $A, B_i \in \mathrm{Diff}^2_\tau(\Omega)$ with real *principal symbols $a_2(x,\xi,\tau)$ and $b_{2,i}(x,\xi,\tau)$. Define*

$$p_2(x,\xi,\tau) = a_2(x,\xi,\tau) + \sum_{i=1}^{k}\frac{b_{2,i}^2(x,\xi,\tau)}{|\xi|^2+\tau^2}, \tag{1.40}$$

and assume that there is $C_0>0$ such that

$$p_2(0,\xi,\tau) \geq C_0(|\xi|^2+\tau^2), \qquad \textit{for all } \xi\in\mathbb{R}^n, \tau\geq 0. \tag{1.41}$$

Then, there exist $r>0$ and C_1, $C_2>0$, so that we have

$$\mathrm{Re}\,(Pu,u)_{L^2} \geq C_1\left\|u\right\|^2_{H^1_\tau} - C_2\left\|u\right\|^2_{L^2}, \qquad \textit{for all } u\in C^\infty_c(B(0,r)), \tau\geq 0. \tag{1.42}$$

In particular, there exist $C,\tau_0>0$ such that

$$\mathrm{Re}\,(Pu,u)_{L^2} \geq C\left\|u\right\|^2_{H^1_\tau}, \qquad \textit{for all } u\in C^\infty_c(B(0,r)), \tau\geq \tau_0.$$

Note that formally, such operators P are "of order 2". The "principal symbol", defined in (1.40) is indeed a homogeneous function of degree 2. Inequality (1.41) is thus a homogeneous inequality, and it is sufficient to assume it on the half-sphere $\mathbb{S}^n_+ := \{(\xi,\tau)\in\mathbb{R}^n\times\mathbb{R}_+, |\xi|^2+\tau^2=1\}$.

The idea of the proof is to "freeze" the coefficients at 0 in order to reduce to the case of a Fourier multiplier, and then use Lemma 1.3.13. For this, we need to estimate the error made by "freezing" the coefficients. This is the aim of Lemma 1.3.15 and Corollary 1.3.16 below. Proposition 1.3.14 is then a consequence of Corollary 1.3.16 together with Lemma 1.3.13.

Lemma 1.3.15. *If $A\in\mathrm{Diff}^2_\tau(\Omega)$ with principal symbol $a_2(x,\xi,\tau)$, then, there exits $C>0$ such that for all $r>0$ such that $B(0,r)\subset\Omega$, we have*

$$|(Au,v)_{L^2} - (a_2(0,D,\tau)u,v)_{L^2}| \leq C\left(r\left\|u\right\|_{H^1_\tau} + \left\|u\right\|_{L^2}\right)\left\|v\right\|_{H^1_\tau},$$

for any $u\in C^\infty_c(B(0,r)), v\in\mathcal{S}(\mathbb{R}^n)$, $\tau\geq 1$.

Corollary 1.3.16. *Let P and p_2 be as in (1.39)-(1.40). Then, for any $\varepsilon>0$, there exits $C>0$ and $r>0$ so that*

$$|(Pu,u)_{L^2} - (p_2(0,D,\tau)u,u)_{L^2}| \leq \varepsilon\left\|u\right\|^2_{H^1_\tau} + C\left\|u\right\|^2_{L^2}$$

for any $u\in C^\infty_c(B(0,r))$, $\tau\geq 1$.

Let us now give the proof of the Gårding inequality of Proposition 1.3.14.

Proof of Proposition 1.3.14. First, assumption (1.41) together with Lemma 1.3.13 implies that

$$\mathrm{Re}\,(p_2(0,D,\tau)u,u)_{L^2} = (p_2(0,D,\tau)u,u)_{L^2} \geq C_0 \|u\|^2_{H^1_\tau}, \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n), \tau \geq 0.$$

Now, Corollary 1.3.16 yields for any $\varepsilon > 0$, the existence of $C > 0$ and $r > 0$ so that

$$|\mathrm{Re}\,(Pu,u)_{L^2} - (p_2(0,D,\tau)u,u)_{L^2}| \leq \varepsilon \|u\|^2_{H^1_\tau} + C\|u\|^2_{L^2}$$

for any $u \in C^\infty_c(B(0,r))$, $\tau \geq 1$. Taking $\varepsilon = \frac{C_0}{2}$ implies the existence of $C > 0$ and $r > 0$ so that

$$\mathrm{Re}\,(Pu,u)_{L^2} \geq \frac{C_0}{2}\|u\|^2_{H^1_\tau} - C\|u\|^2_{L^2}, \qquad \text{for all } u \in C^\infty_c(B(0,r)), \tau \geq 1.$$

which is the first statement. The second statement comes from the fact that $\|u\|_{L^2} \leq \frac{1}{\tau}\|u\|_{H^1_\tau}$ and taking τ large enough in this inequality. □

Before proving the lemma and its corollary, we state (see Exercise 1 in Section 1.4 for a proof) a converse of Proposition 1.3.14, showing that the positivity of the principal symbol is also a necessary condition for the positivity of the operator.

Proposition 1.3.17 (Converse of the Gårding inequality)**.** *Let P and p_2 be as in (1.39)-(1.40). Assume that there exist $r, \tau_0, C_1, C_2 > 0$, so that we have*

$$\mathrm{Re}\,(Pu,u)_{L^2} \geq C_1\|u\|^2_{H^1_\tau} - C_2\|u\|^2_{L^2}, \qquad \textit{for all } u \in C^\infty_c(B(0,r)), \tau \geq \tau_0.$$

Then we have

$$p_2(0,\xi,\tau) \geq C_1(|\xi|^2 + \tau^2), \qquad \textit{for all } \xi \in \mathbb{R}^n, \tau \geq 0.$$

Proof of Lemma 1.3.15. We first write $A \in \mathrm{Diff}^2_\tau$ under the form

$$\begin{aligned} A &= \sum_{i,j=1}^n a_2^{ij}(x)D_iD_j + \sum_{i=1}^n a_1^i(x)\tau D_i + a_0(x)\tau^2 + R, \quad R \in \mathrm{Diff}^1_\tau \\ &= A' + R', \end{aligned} \tag{1.43}$$

where

$$A' = \sum_{i,j=1}^n D_i a_2^{ij}(x)D_j + \sum_{i=1}^n a_1^i(x)\tau D_i + a_0(x)\tau^2 \in \mathrm{Diff}^2_\tau, \quad R' = R - \sum_{i,j=1}^n (D_i a_2^{ij})D_j \in \mathrm{Diff}^1_\tau$$

We estimate the lower order term involving R' using Propositions 1.3.9 and 1.3.5 as

$$|(R'u,v)_{L^2}| = |(u,(R')^*v)_{L^2}| \leq C\|u\|_{L^2}\|v\|_{H^1_\tau}. \tag{1.44}$$

Thus, it only remains to estimate the quantity $|((A' - a_2(0,D,\tau))u,v)_{L^2}|$. By linearity, it is enough to do it term-by-term. Let us consider the most difficult terms, namely $D_i a_2^{ij}(x)D_j$ (terms under the form $a_1^i(x)\tau D_i$ and $a_0(x)\tau^2$ are simpler to treat). Using $a_2^{ij}(0)D_iD_j = D_i a_2^{ij}(0)D_j$ and an integration by parts, we obtain

$$\Big((D_i a_2^{ij}(x)D_j - a_2^{ij}(0)D_iD_j)u,v\Big)_{L^2} = \Big((a_2^{ij}(x) - a_2^{ij}(0))D_ju, D_iv\Big)_{L^2}.$$

For $u \in C^\infty_c(B(0,r))$ this can thus be estimated as

$$\left|\Big((D_i a_2^{ij}(x)D_j - a_2^{ij}(0)D_iD_j)u,v\Big)_{L^2}\right| \leq \left\|a_2^{ij}(x) - a_2^{ij}(0)\right\|_{L^\infty(B(0,r))}\|u\|_{H^1_\tau}\|v\|_{H^1_\tau},$$

where, using the mean value theorem, we have

$$\left\|a_2^{ij}(\cdot) - a_2^{ij}(0)\right\|_{L^\infty(B(0,r))} \leq r\left\|da_2^{ij}\right\|_{L^\infty(B(0,r))} \leq Cr.$$

We have thus obtained

$$\left|\left((D_i a_2^{ij}(x)D_j - a_2^{ij}(0)D_iD_j)u, v\right)_{L^2}\right| \leq Cr \|u\|_{H^1_\tau} \|v\|_{H^1_\tau}, \quad \text{for all } u \in C_c^\infty(B(0,r)), v \in \mathcal{S}(\mathbb{R}^n).$$

A similar computation holds for all terms under the form $a_1^i(x)\tau D_i$ and $a_0(x)\tau^2$, which eventually yields to

$$|(A'u, v)_{L^2} - (a_2(0, D, \tau)u, v)_{L^2}| \leq Cr \|u\|_{H^1_\tau} \|v\|_{H^1_\tau},$$

for all $r, \tau > 0$ and all $u \in C_c^\infty(B(0,r)), v \in \mathcal{S}(\mathbb{R}^n)$. Combined with (1.43)-(1.44), this concludes the proof of the lemma. □

Proof of Corollary 1.3.16. We have

$$(Pu, u)_{L^2} = (Au, u)_{L^2} + \sum_{i=1}^k \left(B_i \circ (-\Delta + \tau^2)^{-1} \circ B_i u, u\right)_{L^2}. \tag{1.45}$$

According to Lemma 1.3.15, applied to $v = u \in C_c^\infty(B(0,r))$, and using $\|u\|_{H^1_\tau} \|u\|_{L^2} \leq \varepsilon \|u\|^2_{H^1_\tau} + \frac{1}{4\varepsilon} \|u\|^2_{L^2}$ for all $\varepsilon > 0$, we obtain the sought result for the difference

$$|((A - a_2(0, D, \tau))u, u)_{L^2}| \leq \varepsilon \|u\|^2_{H^1_\tau} + C_\varepsilon \|u\|^2_{L^2}, \quad u \in C_c^\infty(B(0, r_\varepsilon)).$$

It thus only remains to prove a similar statement for each of the terms in the sum of (1.45), which we shall for simplicity denote $\left(B \circ (-\Delta + \tau^2)^{-1} \circ Bu, u\right)_{L^2}$ here (removing the subscript i). We shall denote b_2 (instead of $b_{2,i}$) for the symbol of B (instead of B_i) accordingly. We write

$$\begin{aligned}\left(B \circ (-\Delta + \tau^2)^{-1} \circ Bu, u\right)_{L^2} &= \left((-\Delta + \tau^2)^{-1} \circ Bu, B^* u\right)_{L^2} \\ &= \left((-\Delta + \tau^2)^{-1} \circ Bu, b_2(0, D, \tau)u\right)_{L^2} + R_1,\end{aligned}$$

where

$$R_1 = \left((-\Delta + \tau^2)^{-1} \circ Bu, (B^* - b_2(0, D, \tau)u\right)_{L^2}.$$

We first estimate R_1, and come back to the main term afterwards. We have

$$|R_1| \leq \left\|(-\Delta + \tau^2)^{-1} \circ Bu\right\|_{H^1_\tau} \|(B^* - b_2(0, D, \tau))u\|_{H^{-1}_\tau}$$

and, using that B sends H^1_τ into H^{-1}_τ, we get

$$\left\|(-\Delta + \tau^2)^{-1} \circ Bu\right\|_{H^1_\tau} \leq \|Bu\|_{H^{-1}_\tau} \leq C \|u\|_{H^1_\tau}.$$

We are thus left to estimate $\|(B^* - b_2(0, D, \tau))u\|_{H^{-1}_\tau}$. For this, we use Lemma 1.3.15 for the differential operator B^*, with principal symbol $b_2(0, \xi, \tau)$ since b_2 is assumed real-valued. This yields

$$|((B^* - b_2(0, D, \tau))u, v)_{L^2}| \leq C \left(r \|u\|_{H^1_\tau} + \|u\|_{L^2}\right) \|v\|_{H^1_\tau},$$

for any $u \in C_c^\infty(B(0,r)), v \in \mathcal{S}(\mathbb{R}^n)$, $\tau \geq 1$. According to the characterization of the H^{-1}_τ norm by duality (see Lemma 1.3.3 above), this implies

$$\|(B^* - b_2(0, D, \tau))u\|_{H^{-1}_\tau} \leq C \left(r \|u\|_{H^1_\tau} + \|u\|_{L^2}\right).$$

Combining the above estimates, we have now proved that

$$|R_1| \leq C \left(r \|u\|_{H^1_\tau} + \|u\|_{L^2}\right) \|u\|_{H^1_\tau}, \quad \text{for all } u \in C_c^\infty(B(0,r)),$$

which is an admissible remainder term (by taking r small enough and using again $ab \leq \varepsilon a^2 + b^2/4\varepsilon$).

To conclude the proof, it only remains to replace $\left((-\Delta+\tau^2)^{-1}\circ Bu, b_2(0,D,\tau)u\right)_{L^2}$ by the same expression with B replaced by $b_2(0,D,\tau)$. That means to estimate

$$\begin{aligned}&\left|\left((-\Delta+\tau^2)^{-1}\circ(B-b_2(0,D,\tau))u, b_2(0,D,\tau)u\right)_{L^2}\right|\\&\quad\le\left\|(-\Delta+\tau^2)^{-1}(B-b_2(0,D,\tau))u\right\|_{H^1_\tau}\|b_2(0,D,\tau)u\|_{H^{-1}_\tau}\\&\quad\le\|(B-b_2(0,D,\tau))u\|_{H^{-1}_\tau}\|u\|_{H^1_\tau}.\end{aligned}$$

This last term has already been estimated with B^* instead of B, but this works the same (since p_2 is real-valued). □

We also have a semiglobal Gårding inequality when replacing the local assumption (1.41) of Proposition 1.3.14 by a semiglobal one.

Proposition 1.3.18 (Semiglobal Gårding inequality)**.** *Let $\Omega\subset\mathbb{R}^n$ be a* bounded *open set, and P be as in* (1.39)*, with $A, B_i\in\mathrm{Diff}^2_\tau(\Omega)$ with* real *principal symbol $a_2(x,\xi,\tau)$ and $b_{2,i}(x,\xi,\tau)$, and $p_2(x,\xi,\tau)$ as in* (1.40)*. Assume there is $C>0$ such that*

$$p_2(x,\xi,\tau)\ge C(|\xi|^2+\tau^2)\qquad\text{for all } (x,\xi,\tau)\in\overline{\Omega}\times\mathbb{R}^n\times\mathbb{R}^+ \tag{1.46}$$

Then, there exist and C_1, $C_2>0$, so that we have

$$\mathrm{Re}\,(Pu,u)_{L^2}\ge C_1\|u\|^2_{H^1_\tau}-C_2\|u\|^2_{L^2},\qquad\text{for all } u\in C^\infty_c(\Omega),\tau\ge 1.$$

In particular, there exist $C,\tau_0>0$,such that

$$\mathrm{Re}\,(Pu,u)_{L^2}\ge C\|u\|^2_{H^1_\tau},\qquad\text{for all } u\in C^\infty_c(\Omega),\tau\ge\tau_0.$$

Note that Assumption (1.46) indeed makes sense up to the boundary of Ω: since all coefficients of the operator have all derivatives uniformly continuous, they can be extended uniquely to $\overline{\Omega}$.

Proof. We may apply Proposition 1.3.14 at any $x_0\in\overline{\Omega}$ ♣ `beware: In the statement of this Proposition,` $x_0\in\Omega$ `(not in the boundary of the open set).` This yields for any $x\in\overline{\Omega}$ the existence of $r^x>0$ and $C_1^x, C_2^x>0$, so that the inequality (1.42) holds for all functions $u\in C^\infty_c(B(x,r^x))$. Since $\overline{\Omega}$ is compact, we can extract from the cover $\overline{\Omega}\subset\bigcup_{x\in\Omega}B(x,r^x)$ a finite cover denoted by $\overline{\Omega}\subset\bigcup_{i\in I}B(x_i,r_i)$. Define a subordinated smooth partition of unity $\chi_i\in C^\infty(B(x_i,r_i))$ so that

$$\sum_i\chi_i^2=1\text{ on }\overline{\Omega}.$$

See e.g. [Hör90, ♣] for the construction of such functions. We thus have $u=\sum_i\chi_i^2u$ and decompose

$$\begin{aligned}(Pu,u)_{L^2}&=\sum_i\left(\chi_i^2Pu,u\right)_{L^2}=\sum_i(\chi_iPu,\chi_iu)_{L^2}\\&=\sum_i(P\chi_iu,\chi_iu)_{L^2}+\sum_i([\chi_i,P]u,\chi_iu)_{L^2}.\end{aligned}$$

Now, the bound furnished by Proposition 1.3.14 yields

$$\sum_i(P\chi_iu,\chi_iu)_{L^2}\ge\left(\min_{i\in I}C_1^{x_i}\right)\sum_{i\in I}\|\chi_iu\|^2_{H^1_\tau}-\left(\max_{i\in I}C_2^{x_i}\right)\sum_{i\in I}\|\chi_iu\|^2_{L^2}.$$

We notice that, for $\tau\ge1$,

$$\|u\|^2_{H^1_\tau}=\left\|\sum_i\chi_i^2u\right\|^2_{H^1_\tau}\le C\sum_i\left\|\chi_i^2u\right\|^2_{H^1_\tau}\le C\sum_i\|\chi_iu\|^2_{H^1_\tau},$$

where we used Proposition 1.3.5 and the fact that the sum is finite. Hence, the above three lines imply the existence of $C_1, C_2 > 0$ such that we have for all $u \in C_c^\infty(\Omega)$,

$$(Pu, u)_{L^2} + \sum_i |([\chi_i, P]u, \chi_i u)_{L^2}| \geq C_1 \|u\|_{H^1_\tau}^2 - C_2 \|u\|_{L^2}^2 . \tag{1.47}$$

It only remains to estimate $|([\chi_i, P]u, \chi_i u)_{L^2}|$. Recalling the form of the operator $P = A + \sum_{j=1}^k B_j \circ (-\Delta + \tau^2)^{-1} \circ B_j$ in (1.39), we have $[\chi_i, A] \in \mathrm{Diff}^1_\tau$, and hence

$$|([\chi_i, A]u, \chi_i u)_{L^2}| \leq C \|u\|_{H^1_\tau} \|u\|_{L^2} ,$$

which is an admissible remainder. We thus only need to examine each $[\chi_i, B_j \circ (-\Delta + \tau^2)^{-1} \circ B_j]$. We remove the indices and write $S = (-\Delta + \tau^2)$ for readability. We have

$$[\chi, BS^{-1}B] = BS^{-1}[\chi, B] + B[\chi, S^{-1}]B + [\chi, B]S^{-1}B.$$

We remark that $[\chi, B] \in \mathrm{Diff}^1_\tau$ and obtain

$$\begin{aligned} \left|\left(BS^{-1}[\chi, B]u, \chi u\right)_{L^2}\right| &= \left|\left([\chi, B]u, S^{-1}B^*\chi u\right)_{L^2}\right| \leq \|[\chi, B]u\|_{H^{-1}_\tau} \left\|S^{-1}B^*\chi u\right\|_{H^1_\tau} \\ &\leq C \|u\|_{L^2} \|B^*\chi u\|_{H^{-1}_\tau} \leq C \|u\|_{L^2} \|u\|_{H^1_\tau} , \\ \left|\left([\chi, B]S^{-1}Bu, \chi u\right)_{L^2}\right| &\leq C \|u\|_{L^2} \|u\|_{H^1_\tau} , \end{aligned}$$

where the second estimate is obtained as the first one. We now rewrite $B[\chi, S^{-1}]B$ using the general fact $[T, S^{-1}] = TS^{-1} - S^{-1}T = S^{-1}[S, T]S^{-1}$, and obtain

$$\begin{aligned} \left|\left(B[\chi, S^{-1}]Bu, \chi u\right)_{L^2}\right| &= \left|\left(BS^{-1}[S, \chi]S^{-1}Bu, \chi u\right)_{L^2}\right| \leq \left\|BS^{-1}[S, \chi]S^{-1}Bu\right\|_{L^2} \|u\|_{L^2} \\ &\leq C \left\|S^{-1}[S, \chi]S^{-1}Bu\right\|_{H^2_\tau} \|u\|_{L^2} \leq C \left\|[S, \chi]S^{-1}Bu\right\|_{L^2} \|u\|_{L^2} \leq C \|u\|_{H^1_\tau} \|u\|_{L^2} . \end{aligned}$$

This together with (1.47) concludes the proof of the proposition. □

Remark that the proof shows and uses that $[P, \chi]$ is "of order 1". However, as P, this operator is not a differential operator. We briefly discuss a more general class of operators containing P, and associated calculus in the next section.

1.3.6 Pseudodifferential operators

♣ `not taught in class`

This section is provided here as a remark: the class of differential operators Diff^m_τ described above can be embedded in the more general class of so-called pseudodifferential operators (depending on a large parameter τ). The latter class has the advantages of being an algebra, and containing both differential operators, nice Fourier multipliers, together with operators of the form (1.39) having symbols like (1.40).

The calculus for differential operators described in the previous section generalizes nicely to this class, with some technicalities. The reader is referred to [Hör85, Hör94] for a description of this theory. We only state here counterparts of the results described in the previous section for such operators. We do not provide proofs of these results, which are beyond the scope of the present introductory book.

Note that introducing these classes of symbols and operators is not needed for the purposes of this book, namely for proving usual Carleman estimates for operators of order 2 with real principal symbols. This section can thus be skipped at first reading. In the proofs in the next chapters, we shall mostly consider differential operators. Yet, we believe that it is good to know that the above operators and results can be embedded in a nice class of operators enjoying nice calculus properties.

A starting point is the remark that Formula (1.29), which to a symbol associates an operator, does not require the symbol $p(x, \xi, \tau)$ to be polynomial. For instance, the operator $(-\Delta + \tau^2)^{-1}$ is well defined for $\tau > 0$ and equal to Formula (1.29) with $p(x, \xi, \tau) = p(\xi, \tau) = \frac{1}{|\xi|^2 + \tau^2}$. We would like to use a class of operators containing also $(-\Delta + \tau^2)^{-1}$, as an operator of "order -2" with symbol $\frac{1}{|\xi|^2 + \tau^2}$ (which is homogeneous of degree -2 in (ξ, τ)). Before introducing the class of pseudodifferential operators which achieve these properties, we need to introduce the class of symbols for which Formula (1.29) will provide with a nice operator, called here the S^m_τ class.

Definition 1.3.19. Given an open set $\Omega \subset \mathbb{R}^n$ and $m \in \mathbb{R}$, we say that $p(x,\xi,\tau)$ belongs to $S^m_\tau(\Omega \times \mathbb{R}^n)$, if it is smooth in (x,ξ) and, for any $\alpha, \beta \in \mathbb{N}^n$, there exists $C_{\alpha,\beta}$ so that

$$\left|\partial_x^\alpha \partial_\xi^\beta p(x,\xi,\tau)\right| \leq C_{\alpha,\beta}\left(\sqrt{|\xi|^2+\tau^2}\right)^{m-|\beta|} \quad \text{for all } (x,\xi) \in \Omega \times \mathbb{R}^n, \tau \geq 1. \tag{1.48}$$

Note that if $m \in \mathbb{N}$ and $p \in \Sigma^m$, then $p \in S^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$. Note also that $\left(\sqrt{|\xi|^2+\tau^2}\right)^m \in S^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$ for any $m \in \mathbb{R}$. Note finally that the symbols useful for later applications to Carleman estimates, given by (1.40), belong to $S^2_\tau(\mathbb{R}^n \times \mathbb{R}^n)$.

Note that in these two examples, the symbol p is either homogeneous of degree m, or a sum of terms which are homogeneous of degree $\leq m$ (more precisely, of the form $p = p_m + p_{m-1} + \cdots$, where p_{m-j} is homogeneous of degree $m-j$). This is no longer the case for general symbols in $S^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$.

For this reason, it is less obvious to define the principal symbol or the principal part of the symbol.

Definition 1.3.20. For $m \in \mathbb{R}$ and $a \in S^m_\tau(\Omega \times \mathbb{R}^n)$, we define the principal symbol a_m of a (or the principal part a_m of the symbol a) to be the equivalence class of a in $S^m_\tau(\Omega \times \mathbb{R}^n)/S^{m-1}_\tau(\Omega \times \mathbb{R}^n)$. We identify a_m with any of its representatives. In case there is a homogeneous representative of a in $S^m_\tau(\Omega \times \mathbb{R}^n)/S^{m-1}_\tau(\Omega \times \mathbb{R}^n)$ (of degree m), then we choose this representative for the principal symbol.

Now, from $p(x,\xi,\tau) \in S^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$, we may define the associated pseudodifferential operator $p(x,D,\tau)$ by mimicking Formula (1.29).

Definition 1.3.21. Given $p \in S^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$, we set

$$(p(x,D,\tau)u)(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi,\tau)\widehat{u}(\xi)d\xi, \quad \text{for } u \in \mathcal{S}(\mathbb{R}^n).$$

We denote by $\Psi^m_\tau(\mathbb{R}^n)$ the set of all such operators for $p \in S^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$.

In quantum mechanics, the formula of this definition is called a quantization formula, for it associates to a "classical observable" (i.e. a function on the classical phase space $\mathbb{R}^n \times \mathbb{R}^n$, i.e. a symbol) a "quantum observable" (i.e. an operator on the quantum Hilbert space $L^2(\mathbb{R}^n)$). Conversely, we say that $p(x,\xi,\tau)$ is the (full) symbol of the operator $p(x,D,\tau)$.

Note that we have $\mathrm{Diff}^m_\tau(\mathbb{R}^n) \subset \Psi^m_\tau(\mathbb{R}^n)$ for $m \in \mathbb{N}$ and $(-\Delta+\tau^2)^{m/2} \in \Psi^m_\tau(\mathbb{R}^n)$ for all $m \in \mathbb{R}$. Note also that $p(x,D,\tau)u$ is well defined and belongs to $\mathcal{S}(\mathbb{R}^n)$ if $u \in \mathcal{S}(\mathbb{R}^n)$. But such an operator has actually better mapping properties, similar to those enjoyed by differential operators.

Theorem 1.3.22 (Action on Sobolev spaces)**.** *Let $P \in \Psi^m_\tau(\mathbb{R}^n)$. Then, for any $s \in \mathbb{R}$, there exits $C > 0$ such that*

$$\|Pu\|_{H^{s-m}_\tau} \leq C\|u\|_{H^s_\tau}, \quad \textit{for all } u \in \mathcal{S}(\mathbb{R}^n) \textit{ and } \tau \geq 1.$$

As a consequence, an operator $P \in \Psi^m_\tau(\mathbb{R}^n)$ can be uniquely extended as a bounded operator from H^s_τ to H^{s-m}_τ uniformly in τ.

Note that the latter part of the proposition follows from the density of $\mathcal{S}(\mathbb{R}^n)$ into H^s_τ for all $s \in \mathbb{R}$. This is the analogue in the more general class $\Psi^m_\tau(\mathbb{R}^n)$ of Proposition 1.3.5 in $\mathrm{Diff}^m_\tau(\mathbb{R}^n)$.

Theorem 1.3.23 (Composition)**.** *Let $m_1, m_2 \in \mathbb{R}$ and $A \in \Psi^{m_1}_\tau(\mathbb{R}^n)$, $B \in \Psi^{m_2}_\tau(\mathbb{R}^n)$ having (full) symbols $a(x,\xi,\tau)$ and $b(x,\xi,\tau)$. Then, the composition $AB \in \Psi^{m_1+m_2}_\tau(\mathbb{R}^n)$ and we have $AB = c(x,D,\tau)$ where, for all $N \in \mathbb{N}$,*

$$c(x,\xi,\tau) = \sum_{\alpha \leq N} \frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha a(x,\xi,\tau)\partial_x^\alpha b(x,\xi,\tau) + r_N(x,\xi,\tau), \quad \textit{with } r_N \in S^{m_1+m_2-N-1}_\tau. \tag{1.49}$$

In particular, the principal symbol of AB is the product $a(x,\xi,\tau)b(x,\xi,\tau)$ (modulo $S^{m_1+m_2-1}_\tau$).

Equivalently, if a_{m_1} and b_{m_2} denote the principal symbols of A and B respectively, the principal symbol of AB is the product $a_{m_1}(x,\xi,\tau)b_{m_2}(x,\xi,\tau)$ (modulo $S^{m_1+m_2-1}_\tau$).

Note that this theorem implies that operators under the form (1.39) (useful for Carleman estimates) belong actually to $\Psi^2_\tau(\mathbb{R}^n)$ and have principal symbol given by (1.40).

As for differential operators, the defect of commutation between A and B is not seen at principal order (both have ab as principal symbol). However, looking carefully at the subprincipal term in the asymptotic expansion (1.50) (i.e. that with $|\alpha| = 1$), one obtains the symbol of the commutator.

Corollary 1.3.24 (Commutator)**.** *Let $m_1, m_2 \in \mathbb{R}$ and $A \in \Psi_\tau^{m_1}(\mathbb{R}^n)$, $B \in \Psi_\tau^{m_2}(\mathbb{R}^n)$ having (full) symbols $a(x,\xi,\tau)$ and $b(x,\xi,\tau)$. Then, the commutator $[A,B] \in \Psi_\tau^{m_1+m_2-1}$ has principal symbol $\frac{1}{i}\{a,b\}(x,\xi,\tau) \in S_\tau^{m_1+m_2-1}$ (modulo $S_\tau^{m_1+m_2-2}$), where the Poisson bracket is defined in Definition 1.3.7.*

The last calculus rule concerns the adjoint operator with respect to the usual $L^2(\mathbb{R}^n, dx)$ inner product, and we have the following generalization of Proposition 1.3.9.

Theorem 1.3.25 (Adjoint)**.** *Let $m \in \mathbb{R}$ and $P \in \Psi_\tau^m(\mathbb{R}^n)$ having (full) symbol $p(x,\xi,\tau)$. Then, there is a unique operator P^* satisfying (1.37). Moreover, we have $P^* \in \Psi_\tau^m(\mathbb{R}^n)$ and $P^* = q(x,D,\tau)$ where, for all $N \in \mathbb{N}$,*

$$q(x,\xi,\tau) = \sum_{\alpha \leq N} \frac{1}{i^{|\alpha|}\alpha!} \partial_x^\alpha \partial_\xi^\alpha \overline{p}(x,\xi,\tau) + r_N(x,\xi,\tau), \quad \text{with } r_N \in S_\tau^{m-N-1}. \tag{1.50}$$

In particular, the principal symbol of P^* is equal to the complex conjugate $\overline{p}(x,\xi,\tau)$ (modulo S_τ^{m-1}).

One may also say that $P^* - \overline{p}(x,D,\tau) \in \Psi_\tau^{m-1}(\mathbb{R}^n)$, if p is the (full) symbol of P, or equivalently that $P^* - \overline{p}_m(x,D,\tau) \in \Psi_\tau^{m-1}(\mathbb{R}^n)$ if p_m denotes the principal symbol of P.

We finally give a Gårding inequality which generalizes that of Proposition (1.3.14), i.e. for operators of the form (1.39), with principal symbol (1.40).

Theorem 1.3.26 (Local Gårding inequality)**.** *Let $x_0 \in \mathbb{R}^n$, $m \in \mathbb{R}$ and $P \in \Psi_\tau^m(\mathbb{R}^n)$ has real principal symbol $p_m(x,\xi,\tau)$ (that is, there is a real-valued representative in the class $S_\tau^m(\mathbb{R}^n \times \mathbb{R}^n)/S_\tau^{m-1}(\mathbb{R}^n \times \mathbb{R}^n)$). Assume that there exist $C_0, R > 0$ such that*

$$\operatorname{Re} p(x_0,\xi,\tau) \geq C_0(\xi^2+\tau^2)^{m/2}, \quad \text{for all } (\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}^+, |(\xi,\tau)| \geq R.$$

Then, there are $C, r, \tau_0 > 0$ such that

$$\operatorname{Re}(Pu,u)_{L^2} \geq C\|u\|^2_{H_\tau^{m/2}}, \quad \text{for all } u \in C_c^\infty(B(x_0,r)), \tau \geq \tau_0.$$

1.4 Exercises on Chapter 1

Exercise 1 (Converse of the Gårding inequality, part of the Exam of May, 2018)**.** The goal of this exercise is to prove the converse of Proposition 1.3.14, namely the statement of Proposition 1.3.17.

We prove in the first place a converse of the Gårding inequality for *differential* operators. We let $\xi \in \mathbb{R}^n$ be fixed, and $\chi \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$ be such that $\chi(0) = 1$, and consider the functions

$$u_\tau(x) = N(\tau)\chi(\sqrt{\tau}x)e^{i\tau x\cdot\xi}, \quad \tau > 0.$$

1. Determine $N(\tau)$ so that $\|u_\tau\|_{L^2(\mathbb{R}^n)} = 1$ for all $\tau > 0$.

2. Give an asymptotic expansion as $\tau \to +\infty$ under the form $\alpha(\tau)+O(\beta(\tau))$ (with α, β to be determined, $\beta = o(\alpha)$) of both quantities: $\frac{1}{\tau}(D_j u_\tau, u_\tau)_{L^2(\mathbb{R}^n)}$ and $(au_\tau, u_\tau)_{L^2(\mathbb{R}^n)}$, where $a \in C^\infty(\mathbb{R}^n)$.

3. We take $P \in \mathrm{Diff}^m_\tau(\mathbb{R}^n)$, with principal symbol p_m. Give an asymptotic expansion as $\tau \to +\infty$ of the quantity
$$\frac{1}{\tau^m}(Pu_\tau, u_\tau)_{L^2(\mathbb{R}^n)}.$$
Hint: one can consider at first the case $P = a(x)\tau^\beta D^\alpha$.

4. Compute $\hat{u}_\tau(\eta)$ in terms of $\hat{\chi}$.

5. Given $s \in \mathbb{R}$, compute an asymptotic equivalent as $\tau \to +\infty$ of the quantity $\frac{1}{\tau^{2s}}\|u_\tau\|^2_{H^s_\tau}$.

6. Assume there exists a neighborhood U of 0 in $\mathbb{R}^n$ and $C_0, \tau_0 > 0$, such that
$$\mathrm{Re}\,(Pu, u)_{L^2} \geq C_0 \|u\|^2_{H^{m/2}_\tau}, \qquad \text{for all } u \in C_c^\infty(U), \tau \geq \tau_0. \tag{1.51}$$
Prove that $\mathrm{Re}(p_m(0, \xi, 1)) \geq C_0(|\xi|^2 + 1)^{m/2}$ for all $\xi \in \mathbb{R}^n$.

7. Deduce that $\mathrm{Re}(p_m(0, \xi, \tau)) \geq C_0(|\xi|^2 + \tau^2)^{m/2}$ for all $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}^+_*$. Conclude.

We now wish to prove a converse of the Gårding inequality for operators of the form (1.39).

8. Give an equivalent as $\tau \to +\infty$ of the quantity $\left((-\Delta + \tau^2)^{-1}u_\tau, u_\tau\right)_{L^2}$.

9. Let $a, b \in C^\infty(\mathbb{R}^n)$. Give an equivalent as $\tau \to +\infty$ of the quantity
$$\left((-\Delta + \tau^2)^{-1}a(x)D^\alpha u_\tau, b(x)D^\beta u_\tau\right)_{L^2}.$$
Hint: one may consider at first the case $a = b = 1$, then the case $\alpha = \beta = 0$, before turning to the general case.

10. Prove Proposition 1.3.17, that is, the converse of the Gårding inequality for operators of the form:
$$P = A + \sum_{i=1}^k B_i \circ (-\Delta + \tau^2)^{-1} \circ B_i, \qquad A, B_i \in \mathrm{Diff}^2_\tau(\mathbb{R}^n).$$

Correction 1. 1. We have $\|u_\tau\|^2_{L^2(\mathbb{R}^n)} = N(\tau)^2 \int_{\mathbb{R}^n} |\chi(\sqrt{\tau}x)|^2 dx = N(\tau)^2\tau^{-n/2}\|\chi\|^2_{L^2} = 1$ for $N(\tau) = \tau^{n/4}\|\chi\|^{-1}_{L^2}$.

2. We have $D_j u_\tau = N(\tau)\frac{1}{i}\left(\sqrt{\tau}\partial_j\chi(\sqrt{\tau}x) + i\tau\xi_j\chi(\sqrt{\tau}x)\right)e^{i\tau x\cdot\xi}$. As a consequence, we have
$$(D_j u_\tau, u_\tau)_{L^2(\mathbb{R}^n)} = \sqrt{\tau}N(\tau)^2\frac{1}{i}\int_{\mathbb{R}^n}\partial_j\chi(\sqrt{\tau}x)\overline{\chi}(\sqrt{\tau}x)dx + \tau\xi_j N(\tau)^2\int_{\mathbb{R}^n}|\chi(\sqrt{\tau}x)|^2 dx,$$

and hence, recalling the choice of $N(\tau)$,

$$\frac{1}{\tau}(D_j u_\tau, u_\tau)_{L^2(\mathbb{R}^n)} = \xi_j + \mathcal{O}\left(\frac{1}{\sqrt{\tau}}\right).$$

Next, we have

$$(au_\tau, u_\tau)_{L^2(\mathbb{R}^n)} = N(\tau)^2 \int_{\mathbb{R}^n} a(x)|\chi(\sqrt{\tau}x)|^2 dx = N(\tau)^2 \tau^{-n/2} \int_{\mathbb{R}^n} a(y/\sqrt{\tau})|\chi(y)|^2 dy.$$

Moreover, we have $a(y/\sqrt{\tau}) = a(0) + \mathcal{O}(1/\sqrt{\tau})$ uniformly for $y \in \operatorname{supp}(\chi)$, so that

$$\begin{aligned}(au_\tau, u_\tau)_{L^2(\mathbb{R}^n)} &= N(\tau)^2 \int_{\mathbb{R}^n} a(x)|\chi(\sqrt{\tau}x)|^2 dx = N(\tau)^2 \tau^{-n/2} \int_{\mathbb{R}^n} \left(a(0) + \mathcal{O}\left(\frac{1}{\sqrt{\tau}}\right)\right) |\chi(y)|^2 dy \\ &= a(0) + \mathcal{O}\left(\frac{1}{\sqrt{\tau}}\right).\end{aligned}$$

3. We first remark that the Leibnitz formula yields

$$D^\alpha \left(\chi(\sqrt{\tau}x)e^{i\tau x\cdot\xi}\right) = \chi(\sqrt{\tau}x) D^\alpha \left(e^{i\tau x\cdot\xi}\right) + r_\tau(x) = \chi(\sqrt{\tau}x)\tau^{|\alpha|}\xi^\alpha e^{i\tau x\cdot\xi} + r_\tau(x),$$

where all terms in the sum $r_\tau(x)$ contain at least one derivative on $\chi(\sqrt{\tau}x)$ and at most $|\alpha| - 1$ derivative on $e^{i\tau x\cdot\xi}$. Hence this term can be written $r_\tau(x) = f_\tau(\sqrt{\tau}x)e^{i\tau x\cdot\xi}$ and estimated roughly as $|f_\tau(y)| \leq \tau^{1/2+|\alpha|-1} = \tau^{|\alpha|-1/2}$ uniformly on $\mathbb{R}^n$ since we have $\operatorname{supp}(f_\tau) \subset \operatorname{supp}(\chi)$. As a consequence, we have

$$\begin{aligned}(a(x)D^\alpha u_\tau, u_\tau)_{L^2(\mathbb{R}^n)} &= N(\tau)^2 \int_{\mathbb{R}^n} a(x) \left(\chi(\sqrt{\tau}x)\tau^{|\alpha|}\xi^\alpha e^{i\tau x\cdot\xi} + r_\tau(x)\right) \chi(\sqrt{\tau}x) e^{-i\tau x\cdot\xi} dx \\ &= \tau^{|\alpha|}\xi^\alpha N(\tau)^2 \int_{\mathbb{R}^n} a(x)|\chi(\sqrt{\tau}x)|^2 dx + N(\tau)^2 \int_{\mathbb{R}^n} a(x) f_\tau(\sqrt{\tau}x)\chi(\sqrt{\tau}x) dx \\ &= \tau^{|\alpha|}\xi^\alpha \left(a(0) + \mathcal{O}\left(\frac{1}{\sqrt{\tau}}\right)\right) + \|\chi\|_{L^2}^{-2} \int_{\mathbb{R}^n} a(y/\sqrt{\tau}) f_\tau(y)\chi(y) dy\end{aligned}$$

Using that $|f_\tau(y)| \leq \tau^{|\alpha|-1/2}$ uniformly on $\mathbb{R}^n$, this implies

$$\left(a(x)\tau^\beta D^\alpha u_\tau, u_\tau\right)_{L^2(\mathbb{R}^n)} = \tau^{\beta+|\alpha|}\xi^\alpha a(0) + \mathcal{O}(\tau^{\beta+|\alpha|-1/2}).$$

Note then that if $P = a(x)\tau^\beta D^\alpha$, then $m = \beta + |\alpha|$, $p_m(x, \xi, \tau) = a(x)\tau^\beta \xi^\alpha$ and this formula reads $(Pu_\tau, u_\tau)_{L^2(\mathbb{R}^n)} = \tau^m p_m(0, \xi, 1) + \mathcal{O}(\tau^{m-1/2})$. By linearity, we thus obtain that for all $P \in \operatorname{Diff}_\tau^m(\mathbb{R}^n)$ with principal symbol $p_m = p_m(x, \xi, \tau)$,

$$\frac{1}{\tau^m}(Pu_\tau, u_\tau)_{L^2(\mathbb{R}^n)} = p_m(0, \xi, 1) + \mathcal{O}(\tau^{-1/2}).$$

4. We have

$$\begin{aligned}\hat{u}_\tau(\eta) &= \int_{\mathbb{R}^n} u_\tau(x)e^{-ix\cdot\eta} dx = N(\tau) \int_{\mathbb{R}^n} \chi(\sqrt{\tau}x)e^{-ix\cdot(\eta-\tau\xi)} dx \\ &= \frac{N(\tau)}{\tau^{n/2}} \int_{\mathbb{R}^n} \chi(y)e^{-i\frac{y}{\sqrt{\tau}}\cdot(\eta-\tau\xi)} dy = \frac{N(\tau)}{\tau^{n/2}} \hat{\chi}\left(\frac{1}{\sqrt{\tau}}(\eta - \tau\xi)\right).\end{aligned}$$

5. As a consequence, we have

$$(2\pi)^n \|u_\tau\|_{H_\tau^s}^2 = \int_{\mathbb{R}^n} (|\eta|^2 + \tau^2)^s |\hat{u}_\tau(\eta)|^2 d\eta = \frac{N(\tau)^2}{\tau^n} \int_{\mathbb{R}^n} (|\eta|^2 + \tau^2)^s \left|\hat{\chi}\left(\frac{1}{\sqrt{\tau}}(\eta - \tau\xi)\right)\right|^2 d\eta.$$

We now set $\zeta = \frac{1}{\sqrt{\tau}}(\eta - \tau\xi)$ and remark that $(|\eta|^2 + \tau^2)^s = \tau^{2s}\left(\left|\xi + \frac{\zeta}{\sqrt{\tau}}\right|^2 + 1\right)^s$ to obtain

$$(2\pi)^n \|u_\tau\|_{H_\tau^s}^2 = \frac{N(\tau)^2}{\tau^{n/2}} \tau^{2s} \int_{\mathbb{R}^n} \left(\left|\xi + \frac{\zeta}{\sqrt{\tau}}\right|^2 + 1\right)^s |\hat{\chi}(\zeta)|^2 d\zeta.$$

Recalling the value of $N(\tau) = \tau^{n/4} \|\chi\|_{L^2}^{-1}$ and using the dominated convergence theorem (note that $\hat{\chi} \in \mathcal{S}(\mathbb{R}^n)$), we finally deduce

$$\begin{aligned}\frac{1}{\tau^{2s}} \|u_\tau\|_{H_\tau^s}^2 &= \frac{1}{\|\chi\|_{L^2}^2} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\left| \xi + \frac{\zeta}{\sqrt{\tau}} \right|^2 + 1 \right)^s |\hat{\chi}(\zeta)|^2 d\zeta \\ &\to_{\tau\to+\infty} \frac{1}{\|\chi\|_{L^2}^2} (|\xi|^2+1)^s \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{\chi}(\zeta)|^2 d\zeta = (|\xi|^2+1)^s,\end{aligned}$$

after having used the Plancherel theorem. That is to say, $\|u_\tau\|_{H_\tau^s} \sim \tau^s (|\xi|^2+1)^{s/2}$.

6. We now assume (1.51), and apply this inequality to $u = u_\tau$ (which also depends on ξ), which satisfies $\text{supp}(u_\tau) \subset U$ for τ sufficiently large (we can alternatively assume that $\text{supp}(\chi) \subset U$). Question 2 with $s = \frac{m}{2}$ implies $\frac{1}{\tau^m} \text{Re}(Pu_\tau, u_\tau)_{L^2(\mathbb{R}^n)} \to \text{Re}(p_m(0,\xi,1))$ and Question 5 implies $\frac{1}{\tau^m} \|u_\tau\|_{H_\tau^{m/2}}^2 \to (|\xi|^2+1)^{m/2}$. As a consequence, dividing (1.51) by τ^m and letting τ go to infinity implies $\text{Re}(p_m(0,\xi,1)) \geq C_0(|\xi|^2+1)^{m/2}$. Since we can choose any $\xi \in \mathbb{R}^n$ in the definition of u_τ, this yields the sought result.

7. We have $\text{Re}(p_m(0,\xi,1)) \geq C_0(|\xi|^2+1)^{m/2}$ for all $\xi \in \mathbb{R}^n$. Applying this inequality to ξ/τ for $\tau > 0$ instead of ξ, we obtain $\text{Re}(p_m(0,\xi/\tau,1)) \geq C_0(|\xi/\tau|^2+1)^{m/2}$. Multiplying this inequality by τ^m and using homogeneity of degree m of both sides, we finally obtain

$$\text{Re}(p_m(0,\xi,\tau)) = \tau^m \text{Re}(p_m(0,\xi/\tau,1)) \geq \tau^m C_0(|\xi/\tau|^2+1)^{m/2} = C_0(|\xi|^2+\tau^2)^{m/2},$$

for all $(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}_*^+$.

We have proved a converse of the Gårding inequality for *differential* operators: if the operator is positive, then its principal symbol has to be positive (in the proper sense). Note that the constant C_0 is the same in the operator inequality and in the symbolic estimate.

8. We have already computed in Question 5 (the computation was valid for all $m \in \mathbb{R}$)

$$\left((-\Delta+\tau^2)^{-1}u_\tau, u_\tau\right)_{L^2} = \|u_\tau\|_{H_\tau^{-1}}^2 \sim_{\tau\to+\infty} \tau^{-2}(|\xi|^2+1)^{-1}.$$

9. We first consider the case $\alpha = \beta = 0$. Notice that we have

$$\begin{aligned}\|au_\tau - a(0)u_\tau\|_{L^2(\mathbb{R}^n)}^2 &= N(\tau)^2 \int_{\mathbb{R}^n} |a(x)-a(0)|^2 |\chi(\sqrt{\tau}x)|^2 dx \\ &= N(\tau)^2 \tau^{-n/2} \int_{\mathbb{R}^n} |a(y/\sqrt{\tau}) - a(0)|^2 |\chi(y)|^2 dy.\end{aligned}$$

Next recalling that $|a(y/\sqrt{\tau}) - a(0)| \leq C/\sqrt{\tau}$ uniformly for $y \in \text{supp}(\chi)$, we obtain

$$\|au_\tau - a(0)u_\tau\|_{L^2(\mathbb{R}^n)}^2 \leq N(\tau)^2 \tau^{-n/2} \int_{\mathbb{R}^n} \frac{C^2}{\tau} |\chi(y)|^2 dy = \frac{C^2}{\tau}. \tag{1.52}$$

As a consequence, we have

$$\begin{aligned}&\left|\left((-\Delta+\tau^2)^{-1}a(x)u_\tau, b(x)u_\tau\right)_{L^2} - a(0)\bar{b}(0)\left((-\Delta+\tau^2)^{-1}u_\tau, u_\tau\right)_{L^2}\right| \\ &\quad\leq \left|\left((-\Delta+\tau^2)^{-1}a(x)u_\tau, (b(x)-b(0))u_\tau\right)_{L^2}\right| + \left|\left((-\Delta+\tau^2)^{-1}(a(x)-a(0))u_\tau, b(0)u_\tau\right)_{L^2}\right| \\ &\quad\leq \|a(x)u_\tau\|_{H_\tau^{-2}} \|(b(x)-b(0))u_\tau\|_{L^2} + \left|b(0)\left((a(x)-a(0))u_\tau, (-\Delta+\tau^2)^{-1}u_\tau\right)_{L^2}\right| \\ &\quad\leq C\|u_\tau\|_{H_\tau^{-2}} \|(b(x)-b(0))u_\tau\|_{L^2} + |b(0)| \|((a(x)-a(0))u_\tau\|_{L^2} \|u_\tau\|_{H_\tau^{-2}}.\end{aligned}$$

Using (1.52) together with Question 5 (yielding $\|u_\tau\|_{H_\tau^{-2}} \sim C\tau^{-2}$), we obtain that this quantity is $\mathcal{O}(\tau^{-2-1/2})$. Together with the previous question, we have obtained

$$\left((-\Delta+\tau^2)^{-1}a(x)u_\tau, b(x)u_\tau\right)_{L^2} \sim a(0)\bar{b}(0)\tau^{-2}(|\xi|^2+1)^{-1}.$$

To turn to the general case, we recall that we proved in Question 2 that $D^\alpha\left(\chi(\sqrt{\tau}x)e^{i\tau x\cdot\xi}\right) = \chi(\sqrt{\tau}x)\tau^{|\alpha|}\xi^\alpha e^{i\tau x\cdot\xi} + r_\tau(x)$ with $r_\tau(x) = f_\tau(\sqrt{\tau}x)e^{i\tau x\cdot\xi}$ with $|f_\tau(y)| \leq \tau^{|\alpha|-1/2}$ uniformly on $\mathbb{R}^n$. Proceeding as in the case $\alpha = \beta = 0$ to estimate all error terms, we deduce

$$\left((-\Delta+\tau^2)^{-1}a(x)D^\alpha u_\tau, b(x)D^\beta u_\tau\right)_{L^2} \sim \left(a(0)\tau^{|\alpha|}\xi^\alpha\right)\left(\bar{b}(0)\tau^{|\beta|}\xi^\beta\right)\tau^{-2}(|\xi|^2+1)^{-1}.$$

10. In particular, if $|\alpha| = |\beta| = 2$, this reads

$$\frac{1}{\tau^2}\left((-\Delta+\tau^2)^{-1}a(x)D^\alpha u_\tau, b(x)D^\beta u_\tau\right)_{L^2} \to \left(a(0)\xi^\alpha\right)\left(\bar{b}(0)\xi^\beta\right)(|\xi|^2+1)^{-1}.$$

If we take $B, C \in \mathrm{Diff}^2_\tau$, with principal symbols $b_2(x,\xi,\tau)$ and $c_2(x,\tau,\xi)$, this formula (together with similar formulae for mononomials of the type $\tau a(x)D_j$, $\tau^2 a(x)$ and lower order terms) implies

$$\frac{1}{\tau^2}\left((-\Delta+\tau^2)^{-1}Bu_\tau, Cu_\tau\right)_{L^2} \to b_2(0,\xi,1)c_2(0,\xi,1)(|\xi|^2+1)^{-1}.$$

We may now deduce from this asymptotic formula that if $B \in \mathrm{Diff}^2_\tau$ with principal symbol $b_2(x,\xi,\tau)$, we have

$$\frac{1}{\tau^2}\left(B\circ(-\Delta+\tau^2)^{-1}\circ Bu_\tau, u_\tau\right)_{L^2} = \frac{1}{\tau^2}\left((-\Delta+\tau^2)^{-1}Bu_\tau, B^*u_\tau\right)_{L^2}$$
$$\to \left(b_2(0,\xi,1)\right)^2(|\xi|^2+1)^{-1}$$

As a consequence, for operators of the form 1.39 with principal symbol of the form (1.40):

$$p_2(x,\xi,\tau) = a_2(x,\xi,\tau) + \sum_{i=1}^k \frac{b^2_{2,i}(x,\xi,\tau)}{|\xi|^2+\tau^2},$$

we have as for differential operators

$$\frac{1}{\tau^2}(Pu,u)_{L^2} \to p_2(0,\xi,1).$$

To prove a converse of the Gårding inequality for such operators, we may now proceed exactly as in Question 7. Assume there exists a neighborhood U of 0 in $\mathbb{R}^n$ and $C_0, \tau_0 > 0$, such that

$$\mathrm{Re}\,(Pu,u)_{L^2} \geq C_0\,\|u\|^2_{H^1_\tau}, \quad \text{for all } u \in C^\infty_c(U), \tau \geq \tau_0.$$

Applying it to $u = u_\tau$ and letting $\tau \to +\infty$ implies $\mathrm{Re}(p_2(0,\xi,1)) \geq C_0(|\xi|^2+1)$. This holds for all $\xi \in \mathbb{R}^n$. Homogeneity of both sides of degree two in (ξ,τ) implies $\mathrm{Re}(p_2(0,\xi,\tau)) \geq C_0(|\xi|^2+\tau)$ for all $\xi \in \mathbb{R}^n$ and $\tau > 0$. This concludes the proof of the converse of the for operators of the form 1.39, that is, of Proposition 1.3.17.

Note: One proves similarly a converse of the local Gårding inequality of Theorem 1.3.26. The only point to check is that for all $p \in S^m_\tau(\mathbb{R}^n\times\mathbb{R}^n)$, we have $\frac{1}{\tau^m}(p(x,D,\tau)u,u)_{L^2} \to p(0,\xi,1)$, which follows from the stationary phase lemma. ♣ `ref ?`

Exercise 2 (Gårding inequality with limited regularity coefficients)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set containing 0. We consider in this exercise operators of the form

$$A = \sum_{i,j=1}^n a_2^{ij}(x)D_iD_j + \sum_{i=1}^n a_1^i(x)\tau D_i + a_0(x)\tau^2 + \sum_{i=1}^n b_1^i(x)D_i + b_0(x)\tau + c(x),$$

with $a_2^{ij}, a_1^i, a_0 \in W^{1,\infty}(\Omega)$ and $b_1^i, b_0, c \in L^\infty(\Omega)$. We shall say that $A \in \mathrm{Diff}^2_{W^{1,\infty}}$, and write $a_2(x,D,\tau) = \sum_{i,j=1}^n a_2^{ij}(x)D_iD_j + \sum_{i=1}^n a_1^i(x)\tau D_i + a_0(x)\tau^2$ and $a_2(x,\xi,\tau) = \sum_{i,j=1}^n a_2^{ij}(x)\xi_i\xi_j + \sum_{i=1}^n a_1^i(x)\tau\xi_i + a_0(x)\tau^2$ (note however that $A \notin \mathrm{Diff}^2_\tau$, which would require C^∞ regularity of the coefficients).

1. Prove that for $A \in \mathrm{Diff}^2_{W^{1,\infty}}$, there exits $C > 0$ such that for all $r > 0$ such that $B(0,r) \subset \Omega$, we have

$$|(Au, v)_{L^2} - (a_2(0, D, \tau)u, v)_{L^2}| \quad \leq C \left(r \|u\|_{H^1_\tau} + \tau^{-1} \|u\|_{H^1_\tau} \right) \|v\|_{H^1_\tau},$$

for any $u \in C^\infty_c(B(0,r)), v \in \mathcal{S}(\mathbb{R}^n)$, $\tau \geq 1$.

2. Prove that if $f \in W^{1,\infty}(\Omega)$, there is $C_f > 0$ such that for all $u \in C^\infty_c(\Omega)$, $\|fu\|_{H^1_\tau} \leq C_f \|u\|_{H^1_\tau}$ and $\|fu\|_{H^{-1}_\tau} \leq C_f \|u\|_{H^{-1}_\tau}$. Deduce that for $A \in \mathrm{Diff}^2_{W^{1,\infty}}$ having *all* coefficients in $W^{1,\infty}(\Omega)$ (including lower order terms), there is $C_A > 0$ such that $\|Au\|_{H^{-1}_\tau} \leq C_A \|u\|_{H^1_\tau}$ for all $u \in C^\infty_c(\Omega)$.

3. We now consider an operator P of the form

$$P = A + \sum_{i=1}^{k} B_i^* \circ (-\Delta + \tau^2)^{-1} \circ B_i \tag{1.53}$$

(note the slight difference with (1.39) in which there is no B_i^*), with $A, B_i \in \mathrm{Diff}^2_{W^{1,\infty}}$, with *real* "principal symbols" $a_2(x,\xi,\tau)$ and $b_{2,i}(x,\xi,\tau)$. Assume further that B_i have *all* coefficients in $W^{1,\infty}(\Omega)$. Define $p_2(x,\xi,\tau)$ as in (1.40) and and assume that there is $C_0 > 0$ such that (1.41) holds (positive symbol). Prove that there exist $r > 0$ and $C, \tau_0 > 0$ so that we have

$$\mathrm{Re}\,(Pu, u)_{L^2} \geq C \|u\|^2_{H^1_\tau}, \qquad \text{for all } u \in C^\infty_c(B(0,r)), \tau \geq \tau_0.$$

Correction 2. ♣ `To be written (one day...)`

Exercise 3 (warm up, part of the Exam of May, 2019)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Given $\mathcal{P} \in \mathrm{Diff}^m_\tau(\Omega)$, we write $P_R = \frac{\mathcal{P}+\mathcal{P}^*}{2}$ and $P_I = \frac{\mathcal{P}-\mathcal{P}^*}{2i}$. We denote by p_R, resp. p_I, the principal symbol of P_R, resp. P_I.

1. Prove that

$$\|\mathcal{P}u\|^2_{L^2(\Omega)} = \|P_R u\|^2_{L^2(\Omega)} + \|P_I u\|^2_{L^2(\Omega)} + (Mu, u)_{L^2(\Omega)}, \qquad \text{for all } u \in C^\infty_c(\Omega),$$

where M is to be expressed in terms of P_R and P_I. Give the order of M, and its principal symbol in terms of p_R, p_I.

2. Prove that

$$\|\mathcal{P}u\|^2_{L^2(\Omega)} = (Lu, u)_{L^2(\Omega)}, \qquad \text{for all } u \in C^\infty_c(\Omega),$$

where L is to be expressed in terms of P_R and P_I. Give the order of L, and its principal symbol in terms of p_R, p_I.

Correction 3 (Correction of Exercise 3)**.** 1. We have $\mathcal{P} = P_R + iP_I$, where both P_R and P_I are formally selfadjoint. As a consequence, for $u \in C^\infty_c(\Omega)$ we have

$$\begin{aligned}
\|\mathcal{P}u\|^2_{L^2(\Omega)} &= ((P_R + iP_I)u, (P_R + iP_I)u)_{L^2} \\
&= \|P_R u\|^2_{L^2(\Omega)} + \|P_I u\|^2_{L^2(\Omega)} + (iP_I u, P_R u)_{L^2} + (P_R u, iP_I u)_{L^2(\Omega)} \\
&= \|P_R u\|^2_{L^2(\Omega)} + \|P_I u\|^2_{L^2(\Omega)} + (iP_R P_I u, u)_{L^2} + (-iP_I P_R u, u)_{L^2(\Omega)} \\
&= \|P_R u\|^2_{L^2(\Omega)} + \|P_I u\|^2_{L^2(\Omega)} + (i[P_R, P_I]u, u)_{L^2(\Omega)},
\end{aligned}$$

which is the sought formula with $M = i[P_R, P_I] \in \mathrm{Diff}^{m+m-1}_\tau(\Omega) = \mathrm{Diff}^{2m-1}_\tau(\Omega)$, with principal symbol $\{p_R, p_I\}$.

2. Using again selfadjointness of P_R and P_I, we obtain

$$\|\mathcal{P}u\|_{L^2(\Omega)}^2 = \left((P_R^2 + P_I^2 + i[P_R, P_I])u, u\right)_{L^2(\Omega)},$$

that is to say, $L = P_R^2 + P_I^2 + i[P_R, P_I]$. We have $P_R \in \mathrm{Diff}_\tau^m(\Omega)$, $P_I \in \mathrm{Diff}_\tau^m(\Omega)$, hence $P_R^2 \in \mathrm{Diff}_\tau^{2m}(\Omega)$, $P_I^2 \in \mathrm{Diff}_\tau^{2m}(\Omega)$. From preceding question, we know $[P_R, P_I] \in \mathrm{Diff}_\tau^{2m-1}(\Omega)$ so this term is lower order. The principal symbol of L in $\mathrm{Diff}_\tau^{2m}(\Omega)$ is thus $p_R^2 + p_I^2$.

Exercise 4 (Elliptic and subelliptic estimates, part of the Exam of May, 2019)**.** This exercice is *not independent* from Exercise 3. In this exercise, we consider on $(-1,1) \subset \mathbb{R}$ the operator

$$\mathcal{P} = D_x + i\tau V(x) + W(x), \quad V \in C^\infty([-1,1]), \text{ real-valued}, \quad W \in C^\infty([-1,1]),$$

with V, W bounded as well as all of their derivatives.

1. To which class does $\mathcal{P}$ belong? Compute the principal symbol p of $\mathcal{P}$. Compute $\mathrm{Re}(p)$, $\mathrm{Im}(p)$, $\{\bar{p}, p\}$, and $\{\mathrm{Re}(p), \mathrm{Im}(p)\}$.
2. Assume $V(0) \neq 0$. Prove that there are $C, r, \tau_0 > 0$ such that $\|\mathcal{P}u\|_{L^2(-1,1)}^2 \geq C \|u\|_{H_\tau^1}^2$ for all $u \in C_c^\infty(-r, r)$ and $\tau \geq \tau_0$.
3. Assume now that $V(0) = 0$ but $V'(0) > 0$. Prove that there are $C, r, \tau_0 > 0$ such that $\|\mathcal{P}u\|_{L^2(-1,1)}^2 \geq C\tau \|u\|_{L^2}^2$ for all $u \in C_c^\infty(-r, r)$ and $\tau \geq \tau_0$.
4. Assume again that $V(0) = 0$. Let $\chi \in C_c^\infty(-1,1)$ such that $\chi = 1$ in a neighborhood of zero. With $v_\tau(x) = \chi(\sqrt{\tau}x)$, give an equivalent of $\|v_\tau\|_{L^2(-1,1)}^2$ and $\|\mathcal{P}v_\tau\|_{L^2(-1,1)}^2$ as $\tau \to +\infty$. Compare with Question 3.
5. We consider the case $W = 0$, $V(0) = 0$ and $V'(0) < 0$.
   (a) We set $F(x) = \int_0^x V(s)ds$ and $w_\tau(x) = \chi(x)e^{\tau F(x)}$ for $\chi \in C_c^\infty(-1,1)$. Compute $\mathcal{P}w_\tau$, and prove that one can choose χ not identically vanishing so that $\|\mathcal{P}w_\tau\|_{L^2(-1,1)} \leq Ce^{-\tau\delta}$ for some $C, \delta > 0$ and all $\tau \geq 1$.
   (b) Prove a polynomial (in terms of τ) lower bound for $\|w_\tau\|_{L^2(-1,1)}$.
   (c) Discuss the possibility of having subelliptic estimates in this case, that is to say, for $\alpha, s \in \mathbb{R}$, $r, \tau_0, C > 0$, having $\|\mathcal{P}u\|_{L^2(-1,1)}^2 \geq C\tau^\alpha \|u\|_{H_\tau^s}^2$ for all $u \in C_c^\infty(-r, r)$ and $\tau \geq \tau_0$.
6. We now consider the case $W = 0$, $V(0) = 0$ and $V'(0) = 0$. What is the best subelliptic estimate to expect in this situation? One may consider the functions $v_\tau(x) = \chi(\tau^\gamma x)$, for γ to be determined.
7. Explain/discuss, in the case of the operator $\mathcal{P}$, the link between subelliptic estimates and the properties of $\mathrm{Re}(p)$, $\mathrm{Im}(p)$, and $\{\mathrm{Re}(p), \mathrm{Im}(p)\}$.

Correction 4 (Correction of Exercise 4)**.** 1. We have $\mathcal{P} \in \mathrm{Diff}_\tau^1((-1,1))$ with principal symbol $p(x,\xi) = \xi + i\tau V(x)$. As a consequence, we have $\mathrm{Re}(p)(x,\xi) = \xi$, $\mathrm{Im}(p)(x,\xi) = \tau V(x)$ and

$$\begin{aligned}\{\mathrm{Re}(p), \mathrm{Im}(p)\}(x,\xi) &= \partial_\xi \mathrm{Re}(p)\partial_x \mathrm{Im}(p)(x,\xi) - \partial_\xi \mathrm{Re}(p)\partial_x \mathrm{Im}(p)(x,\xi) = \tau V'(x), \\ \{\bar{p}, p\} &= \{\mathrm{Re}(p) - i\,\mathrm{Im}(p), \mathrm{Re}(p) + i\,\mathrm{Im}(p)\} = i\{\mathrm{Re}(p), \mathrm{Im}(p)\} - i\{\mathrm{Im}(p), \mathrm{Re}(p)\} \\ &= 2i\{\mathrm{Re}(p), \mathrm{Im}(p)\} = 2i\tau V'.\end{aligned}$$

Note that this symbol is homogeneous of degree 1 with respect to (ξ, τ), which is consistent with the result of Exercice 3.

2. According to Exercice 3, we have $\|\mathcal{P}u\|^2_{L^2(-1,1)} = (Lu,u)_{L^2(-1,1)}$ with L having for principal symbol

$$\ell(x,\xi) = \mathrm{Re}(p)^2(x,\xi) + \mathrm{Im}(p)^2(x,\xi) = \xi^2 + \tau^2 V^2(x).$$

The assumption $V(0) \neq 0$ implies that $\ell(0,\xi) = \xi^2 + \tau^2 V^2(0) \geq \min\{1, V^2(0)\}(\xi^2+\tau^2)$. The Gårding inequality (for differential operators) implies the existence of $r \in (0,1)$, $C, \tau_0 > 0$ such that $\|\mathcal{P}u\|^2_{L^2(-1,1)} \geq C\|u\|^2_{H^1_\tau}$ for all $u \in C^\infty_c(-r,r)$ and $\tau \geq \tau_0$.

3. According to Exercice 3, we have, for all $u \in C^\infty_c(-1,1)$,

$$\|\mathcal{P}u\|^2_{L^2(-1,1)} = \|P_R u\|^2_{L^2(-1,1)} + \|P_I u\|^2_{L^2(-1,1)} + (Mu,u)_{L^2(-1,1)} \geq (Mu,u)_{L^2(-1,1)},$$

where M has principal symbol $\{\mathrm{Re}(p), \mathrm{Im}(p)\}(x,\xi) = \tau V'(x)$. Since we assume here $V'(0) > 0$, there are $r \in (-1,1)$ and $C > 0$ such that $\|\mathcal{P}u\|^2_{L^2(-1,1)} \geq C\tau\|u\|^2_{L^2}$ for all $u \in C^\infty_c(-r,r)$ and $\tau \geq 1$ (which is stronger than the sought statement).

4. Note first that the function v_τ "concentrates towards the frequency $\xi = 0$ and the point $x = 0$" in the limit $\tau \to +\infty$ (when considering operators in Diff^m_τ). That is, precisely at the point $(x,\xi) = (0,0)$ where we assume $p(x,\xi) = 0$. On the one hand, a change of variables yields $\|v_\tau\|^2_{L^2(-1,1)} = \frac{1}{\sqrt{\tau}}\|\chi\|^2_{L^2(-1,1)}$. On the other hand, we have

$$(\mathcal{P}v_\tau)(x) = (D_x + i\tau V(x) + W(x))\chi(\sqrt{\tau}x) = -i\sqrt{\tau}\chi'(\sqrt{\tau}x) + i\tau V(x)\chi(\sqrt{\tau}x) + W(x)\chi(\sqrt{\tau}x).$$

This implies

$$\begin{aligned}\|\mathcal{P}v_\tau\|^2_{L^2(-1,1)} &= \int_{\mathbb{R}} \left|-i\sqrt{\tau}\chi'(\sqrt{\tau}x) + i\tau V(x)\chi(\sqrt{\tau}x) + W(x)\chi(\sqrt{\tau}x)\right|^2 dx \\ &= \frac{1}{\sqrt{\tau}}\int_{\mathbb{R}} \left|-i\sqrt{\tau}\chi'(y) + i\tau V(\frac{y}{\sqrt{\tau}})\chi(y) + W(\frac{y}{\sqrt{\tau}})\chi(y)\right|^2 dy.\end{aligned}$$

Recalling the assumption $V(0) = 0$, we write $V(s) = sV'(0) + O(s^2)$ uniformly on $\mathrm{supp}(\chi)$. This yields

$$\begin{aligned}\|\mathcal{P}v_\tau\|^2_{L^2(-1,1)} &= \frac{1}{\sqrt{\tau}}\int_{\mathbb{R}} \left|-i\sqrt{\tau}\chi'(y) + i\sqrt{\tau}yV'(0)\chi(y) + O(1)\chi(y)\right|^2 dy \\ &= \frac{1}{\sqrt{\tau}}\int_{\mathbb{R}} \tau\left|\chi'(y) - yV'(0)\chi(y)\right|^2 dy + O\left(\frac{1}{\sqrt{\tau}}\right).\end{aligned}$$

Hence, assuming χ is such that $c_0 := \int_{\mathbb{R}} |\chi'(y) - yV'(0)\chi(y)|^2 dy > 0$ we have $\|\mathcal{P}v_\tau\|^2_{L^2(-1,1)} \sim \frac{1}{\sqrt{\tau}}c_0\tau$. In any case, there are constants $C, \tau_0 > 0$ such that we have $\|\mathcal{P}v_\tau\|^2_{L^2(-1,1)} \leq C\frac{1}{\sqrt{\tau}}\tau$ for $\tau \geq \tau_0$.

Recalling the norm of v_τ, we have obtained that $\|\mathcal{P}v_\tau\|^2_{L^2(-1,1)} \sim c_1\tau\|v_\tau\|^2_{L^2(-1,1)}$ with $c_1 = c_0\|\chi\|^{-2}_{L^2(-1,1)}$ in the first case and $\|\mathcal{P}v_\tau\|^2_{L^2(-1,1)} \leq C\tau\|v_\tau\|^2_{L^2(-1,1)}$ in any case. This proves that as soon as $V(0) = 0$, one cannot hope to obtain a greater power of τ than that obtained in Question 3.

5. (a) Since $W = 0$, we have

$$\mathcal{P}w_\tau(x) = \left(-i\chi'(x) - i\tau F'(x)\chi(x) + i\tau V(x)\chi(x)\right)e^{\tau F(x)} = -i\chi'(x)e^{\tau F(x)},$$

where we have used $F' = V$. Writing again the Taylor expansion of V at zero and using $V(0) = 0$, we obtain $F(x) = \int_0^x V(s)ds = \int_0^x sV'(0) + O(s^2)ds = V'(0)\frac{x^2}{2} + O(x^3)$, uniformly on $[-1,1]$. We now set $d_0 := -V'(0) > 0$ by assumption. There exists $r \in (0,1)$ such that $F(x) \leq -\frac{d_0 x^2}{4}$ for all $x \in (-r,r)$. We write

$$\|\mathcal{P}w_\tau\|^2_{L^2(-1,1)} = \left\|\chi' e^{\tau F}\right\|^2_{L^2(-1,1)} = \int_{-1}^1 |\chi'(x)|^2 e^{2\tau F(x)} dx.$$

Assuming that $\mathrm{supp}(\chi) \subset (-r, r)$, we thus deduce

$$\|\mathcal{P}w_\tau\|^2_{L^2(-1,1)} \leq \int_{-1}^{1} |\chi'(x)|^2 e^{-\tau \frac{d_0 x^2}{2}} dx.$$

Now, if we choose further that $\chi = 1$ on $(-r/2, r/2)$, we have $\mathrm{supp}(\chi') \subset (-r, r) \setminus (-r/2, r/2)$ and hence $\frac{x^2 d_0}{2} \geq \frac{d_0 r^2}{4}$ on $\mathrm{supp}(\chi')$. This finally implies $\|\mathcal{P}w_\tau\|^2_{L^2(-1,1)} \leq \|\chi'\|_{L^2(-1,1)} e^{-\tau \frac{d_0 r^2}{4}}$, with $d_0 = -V'(0) > 0$, which is the sought result.

(b) Since $F(x) = \int_0^x V(s)ds = \int_0^x sV'(0) + O(s^2)ds = -d_0 \frac{x^2}{2} + O(x^3)$, we can also assume (up to reducing r) that $F(x) \geq -d_0 \frac{x^2}{4}$ on $(-r, r)$. We then have

$$\|w_\tau\|^2_{L^2(-1,1)} \geq \int_{-1}^{1} |\chi(x)|^2 e^{-\tau \frac{d_0 x^2}{2}} dx \geq \int_{-r/2}^{r/2} e^{-\tau \frac{d_0 x^2}{2}} dx = \int_{\mathbb{R}} e^{-\tau \frac{d_0 x^2}{2}} dx - \int_{|x| \geq r/2} e^{-\tau \frac{d_0 x^2}{2}} dx.$$

Now, recalling that $\int_{\mathbb{R}} e^{-\tau \frac{d_0 x^2}{2}} dx = \sqrt{\frac{2\pi}{d_0 \tau}}$ and

$$\begin{aligned} \int_{-r/2}^{r/2} e^{-\tau \frac{d_0 x^2}{2}} dx = 2 \int_{r/2}^{\infty} e^{-\tau \frac{d_0 x^2}{2}} dx \leq 2 \int_{r/2}^{\infty} e^{-\tau \frac{d_0 (r/2)^2}{4}} e^{-\tau \frac{d_0 x^2}{4}} dx \\ \leq e^{-\tau \frac{d_0 r^2}{16}} \int_{\mathbb{R}} e^{-\tau \frac{d_0 x^2}{2}} dx = \sqrt{\frac{2\pi}{d_0 \tau}} e^{-\tau \frac{d_0 r^2}{16}}. \end{aligned}$$

Combining the above three identities, we have obtained $\|w_\tau\|^2_{L^2(-1,1)} \geq \frac{1}{2}\sqrt{\frac{2\pi}{d_0 \tau}}$ for τ sufficiently large.

(c) In the present setting, we have constructed for all $r > 0$ a $\tau-$depending family of functions $w_\tau \in C_c^\infty(-r, r)$ with the following properties: there are $C, \delta, \tau_0 > 0$ (depending on r) such that for all $\tau \geq \tau_0$

$$\|\mathcal{P}w_\tau\|^2_{L^2(-1,1)} \leq C e^{-\delta \tau}, \quad \|w_\tau\|^2_{L^2(-1,1)} \geq \frac{C}{\sqrt{\tau}}.$$

In particular, this prevents the possibility of any subelliptic estimate to be true: applied to w_τ, it would yield $\frac{C}{\sqrt{\tau}} \tau^\alpha \leq C e^{-\delta \tau}$ for all τ sufficiently large.

6. We proceed as in Question 4, except that the appropriate scale of concentration is to be determined. On the one hand, we have $\|v_\tau\|^2_{L^2(-1,1)} = \frac{1}{\tau^\gamma} \|\chi\|^2_{L^2(-1,1)}$. On the other hand, we have

$$(\mathcal{P}v_\tau)(x) = (D_x + i\tau V(x))\chi(\tau^\gamma x) = -i\tau^\gamma \chi'(\tau^\gamma x) + i\tau V(x)\chi(\tau^\gamma x).$$

This implies

$$\|\mathcal{P}v_\tau\|^2_{L^2(-1,1)} = \int_{\mathbb{R}} |\tau^\gamma \chi'(\tau^\gamma x) - \tau V(x)\chi(\tau^\gamma x)|^2 dx = \frac{1}{\tau^\gamma} \int_{\mathbb{R}} \left|\tau^\gamma \chi'(y) - \tau V(\frac{y}{\tau^\gamma})\chi(y)\right|^2 dy.$$

Recalling the assumption $V(0) = V'(0) = 0$, we write $V(s) = O(s^2)$ uniformly on $\mathrm{supp}(\chi)$. This implies the rough estimate

$$\begin{aligned} \|\mathcal{P}v_\tau\|^2_{L^2(-1,1)} &\leq \frac{1}{\tau^\gamma} \int_{\mathbb{R}} 2\tau^{2\gamma} |\chi'(y)|^2 + 2 \left(\tau C \left|\frac{y}{\tau^\gamma}\right|^2 \chi(y)\right)^2 dy \\ &\leq \frac{C}{\tau^\gamma} \left(\tau^{2\gamma} + \tau^{2-4\gamma}\right) \leq C \left(\tau^{2\gamma} + \tau^{2-4\gamma}\right) \|v_\tau\|^2_{L^2(-1,1)}. \end{aligned}$$

The growth of $\tau^{2\gamma} + \tau^{2-4\gamma}$ is minimal when $2\gamma = 2 - 4\gamma$, that is $\gamma = 1/3$.

Hence, in this case, one cannot hope to have a better estimate than $\|\mathcal{P}u\|^2_{L^2(-1,1)} \geq C\tau^{2/3} \|u\|^2_{L^2(-1,1)}$ if $V(0) = V'(0) = 0$.

7. Here, we interpret the results of the above questions in a more intrinsic way. The operator $\mathcal{P}$ is in $\text{Diff}^1_\tau(-1,1)$, with nontrivial real and imaginary parts (in particular, its principal symbol $p(x,\xi) = \xi + i\tau V(x)$ has nontrivial real and imaginary parts). We have proved:

   - If $\text{Im}(p)(0,0) \neq 0$ (but $\text{Re}(p)(0,0) = 0$), then $\|\mathcal{P}u\|_{L^2(-1,1)} \geq C\|u\|_{H^1_\tau} \geq \tau\|u\|_{L^2}$. This is an elliptic estimate: an operator of order one dominates an H^1_τ norm.
   - If $\text{Re}(p)(0,0) = \text{Im}(p)(0,0) = 0$ but $\{\text{Re}(p), \text{Im}(p)\}(0,0) > 0$, then $\|\mathcal{P}u\|_{L^2(-1,1)} \geq C\sqrt{\tau}\|u\|_{L^2}$ (this is Question 3). This is a subelliptic estimate with loss of half a derivative: an operator of order one dominates essentially the $H^{1/2}_\tau$ norm. Moreover, this estimate is optimal, as shown by Question 4.
   - If $\text{Re}(p)(0,0) = \text{Im}(p)(0,0) = 0$ and $\{\text{Re}(p), \text{Im}(p)\}(0,0) < 0$, then there is no hope of obtaining any subelliptic estimate (this is Question 5).
   - In the (very degenerate) case: $\text{Re}(p)(0,0) = \text{Im}(p)(0,0) = 0$ and $\{\text{Re}(p), \text{Im}(p)\}(0,0) = 0$, the best subelliptic estimate one can expect is $\|\mathcal{P}u\|_{L^2(-1,1)} \geq C\tau^{1/3}\|u\|_{L^2}$ (this is Question 6). Such an estimate would be called a subelliptic estimate with loss of $2/3$ derivatives: an operator of order one dominates essentially the $H^{1/3}_\tau$ norm.

Chapter 2

Classical estimates and applications

This chapter is devoted to classical unique continuation results under a pseudoconvexity condition. These results are stated here in the particular situation of second order operators with real principal symbol (in which case the statement has a simpler geometric interpretation).

Quantitative estimates are derived in the case of elliptic operators, and applications are given to eigenfunctions of Laplace-Beltrami operators, and observability properties for the heat equation.

As already mentioned, these results rely on a Carleman estimate, which we now state.

2.1 The Carleman estimate

Here, we recall that, $P \in \mathrm{Diff}^2(\Omega)$ (with principal symbol $p_2(x,\xi)$) and $\Phi \in C^\infty(\Omega;\mathbb{R})$ being given, the conjugated operator is $P_\Phi = e^{\tau\Phi} P e^{-\tau\Phi} \in \mathrm{Diff}^2_\tau(\Omega)$ and its principal symbol is $p_\Phi(x,\xi,\tau) = p_2(x,\xi+id\Phi(x))$ (computed in Lemma 1.3.10 and Example 1.3.12). We write $p_\Phi = \mathrm{Re}(p_\Phi) + i\,\mathrm{Im}(p_\Phi)$.

2.1.1 Carleman estimate under subellipticity condition

As we have seen in the Introduction in Section 1.2.5, our goal is to obtain estimates of the type of (1.27). In this section, we prove a Carleman estimate (Theorem 2.1.1) under a symbolic condition usually called "Hörmander subellipticity condition" (namely (2.3)). Yet, this assumptions might seem not so natural at first sight. The next sections link this condition to the geometry of the operator.

Theorem 2.1.1 (Local Carleman estimate)**.** *Let Ω be an open subset of $\mathbb{R}^n$ and $x_0 \in \Omega$. Let $P \in \mathrm{Diff}^2(\Omega)$ be a (classical) differential operator with* real-valued principal symbol *p_2 and $\Phi \in C^\infty(\overline{\Omega};\mathbb{R})$.*

Then, the following statements are equivalent:

1. *There exist $C, r, \tau_0 > 0$ so that we have the following estimate*

$$\tau^3 \left\|e^{\tau\Phi}u\right\|_{L^2}^2 + \tau \left\|e^{\tau\Phi}\nabla u\right\|_{L^2}^2 \le C \left\|e^{\tau\Phi}Pu\right\|_{L^2}^2, \quad \text{for all } u \in C^\infty_c(B(x_0,r)), \tau \ge \tau_0; \tag{2.1}$$

2. *There exist $C, r, \tau_0 > 0$ so that we have the following estimate*

$$\tau \|v\|_{H^1_\tau}^2 \le C \|P_\Phi v\|_{L^2}^2, \quad \text{for all } v \in C^\infty_c(B(x_0,r)), \tau \ge \tau_0; \tag{2.2}$$

3. *There exist $C_1, C_2 > 0$ such that for all $(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}^*_+$,*

$$\frac{C_1}{|\xi|^2+\tau^2}\left[(\mathrm{Re}\, p_\Phi)^2 + (\mathrm{Im}\, p_\Phi)^2\right] + \frac{1}{\tau}\left\{\mathrm{Re}\, p_\Phi, \mathrm{Im}\, p_\Phi\right\} \ge C_2\left(|\xi|^2+\tau^2\right), \tag{2.3}$$

   where the symbols are taken at the point (x_0,ξ,τ).

Notice that $\frac{\mathrm{Im}\, p_\Phi}{\tau} = 2\widetilde{p_2}(x,\xi,d\Phi(x))$ (see Example 1.3.12) is smooth, so this is not a problem to divide by τ in (2.3), even when $\tau \to 0$.

Before proceeding to the proof of this result, several comments are in order. First, the statement (2.1) is that useful for applications (to unique continuation in particular). The statement (2.2) is only a reformulation in terms of the conjugated operator, which belongs to Diff^2_τ, and is thus analyzable with the tools developed in Section 1.3. The statement (2.3), as opposed to the previous ones, is concerned with a "symbolic estimate", concerning only the principal symbol of the conjugated operator. The interest of this result is that it reduces the problem of proving a Carleman estimate to a checkable property on the principal symbol of the conjugated operator. The question of rephrasing the condition (2.3) in geometric terms is addressed in Sections 2.1.2 and 2.2 below. The useful information in this theorem is (2.3) $\implies$ (2.1). However, that the converse is true indicates the limit of this classical Carleman approach. This point is slightly more technical and we omit the proof and refer the reader to [Hör94, Section 28.2].

Exercise 3 in Section 1.4 presents the key computation in the proof of Carleman estimates.

Proof. The equivalence between (2.1) and (2.2) comes from the change of unknown $v = e^{\tau\Phi}u$. This yields $P_\Phi v = e^{\tau\Phi}Pe^{-\tau\Phi}v = e^{\tau\Phi}Pu$. Moreover, we have $\nabla u = \nabla(e^{-\tau\Phi}v) = e^{-\tau\Phi}(\nabla v - \tau v\nabla\Phi)$ so that

$$\tau^2 \left\|e^{\tau\Phi}u\right\|_{L^2}^2 + \left\|e^{\tau\Phi}\nabla u\right\|_{L^2}^2 \leq \tau^2 \|v\|_{L^2}^2 + 2\|\nabla v\|_{L^2}^2 + 2\|\tau v\nabla\Phi\|_{L^2}^2 \leq C\|v\|_{H^1_\tau}^2,$$

and thus (2.2) implies (2.1). Conversely, we have $\nabla v = \nabla(e^{\tau\Phi}u) = e^{\tau\Phi}(\nabla u + \tau u\nabla\Phi)$ so that

$$\begin{aligned}\|v\|_{H^1_\tau}^2 = \|\nabla v\|_{L^2}^2 + \tau^2\|v\|_{L^2}^2 &\leq 2\left\|e^{\tau\Phi}\nabla u\right\|_{L^2}^2 + 2\left\|e^{\tau\Phi}\tau u\nabla\Phi\right\|_{L^2}^2 + \tau^2\left\|e^{\tau\Phi}u\right\|_{L^2}^2 \\ &\leq C\left(\tau^2\left\|e^{\tau\Phi}u\right\|_{L^2}^2 + \left\|e^{\tau\Phi}\nabla u\right\|_{L^2}^2\right),\end{aligned}$$

and (2.1) implies (2.2).

We now want to prove that (2.3) implies (2.2). Before going further, let us notice that Lemma 1.3.10 only depends on the leading order of the operator P. More precisely, if $\tilde{P} \in \mathrm{Diff}^2(\Omega)$ has the same principal symbol as P, then $P - \tilde{P} \in \mathrm{Diff}^1(\Omega)$ and $P_\Phi - \tilde{P}_\Phi \in \mathrm{Diff}^1_\tau(\Omega)$, i.e.

$$P_\Phi = \tilde{P}_\Phi + R, \quad \text{with} \quad R \in \mathrm{Diff}^1_\tau.$$

Henceforth, assuming the Carleman inequality (2.2) for $\tilde{P}_\Phi$,

$$\tau\|v\|_{H^1_\tau}^2 \leq C\left\|\tilde{P}_\Phi v\right\|_{L^2}^2 \tag{2.4}$$

yields

$$\tau\|v\|_{H^1_\tau}^2 \leq C\|(P_\Phi - R)v\|_{L^2}^2 \leq C\|P_\Phi v\|_{L^2}^2 + C\|Rv\|_{L^2}^2 \leq C\|P_\Phi v\|_{L^2}^2 + D\|v\|_{H^1_\tau}^2$$

with Proposition 1.3.5. Then, for τ large enough, we have $\tau - D \geq \tau/2$, and the last term can then be absorbed in the left hand-side, yielding the sought Carleman inequality (2.2) for P_Φ, with different constants C and τ_0.

Since the operator P has real principal symbol p_2, we shall choose $\tilde{P} = \frac{P+P^*}{2}$, which is selfadjoint and has the same principal symbol p_2. Note that if $P = \sum_{i,j=1}^n a^{ij}(x)D_iD_j$ with $a^{ij} = a^{ji}$ real-valued (which is, modulo $\mathrm{Diff}^1(\Omega)$, the general expression for $P \in \mathrm{Diff}^2(\Omega)$ with real-valued principal symbol, see Example 1.38), then we have $\tilde{P} = \sum_{i,j=1}^n D_ia^{ij}(x)D_j$ modulo a (selfadjoint) first order operator.

We may thus focus on $\tilde{P}_\Phi = e^{\tau\Phi}\tilde{P}e^{-\tau\Phi}$ and prove (2.4). To this aim, we decompose the operator $\tilde{P}_\Phi$ as

$$\tilde{P}_\Phi = Q_R + iQ_I, \tag{2.5}$$

with

$$Q_R = \frac{\tilde{P}_\Phi + \tilde{P}_\Phi^*}{2}; \quad Q_I = \frac{\tilde{P}_\Phi - \tilde{P}_\Phi^*}{2i}.$$

Note that both Q_R and Q_I are formally selfadjoint ($Q_R^* = Q_R$ and $Q_I^* = Q_I$), and, according to Proposition 1.3.9, we have $Q_R, Q_I \in \mathrm{Diff}^2_\tau$ with principal symbols (see Example 1.3.12)

$$\begin{aligned}q_R(x,\xi,\tau) &= \frac{p_\Phi + \overline{p_\Phi}}{2}(x,\xi,\tau) = \operatorname{Re} p_\Phi(x,\xi,\tau) = p_2(x,\xi) - \tau^2 p_2(x, d\Phi(x)), \\ q_I(x,\xi,\tau) &= \frac{p_\Phi - \overline{p_\Phi}}{2i}(x,\xi,\tau) = \operatorname{Im} p_\Phi(x,\xi,\tau) = 2\tau\widetilde{p_2}(x,\xi,d\Phi(x)).\end{aligned}$$

Moreover (this is a key point), the operator $\tilde{P}_\Phi$ is a second order polynomial in (D, τ), such that $\tilde{P}_\Phi = \tilde{P}$ when $\tau = 0$. If $\tilde{P} = \sum_{i,j=1}^n D_i a^{ij}(x) D_j$, then $\tilde{P}_\Phi = \sum_{i,j=1}^n (D_i + i\tau\partial_i\Phi)a^{ij}(x)(D_j + i\tau\partial_j\Phi)$. This implies that $\tilde{P}_\Phi = \tilde{P} + \tau M$ for some $M \in \mathrm{Diff}^1_\tau(\Omega)$, and, since $\tilde{P}$ is chosen to be selfadjoint, this implies that

$$Q_I = \frac{\tilde{P}_\Phi - \tilde{P}_\Phi^*}{2i} = \frac{\tau M - \tau M^*}{2i} = \tau \tilde{Q}_I, \quad \text{with} \quad \tilde{Q}_I = \frac{M - M^*}{2i} \in \mathrm{Diff}^1_\tau(\Omega), \tag{2.6}$$

i.e. τ may be factorized in the skewadjoint part of the operator.

Using (2.5), the central computation is now as follows, for $v \in C^\infty_c(\Omega)$,

$$\begin{aligned}\left\|\tilde{P}_\Phi v\right\|_{L^2}^2 &= \left(\tilde{P}_\Phi v, \tilde{P}_\Phi v\right)_{L^2} = ((Q_R + iQ_I)v, (Q_R + iQ_I)v) \\ &= (Q_R v, Q_R v) + (iQ_I v, iQ_I v) + (Q_R v, iQ_I v) + (iQ_I v, Q_R v) \\ &= \|Q_R v\|_{L^2}^2 + \|Q_I v\|_{L^2}^2 - i\,(Q_I \circ Q_R v, v) + i\,(Q_R \circ Q_I v, v) \\ &= \|Q_R v\|_{L^2}^2 + \|Q_I v\|_{L^2}^2 + (i[Q_R, Q_I]v, v)\,. \end{aligned} \tag{2.7}$$

Now, we have 2 kinds of terms

- the one with $\|Q_R v\|_{L^2}^2$ (and resp. $\|Q_I v\|_{L^2}^2$) that corresponds to $\left(Q_R^2 v, v\right)$ where Q_R^2 is of order 4 with principal symbol $(\mathrm{Re}\, p_\Phi)^2$ (resp. $(\mathrm{Im}\, p_\Phi)^2$);
- the one with $i[Q_R, Q_I]$ which is of order $2 + 2 - 1 = 3$ and principal symbol $\{\mathrm{Re}\, p_\Phi, \mathrm{Im}\, p_\Phi\}$ by Proposition 1.3.8.

The first two operators have stronger order (4) but they can cancel and are therefore not sufficient to obtain the "coercivity" estimate. The idea is thus to use the commutator where both q_R and q_I cancel. However, to compare these terms, we need to bring them to the same order and "sacrifice" this main order 4. More precisely, let $C_1 > 0$ be as in Assumption (2.3) (that this is the right constant will appear in (2.9) below). For $\tau \geq C_1$, we have

$$\frac{1}{\tau^{1/2}} \geq \frac{C_1^{1/2}}{\tau} \geq \frac{C_1^{1/2}}{(|\xi|^2 + \tau^2)^{1/2}} \quad \text{for all } \xi \in \mathbb{R}^n.$$

This implies (using again the Plancherel Theorem)

$$\begin{aligned}\frac{1}{\tau}\|Q_R v\|_{L^2}^2 = \left\|\frac{Q_R v}{\tau^{1/2}}\right\|_{L^2}^2 &\geq \left\|C_1^{1/2}(-\Delta + \tau^2)^{-1/2} Q_R v\right\|_{L^2}^2 \\ &\geq C_1 \left((-\Delta + \tau^2)^{-1/2} Q_R v, (-\Delta + \tau^2)^{-1/2} Q_R v\right) \\ &\geq C_1 \left(Q_R(-\Delta + \tau^2)^{-1} Q_R v, v\right). \end{aligned} \tag{2.8}$$

The same estimate applies to Q_I. Combining (2.7) with (2.8), we have now proved

$$\frac{1}{\tau}\left\|\tilde{P}_\Phi v\right\|_{L^2}^2 \geq (Lv, v)_{L^2}\,, \tag{2.9}$$

with

$$L = C_1\Big(Q_R(-\Delta + \tau^2)^{-1}Q_R + Q_I(-\Delta + \tau^2)^{-1}Q_I\Big) + i\left[Q_R, \frac{Q_I}{\tau}\right].$$

But we have proved in (2.6) that $Q_I = \tau\tilde{Q}_I$ with $\tilde{Q}_I \in \mathrm{Diff}^1_\tau(\Omega)$. This implies that $\left[Q_R, \frac{Q_I}{\tau}\right] \in \mathrm{Diff}^2_\tau$ as well. The operator L is thus precisely of the form of that in Proposition 1.3.14, is moreover selfadjoint, and has principal symbol (in the sense of Proposition 1.3.14)

$$\frac{C_1}{|\xi|^2 + \tau^2}\left((\mathrm{Re}\, p_\Phi)^2 + (\mathrm{Im}\, p_\Phi)^2\right) + \left\{\mathrm{Re}\, p_\Phi, \frac{\mathrm{Im}\, p_\Phi}{\tau}\right\},$$

which satisfies (2.3). Hence, the Gårding inequality of Proposition 1.3.14 applies and yields the existence of $C, \tau_0, r > 0$ such that

$$(Lv, v)_{L^2} \geq C \|v\|_{H^1_\tau}^2, \quad \text{for all } v \in C_c^\infty(B(x_0, r)), \quad \tau \geq \tau_0,$$

which, in view of (2.9), yields (2.4) and concludes the proof of the Carleman estimate (2.2).

□

Note that in (2.8), since Q_R is only defined on Ω, and since $(-\Delta+\tau^2)^{-1}Q_R v \notin C_c^\infty(\Omega)$, the expression $Q_R(-\Delta+\tau^2)^{-1}Q_R v$ is not well-defined. However, its pairing with the function $v \in C_c^\infty(B(x_0, r))$ is well defined (e.g. as $\big(\chi Q_R \tilde{\chi}(-\Delta+\tau^2)^{-1}Q_R v, v\big)$ with $\chi \in C^\infty(\Omega)$ such that $\chi = 1$ on a neighborhood of $B(x_0, r)$, and $\tilde{\chi} \in C^\infty(\Omega)$ with $\tilde{\chi} = 1$ on a neighborhood of $\operatorname{supp}(\chi)$).

Remark 2.1.2 (Lower order terms)**.** As seen in the proof, an important feature of the Carleman estimates (2.1) is its insensitivity with respect to lower order terms. More precisely, if (2.1) is satisfied for an operator P, then it also holds for $P' := P + \sum_{k=1}^n b_k(x)D_k + c(x)$ as soon as $b_k, c \in L^\infty(\Omega)$. Indeed, applying (2.1) for P yields

$$\begin{aligned} \tau^3 \left\|e^{\tau\Phi}u\right\|_{L^2}^2 + \tau \left\|e^{\tau\Phi}\nabla u\right\|_{L^2}^2 &\leq C \Big\|e^{\tau\Phi}\Big(P' - \sum_{k=1}^n b_k(x)D_k + c(x)\Big)u\Big\|_{L^2}^2 \\ &\leq C \left\|e^{\tau\Phi}P'\right\|_{L^2}^2 + C \left\|e^{\tau\Phi}\nabla u\right\|_{L^2}^2 + C \left\|e^{\tau\Phi}u\right\|_{L^2}^2, \end{aligned}$$

and the last two terms can be absorbed in the left handside for τ large enough. Note in particular that no regularity is required on the lower order terms when proceeding that way.

Remark 2.1.3. The quantitative subelliptic condition (2.3) can actually be replaced by the qualitative assumption (writing $K = \{(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}^+, |\xi|^2 + \tau^2 = 1\}$)

$$p_\Phi(x_0, \xi, \tau) = 0, \quad \Longrightarrow \left\{\operatorname{Re} p_\Phi, \frac{\operatorname{Im} p_\Phi}{\tau}\right\}(x_0, \xi, \tau) > 0 \quad \text{for all } (\xi, \tau) \in K.$$

This is proved using Lemma 2.1.8 below, as in the proof of Proposition 2.1.7. This will however not be used here.

Remark 2.1.4 (Estimate with loss of half a derivative)**.** Estimates like the Carleman estimate (2.2) are often called subelliptic estimates. Indeed, if the operator P_Φ were elliptic in the (ξ, τ) variables, we would have an estimate from below with the norm H^2_τ instead of H^1_τ. But here the principal symbol of P_Φ, namely $p_2(x, \xi + i\tau d\Phi)$ may vanish (with $(\xi, \tau) \neq 0$), even if p_2 is elliptic.

Take for instance the Laplace operator described in Example 1.3.11. The principal symbol of p_Φ is $|\xi|^2 - \tau^2|\nabla\Phi|^2 + 2i\tau\xi \cdot \nabla\Phi$. It cancels if we take $\xi \perp \nabla\Phi$ and $\tau^2 = |\xi|^2/|\nabla\Phi|^2$, which is always possible if $n \geq 2$. This actually happens on a conic set.

Note that it can seem surprising since for fixed τ, the operator P_Φ is elliptic in the ξ variable. We could expect an inequality of the form

$$\|u\|_{H^2_\tau} \leq C_\tau \|P_\Phi u\|_{L^2}.$$

It is indeed possible if P is elliptic, but the constant C_τ will then blow-up as $\tau^{1/2}$. This expresses a loss of "half a derivative" w.r.t. elliptic estimates. It has the same homogeneity as the H^1_τ estimate in the general case.

2.1.2 Carleman estimate for pseudoconvex functions

We now reduce the quantitative symbolic Assumption (2.3) of the Carleman estimate to a qualitative convexity condition on the weight function Φ (with respect to the symbol p_2).

Definition 2.1.5 (Pseudoconvexity for functions)**.** Let $\Omega \ni x_0$ be an open set, $P \in \operatorname{Diff}^2(\Omega)$ be a (classical) differential operator with real-valued principal symbol p_2 and $\Phi \in C^\infty(\Omega)$ real-valued.

We say that the function Φ is pseudoconvex with respect to P at x_0 if it satisfies

$$\{p_2, \{p_2, \Phi\}\}(x_0, \xi) > 0, \qquad \text{if } p_2(x_0, \xi) = 0 \text{ and } \xi \neq 0; \tag{2.10}$$

$$\frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\}(x_0, \xi, \tau) > 0, \qquad \text{if } p_\Phi(x_0, \xi, \tau) = 0 \text{ and } \tau > 0, \tag{2.11}$$

where $p_\Phi(x, \xi, \tau) = p_2(x, \xi + i\tau d\Phi(x))$.

Note that in some sense, we could say that for real operators, the first line is the limit of the second line as τ tends to 0, as shows Lemma 2.1.6:

Lemma 2.1.6. *Let p be a real-valued smooth function on $\Omega \times \mathbb{R}^n$. Then, we have $\lim_{\tau \to 0} \frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\}(x, \xi, \tau) = 2\{p\{p, \Phi\}\}(x, \xi)$ for all $(x, \xi) \in \Omega \times \mathbb{R}^n$.*

We now state the equivalence between Definition 2.1.5 and the Hörmander subellipticity condition (2.3) (itself equivalent to the Carleman estimate (2.1)).

Proposition 2.1.7. *Let $\Omega \ni x_0$ be an open set, $P \in \mathrm{Diff}^2(\Omega)$ with real-valued principal symbol p_2 and $\Phi \in C^\infty$ real-valued. If Φ is pseudoconvex with respect to P at x_0, then the subellipticity condition (2.3) is satisfied at x_0.*

And hence, if Φ is a pseudoconvex function in the sense of Definition 2.1.5, the Carleman estimate of Theorem 2.1.1 holds with weight Φ.

The proof uses the following (elementary but very useful) lemma (in the simpler case $h = 0$; the general case will be used later on).

Lemma 2.1.8. *Let K be a compact topological space and f, g, h three continuous real-valued functions on K. Assume that $f \geq 0$ on K, and $g > 0$ on $\{f = 0\}$. Then, there exists $A_0, C > 0$ such that for all $A \geq A_0$, we have $g + Af - \frac{1}{A}h \geq C$ on K.*

We prove the proposition from the two lemmata and then prove the lemmata.

Proof of Proposition 2.1.7. Note first that since $\{f, f\} = 0$ and $\{f, g\} = -\{g, f\}$ for any f and g, we have

$$\begin{aligned}
\frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\} &= \frac{1}{i\tau}\{\operatorname{Re} p_\Phi - i \operatorname{Im} p_\Phi, \operatorname{Re} p_\Phi + i \operatorname{Im} p_\Phi\} \\
&= \frac{1}{\tau}\{\operatorname{Re} p_\Phi, \operatorname{Im} p_\Phi\} - \frac{1}{\tau}\{\operatorname{Im} p_\Phi, \operatorname{Re} p_\Phi\} \\
&= \frac{2}{\tau}\{\operatorname{Re} p_\Phi, \operatorname{Im} p_\Phi\}.
\end{aligned}$$

Moreover, we recall that $\frac{\operatorname{Im} p_\Phi}{\tau} = 2\tilde{p}_2(x_0, \xi, d\Phi(x_0))$ is smooth (note that this could also be seen as a consequence of the fact that p is real and $p_\Phi = p$ on the set $\{\tau = 0\}$, so we can factorize $\operatorname{Im} p_\Phi$ by Taylor expansion).

We notice that all terms in (2.3) are homogeneous in (ξ, τ) of order 2 and continuous thanks to the previous remark. Therefore, it is enough to prove (2.3) on the set $K = \{(\xi, \tau), |\xi|^2 + \tau^2 = 1; \tau \geq 0\}$. On this compact set, the result is a consequence of Lemma 2.1.8 with $f = (\operatorname{Re} p_\Phi)^2 + (\operatorname{Im} p_\Phi)^2$, $g = 2\{\operatorname{Re} p_\Phi, \frac{\operatorname{Im} p_\Phi}{\tau}\}$ and $h = 0$ (the function h will be useful for another application in the next chapter).

Lemma 2.1.6 then proves that the first assumption in Definition 2.1.5 is the limit of the second one on the set $\{\tau = 0\}$. Hence, we have $g > 0$ on $\{f = 0\}$ on the whole K, up to the set $\{\tau = 0\} \cap \{|\xi|^2 + \tau^2 = 1\}$.

Lemma 2.1.8 then concludes the proof of the subellipticity condition (2.3). □

Proof of Lemma 2.1.6. We first notice that for $\tau = 0$, $\{\overline{p_\Phi}, p_\Phi\} = \{\overline{p}, p\}$ so since p is real, $\{\overline{p_\Phi}, p_\Phi\} = 0$ for $\tau = 0$. The definition of the derivative in $\tau = 0$ then yields

$$\lim_{\tau \to 0} \frac{1}{\tau}\{\overline{p_\Phi}, p_\Phi\} = \frac{\partial}{\partial\tau}\{\overline{p_\Phi}, p_\Phi\}\Big|_{\tau=0}. \tag{2.12}$$

Also, we have $\partial_\tau(\{\overline{p_\Phi}, p_\Phi\}) = \{\partial_\tau \overline{p_\Phi}, p_\Phi\} + \{\overline{p_\Phi}, \partial_\tau p_\Phi\}$.

But since p is real, $\overline{p_\Phi} = p(x, \xi - i\tau d\Phi(x))$, so that

$$\partial_\tau p_\Phi(x,\xi,\tau) = id\Phi \cdot \partial_\xi p(x, \xi + i\tau d\Phi) = i\{p_\Phi, \Phi\}(x,\xi,\tau)$$
$$\partial_\tau \overline{p_\Phi} = -id\Phi \cdot \partial_\xi p(x, \xi - i\tau d\Phi) = -i\{\overline{p_\Phi}, \Phi\}(x,\xi,\tau).$$

So, we get $\partial_\tau(\{\overline{p_\Phi}, p_\Phi\}) = -i\{\{\overline{p_\Phi}, \Phi\}, p_\Phi\} + i\{\overline{p_\Phi}, \{p_\Phi, \Phi\}\}$. When specified for $\tau = 0$, we obtain

$$\frac{\partial}{\partial\tau}\{\overline{p_\Phi}, p_\Phi\}\Big|_{\tau=0} = -i\{\{p,\Phi\},p\} + i\{p,\{p,\Phi\}\} = 2i\{p,\{p,\Phi\}\}.$$

Together with (2.12), this concludes the proof of the lemma. □

Proof of Lemma 2.1.8. The set $N = \{f = 0\} \cap K$ is compact. Since g is continuous, its minimum on the set N is reached. So, we have $g \geq \min_N g = C_1 > 0$ on N. Since g is continuous and N compact, there exists an open neighborhood V of N so that $g \geq C_1/2$ on V. Now, $K \setminus V$ is closed in K and therefore compact. So, f reaches its minimum on $K \setminus V$. But since $(K \setminus V) \cap N = \emptyset$, we have $f \neq 0$ on $K \setminus V$ and hence $f > 0$ on that set, that is $C_2 = \min_{K\setminus V} f > 0$. Define now $C_3 = \min_{K\setminus V} g$ and $C_4 = \max_K |h|$.

We are in the following situation, for some A still to be chosen:

- on V, we have $g + Af - \frac{1}{A}h \geq \frac{C_1}{2} - \frac{1}{A}C_4$.
- on $K \setminus V$, we have $g + Af - \frac{1}{A}h \geq C_3 + AC_2 - \frac{1}{A}C_4$.

So, we need to choose A so that it leads to positive lower bound. If we want the final estimate with $C = C_1/4$, for instance, we need

$$A \geq \frac{4C_4}{C_1}, \quad \text{and} \quad A^2C_2 + A\left(C_3 - \frac{C_1}{4}\right) - C_4 > 0.$$

Since $C_2 > 0$, the last case is fulfilled if A is large enough since the polynomial of order 2 converges to $+\infty$ as A goes to ∞. □

A very important drawback to Definition 2.1.5 is that, it is not only dependent on the level set of the functions, but also on the "convexity with respect to the level sets". This is not a geometric assumption (in general, $g''(x_0)$ is a geometric quantity only if $g'(x_0) = 0$). We now need to link this definition to geometric quantities, so that to be able to formulate a result with, at least, a geometric assumption (that is invariant by diffeomorphisms). Before that, let us stress an important stability feature of the pseudoconvexity assumption of Definition 2.1.5.

2.1.3 Stability of the pseudoconvexity assumption

We prove that the pseudoconvexity condition of Definition 2.1.5 is stable by small C^2 perturbations of the weight function Φ. This will be very useful for perturbing the surface across which to prove unique continuation.

Proposition 2.1.9 (Stability and Geometric convexification)**.** *Let $\Omega \ni x_0$ such that $\overline{\Omega}$ is compact. Assume $P \in \mathrm{Diff}^2(\Omega)$ has real-valued principal symbol, and $\Phi \in C^\infty$ is pseudoconvex with respect to P at x_0 (in the sense of Definition 2.1.5). Then there exists $\varepsilon_0 > 0$ so that any $\Phi_\varepsilon \in C^2(\overline{\Omega})$ with $\|\Phi - \Phi_\varepsilon\|_{C^2(\overline{\Omega})} < \varepsilon_0$ is pseudoconvex with respect to P at x_0.*

Note that modifying Φ allows to slightly change its level sets. For instance, taking $\Phi_\varepsilon(x) = \Phi(x) - \varepsilon|x - x_0|^2$ (which shall be very useful for applications to unique continuation), the level set $\{\Phi_\varepsilon = 0\}$ is slightly bended (except at x_0) into the set $\{\Phi > 0\}$ (where u will be assumed to be zero). This slight change will be crucial for the proof of the unique continuation theorem. ♣ `faire un dessin`

Proof. First, we notice that we can prove as in the proof of Proposition 2.1.7 (still using Lemma 2.1.8 combined with Lemma 2.1.6 for the limit when $\tau = 0$) that Definition 2.1.5 implies (and is actually equivalent to) the existence of an inequality of the form

$$c_\Phi(\xi,\tau) + C_1 \frac{|p_\Phi(x_0,\xi,\tau)|^2}{|\xi|^2+\tau^2} \geq C_2(|\xi|^2+\tau^2),$$

uniformly for (ξ,τ) with $|\xi|^2+\tau^2 = 1$, $\tau \geq 0$ (see Lemma 2.1.6), where

$$c_\Phi(\xi,\tau) = \frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\}(x_0,\xi,\tau), \text{ for } \tau > 0 \quad \text{ and } \quad c_\Phi(\xi,0) = 2\{p_2,\{p_2,\Phi\}\}(x_0,\xi).$$

We then remark that all quantities in the above estimate only involve derivatives of Φ of order at most 2 (as a consequence of Lemma 2.1.6) at the point x_0. It is therefore stable by the addition of a function small for the C^2 norm around x_0. □

2.2 Strongly pseudoconvex surfaces

Until this point, we have proved a Carleman estimate with weight Φ provided Φ satisfies a (weird?) pseudoconvexity condition (see Definition 2.1.5). The main purpose of this section is to provide a geometric characterization of surfaces S for which we can find a function Φ having S as a level set and being appropriate for the Carleman estimate (that is, satisfies Definition 2.1.5).

Definition 2.2.1 (Usual pseudoconvexity for surfaces)**.** Let $\Omega \ni x_0$ be an open set, $P \in \text{Diff}^2(\Omega)$ with real-valued principal symbol p_2 and $\Psi \in C^\infty(\Omega)$ real-valued. We say that the *oriented hypersurface* $S = \{\Psi = \Psi(x_0)\} \ni x_0$ is strongly pseudoconvex with respect to P at x_0 if

$$\{p_2,\{p_2,\Psi\}\}(x_0,\xi) > 0, \qquad \text{if } p_2(x_0,\xi) = \{p_2,\Psi\}(x_0,\xi) = 0 \text{ and } \xi \neq 0; \tag{2.13}$$

$$\frac{1}{i\tau}\{\overline{p_\Psi}, p_\Psi\}(x_0,\xi,\tau) > 0, \quad \text{if } p_\Psi(x_0,\xi,\tau) = \{p_\Psi,\Psi\}(x_0,\xi,\tau) = 0 \text{ and } \tau > 0, \tag{2.14}$$

where $p_\Psi(x,\xi,\tau) = p_2(x,\xi + i\tau d\Psi(x))$.

Note that the definition seems to depend on the defining function Ψ for the surface S, and not only on the oriented hypersurface S itself. Lemma 2.2.2 shows this is not the case, and hence justifies the definition.

Lemma 2.2.2. *Assume $S = \{\Psi_1 = \Psi_1(x_0)\} = \{\Psi_2 = \Psi_2(x_0)\}$ with $d\Psi_j(x_0) \neq 0$, $j = 1,2$ and $d\Psi_2(x_0) = \lambda d\Psi_1(x_0)$ for some $\lambda > 0$ (same orientation). Then Ψ_1 satisfies (2.13) if and only if Ψ_2 satisfies (2.13), and Ψ_1 satisfies (2.14) if and only if Ψ_2 satisfies (2.14).*

Before proving Lemma 2.2.2, we need the following two lemmata.

Lemma 2.2.3. *Let $\Omega \subset \mathbb{R}^n$, $p \in C^\infty(\Omega \times \mathbb{R}^n)$ real-valued, and $\Psi \in C^\infty(\Omega;\mathbb{R})$. For all $(x,\xi) \in \Omega \times \mathbb{R}^n$ and $\tau > 0$, we have*

$$\begin{aligned}\frac{1}{i\tau}\{\overline{p_\Psi}, p_\Psi\}(x,\xi,\tau) = &\frac{2}{\tau}\operatorname{Im}\left[\partial_\xi p(x,\xi - i\tau d\Psi(x)) \cdot \partial_x p(x,\xi + i\tau d\Psi(x))\right] \\ &+ 2\operatorname{Hess}(\Psi)(x)\left[\partial_\xi p(x,\xi - i\tau d\Psi(x)); \partial_\xi p(x,\xi + i\tau d\Psi(x))\right],\end{aligned}$$

Lemma 2.2.4. *Condition (2.14) (for all $\tau > 0$) is equivalent to Condition (2.14) for $\tau = 1$, that is, for all $\xi \in \mathbb{R}^n$,*

$$\begin{aligned}&\frac{1}{i}\{p_2(x,\xi - id\Psi), p_2(x,\xi + id\Psi)\}(x = x_0,\xi) > 0, \\ &\textit{if } p_2(x_0,\xi + id\Psi(x_0)) = \{p_2,\Psi\}(x_0,\xi + id\Psi(x_0)) = 0.\end{aligned} \tag{2.15}$$

This is more intrinsic reformulation of the condition which removes the unnecessary dependence with respect to the parameter τ. Note however that the τ depending version of the assumption is useful in applications to Carleman estimates.

Proof of Lemma 2.2.4. We write p instead of p_2 for simplicity. We get rid of the parameter $\tau > 0$ by homogeneity. Namely, we have, for $\tau > 0$,

$$\begin{aligned}
p_\Psi(x_0,\xi,\tau) &= p(x_0,\xi + i\tau d\Psi(x_0)) = \tau^2 p(x_0,\xi/\tau + id\Psi(x_0)) \\
&= \tau^2 p_\Psi(x_0,\xi/\tau,1), \\
\{p_\Psi,\Psi\}(x_0,\xi,\tau) &= \partial_\xi p(x_0,\xi + i\tau d\Psi(x_0))\cdot\partial_x\Psi(x_0) = \tau\partial_\xi p(x_0,\xi/\tau + id\Psi(x_0))\cdot\partial_x\Psi(x_0) \\
&= \tau\{p_\Psi,\Psi\}(x_0,\xi/\tau,1), \\
\frac{1}{i\tau}\{\overline{p_\Psi},p_\Psi\}(x_0,\xi,\tau) &= \frac{2}{\tau}\tau^3\,\mathrm{Im}\left[\partial_\xi p(x,\xi/\tau - id\Psi(x))\cdot\partial_x p(x,\xi/\tau + id\Psi(x))\right] \\
&\quad + 2\tau^2\,\mathrm{Hess}(\Psi)(x)\left[\partial_\xi p(x,\xi/\tau - id\Psi(x));\partial_\xi p(x,\xi/\tau + id\Psi(x))\right] \\
&= \frac{\tau^2}{i}\{\overline{p_\Psi},p_\Psi\}(x_0,\xi/\tau,1),
\end{aligned}$$

after having used Lemma 2.2.3. As a consequence, Condition (2.14) for all $\xi \in \mathbb{R}^n$ and all $\tau > 0$ is equivalent to Condition (2.15) for all $\xi \in \mathbb{R}^n$. ☐

Proof of Lemma 2.2.3. This is the following computation

$$\begin{aligned}
\frac{1}{i\tau}\{\overline{p_\Psi},p_\Psi\}(x_0,\xi,\tau) &= \frac{1}{i\tau}\partial_\xi p(x_0,\xi - i\tau d\Psi(x_0))\cdot\partial_x p(x_0,\xi + i\tau d\Psi(x_0)) \\
&\quad + \mathrm{Hess}(\Psi)(x_0)\left[\partial_\xi p(x_0,\xi - i\tau d\Psi(x_0));\partial_\xi p(x_0,\xi + i\tau d\Psi(x_0))\right] \\
&\quad - \frac{1}{i\tau}\partial_x p(x_0,\xi - i\tau d\Psi(x_0))\cdot\partial_\xi p(x_0,\xi + i\tau d\Psi(x_0)) \\
&\quad + \mathrm{Hess}(\Psi)(x_0)\left[\partial_\xi p(x_0,\xi - i\tau d\Psi(x_0));\partial_\xi p(x_0,\xi + i\tau d\Psi(x_0))\right] \\
&= \frac{2}{\tau}\,\mathrm{Im}\left[\partial_\xi p(x_0,\xi - i\tau d\Psi(x_0))\cdot\partial_x p(x_0,\xi + i\tau d\Psi(x_0))\right] \\
&\quad + 2\,\mathrm{Hess}(\Psi)(x_0)\left[\partial_\xi p(x_0,\xi - i\tau d\Psi(x_0));\partial_\xi p(x_0,\xi + i\tau d\Psi(x_0))\right],
\end{aligned}$$

where both equalities use the fact that p is a real-valued symbol (and Ψ a real-valued function). ☐

Proof of Lemma 2.2.2. Let us first proof that under these assumptions, we may write $\Psi_2(x) = \mu(x)\Psi_1(x)$ for x in a neighborhood of x_0, with $\mu(x_0) > 0$. The statement of the lemma will be proved in a second place.

First assume to simplify notations that $x_0 = 0$. Since $d\Psi_1(0) \neq 0$, there is $k \in \{1,\cdots,n\}$ such that $\partial_{x_k}\Psi_1(0) \neq 0$. Assume e.g. $k = n$. The implicit function theorem implies that we may write locally $S = \{(x',x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}, x_n = f_1(x')\}$, with $f_1(0) = 0$ (since $0 \in S$). We perform the following local change of variables: $\chi(x',x_n) = (x',x_n - f_1(x'))$, which is a local diffeomorphism near 0, such that $\chi(S) = \{(x',0), x' \in \mathbb{R}^{n-1}\}$ locally. Denoting $\tilde{\Psi}_j = \Psi_j \circ \chi^{-1}$, $j = 1,2$, and remarking that $\tilde{\Psi}_j(y) = 0$ iff $\chi^{-1}(y) \in S$, we have locally

$$\{x_n = 0\} = \chi(S) = \{\tilde{\Psi}_1 = 0\} = \{\tilde{\Psi}_2 = 0\}.$$

This, together with the assumption of the lemma, implies in particular that $d\tilde{\Psi}_j(0) = \lambda_j dx_n$ with $\lambda_j \neq 0, \lambda_2/\lambda_1 = \lambda > 0$. The Taylor formula together with the fact that $\tilde{\Psi}_j(x',0) = 0$ writes $\tilde{\Psi}_j(x',x_n) = x_n G_j(x',x_n)$, with $G_j(x',x_n) = \int_0^1 \partial_{x_n}\tilde{\Psi}_j(x',tx_n)dt$. In particular, we have $G_j(0,0) = \partial_{x_n}\tilde{\Psi}_j(0,0) = \lambda_j \neq 0$. By continuity, we have $G_j(x',x_n) \neq 0$ in a whole neighborhood of zero, and, in this neighborhood, we have $\tilde{\Psi}_2 = \frac{G_2}{G_1}\tilde{\Psi}_1$, with $\frac{G_2}{G_1}(0) = \frac{\lambda_2}{\lambda_1} = \lambda > 0$. Coming back to the original variables yields $\Psi_2(x) = \mu(x)\Psi_1(x)$, $\mu(x) = \frac{G_2}{G_1}\circ\chi$, for x in a neighborhood of 0, with $\mu(0) > 0$.

We are now in position to prove the main statement of the lemma. We write p instead of p_2 for short. With $\Psi_2(x) = \mu(x)\Psi_1(x)$, we have

$$\{p,\Psi_2\}(x_0,\xi,\tau) = \mu(x_0)\{p,\Psi_1\}(x_0,\xi,\tau) + \Psi_1(x_0)\{p,\mu\}(x_0,\xi,\tau) = \mu(x_0)\{p,\Psi_1\}(x_0,\xi,\tau),$$

since $\Psi_1(x_0) = 0$. Also, using again $\Psi_1(x_0) = 0$, we have

$$\{p,\{p,\Psi_2\}\}(x_0,\xi,\tau) = \mu(x_0)\{p,\{p,\Psi_1\}\}(x_0,\xi,\tau) + 2\{p,\Psi_1\}(x_0,\xi,\tau)\{p,\mu\}(x_0,\xi,\tau).$$

From these two lines and the fact that $\mu(x_0) > 0$, we deduce that Ψ_1 satisfies (2.13) if and only if Ψ_2 satisfies (2.13).

We now check for Condition (2.14) for $\tau = 1$, that is Condition (2.15) (which is sufficient by Lemma 2.2.4). With $\Psi_2(x) = \mu(x)\Psi_1(x)$ and $\Psi_1(x_0) = 0$, we have $d\Psi_2(x_0) = \mu(x_0) d\Psi_1(x_0)$ together with

$$\begin{aligned} p(x_0, \xi + i d\Psi_2(x_0)) &= p(x_0, \xi + i\mu(x_0) d\Psi_1(x_0)), \\ \{p, \Psi_2\}(x_0, \xi + i d\Psi_2(x_0)) &= \mu(x_0)\{p, \Psi_1\}(x_0, \xi + i\mu(x_0) d\Psi_1(x_0)). \end{aligned}$$

Finally using Lemma 2.2.3, we have

$$\begin{aligned} \frac{1}{i}\{\overline{p_{\Psi_2}}, p_{\Psi_2}\}(x_0, \xi, 1) = &\, 2\operatorname{Im}\left[\partial_\xi p(x_0, \xi - i d\Psi_2(x_0)) \cdot \partial_x p(x_0, \xi + i d\Psi_2(x_0))\right] \\ &+ 2\operatorname{Hess}(\Psi_2)(x_0)\left[\partial_\xi p(x_0, \xi - i d\Psi_2(x_0)); \partial_\xi p(x_0, \xi + i d\Psi_2(x_0))\right]. \end{aligned}$$

We compute

$$\operatorname{Hess}(\Psi_2) = \Psi_1 \operatorname{Hess}(\mu) + 2\partial\mu \otimes \partial\Psi_1 + \mu \operatorname{Hess}(\Psi_1),$$

and hence

$$\operatorname{Hess}(\Psi_2)(x_0) = 2\partial\mu(x_0) \otimes \partial\Psi_1(x_0) + \mu(x_0)\operatorname{Hess}(\Psi_1)(x_0),$$

♣ ♣ □

Remark that Definition 2.2.1 looks very similar to Definition 2.1.5. It is just slightly weaker because the positivity condition is assumed only under the additional conditions $\{p_2, \Phi\} = 0$ and $\{p_\Phi, \Phi\} = 0$. In particular, the level sets of a pseudoconvex functions are pseudoconvex oriented surfaces. This is however not useful since Definition 2.1.5 is not geometric (but rather linked to Carleman estimates).

The importance of Definition 2.2.1 is twofold:

- It is a purely geometric definition: this comes from Lemma 2.2.2 and the fact that Conditions (2.13)-(2.14) are invariant by diffeomorphisms ♣ `prove that the Poisson bracket is invariant by diffeomorphism: discuss action of a diffeomorphism on symbols/differential operators`
- Once Ψ satisfies this geometric condition, one can produce a function Φ having the same levelsets (hence keeping the geometry unchanged), and that satisfies the stronger pseudoconvexity condition of Definition 2.1.5. This is the goal of the next section.

Note that, once again, Condition (2.13) (on the real domain) is the limit as $\tau \to 0^+$ of Condition 2.14 (on the complex domain). This follows both from Lemma 2.1.6 and the fact that

$$\begin{aligned} \{p_\Psi, \Psi\}(x, \xi, \tau) &= \partial_\xi\big(p(x, \xi + i\tau d\Psi(x)\big) \cdot \partial_x\Psi(x) = (\partial_\xi p)(x, \xi + i\tau d\Psi(x)) \cdot \partial_x\Psi(x) \\ &= \{p_2, \Psi\}(x, \xi + i\tau d\Psi(x)) \to \{p_2, \Psi\}(x, \xi), \quad \text{as } \tau \to 0^+. \end{aligned} \tag{2.16}$$

2.2.1 (Analytic) convexification

Proposition 2.2.5 (Analytic convexification)**.** *Let $\Omega \ni x_0$ be an open set, $P \in \operatorname{Diff}^2(\Omega)$ with real-valued principal symbol p_2 and $\Psi \in C^\infty(\Omega)$ real-valued. Assume the oriented hypersurface $S = \{\Psi = \Psi(x_0)\}$ is strongly pseudoconvex with respect to P at x_0 (Definition 2.2.1). Then there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, the function $\Phi = e^{\lambda\Psi}$ is pseudoconvex with respect to P at x_0 (Definition 2.1.5).*

Hence, the Carleman estimate of Theorem 2.1.1 holds with weight Φ.

Note that the geometry of the level-sets of Φ and Ψ are actually the same: only the values of the level sets of Φ are stretched. Here, for any strongly pseudoconvex oriented surface $S = \{\psi = \Psi(x_0)\}$ (which will be the relevant geometric condition for the unique continuation result under consideration), this proposition produces an admissible Carleman weight (that is, a pseudoconvex function) Φ having exactly the same geometric properties.

In order to simplify the notation for the proof, we recall that x_0 is fixed and remark that changing the function Ψ by a constant does not change the assumption. We may thus assume that

$$\Psi(x_0) = 0, \quad \text{and hence} \quad \Phi(x_0) = 1 \quad \text{and} \quad d\Phi(x_0) = \lambda d\Psi(x_0). \tag{2.17}$$

We also denote

$$c_\Psi(\xi,\tau) = \frac{1}{i\tau}\{\overline{p_\Psi}, p_\Psi\}(x_0,\xi,\tau), \text{ for } \tau > 0 \quad \text{ and } \quad c_\Psi(\xi,0) = 2\{p_2,\{p_2,\Psi\}\}(x_0,\xi),$$

with a similar definition for $c_\Phi(\xi,\tau)$. According to Lemma 2.1.6, $c_\Psi(\xi,\tau)$ and $c_\Phi(\xi,\tau)$ are continuous on the whole $\mathbb{R}^n \times \mathbb{R}^+$. The proof of Proposition 2.2.5 is then based on the following computation.

Lemma 2.2.6. *Assume $\Phi = e^{\lambda\Psi}$. For all $(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}^+$ and all $\lambda > 0$, we have*

$$c_\Phi(\xi,\tau) = \lambda c_\Psi(\xi,\lambda\tau) + 2\lambda^2 \left|\{p_\Psi,\Psi\}(x_0,\xi,\lambda\tau)\right|^2.$$

That is this additional term that comes from the convexification that will allow to get extra positivity when $\{p_\Psi,\Psi\} \neq 0$. The positivity when $\{p_\Psi,\Psi\} \neq 0$ being ensured by the assumptions on Ψ.

We first prove the proposition from the lemma and then prove the lemma.

Proof of Proposition 2.2.5 from Lemma 2.2.6. Using Lemma 2.1.8 (combined with Lemma 2.1.6 and (2.16) in the limit $\tau \to 0^+$), Properties (2.13)-(2.14) imply the existence of $C_1, C_2 > 0$ so that

$$c_\Psi(\xi,\tau) + C_1\left|\{p_\Psi,\Psi\}(x_0,\xi,\tau)\right|^2 + C_1\frac{|p_\Psi(x_0,\xi,\tau)|^2}{|\xi|^2+\tau^2} \geq C_2(|\xi|^2+\tau^2).$$

for any $\tau \geq 0$, $|\xi|^2+\tau^2 = 1$ (note that this takes into account the limit $\tau \to 0^+$). Replacing τ by $\lambda\tau$ for $\lambda \geq 1$ and using homogeneity, this can be reformulated as

$$c_\Psi(\xi,\lambda\tau) + C_1\left|\{p_\Psi,\Psi\}(x_0,\xi,\lambda\tau)\right|^2 + C_1\frac{|p_\Psi(x_0,\xi,\lambda\tau)|^2}{|\xi|^2+\lambda^2\tau^2} \geq C_2(|\xi|^2+\lambda^2\tau^2). \tag{2.18}$$

for any $(\xi,\tau) \neq (0,0)$ with $\tau \geq 0$.

Moreover, using Lemma 2.2.6 and noticing (see (2.17)) that

$$p_\Psi(x_0,\xi,\lambda\tau) = p_2(x_0,\xi+i\lambda\tau d\Psi(x_0)) = p_2(x_0,\xi+i\tau d\Phi(x_0)) = p_\Phi(x_0,\xi,\tau),$$

we obtain

$$\begin{aligned} c_\Phi(\xi,\tau) + C_1\lambda\frac{|p_\Phi(x_0,\xi,\tau)|^2}{|\xi|^2+\tau^2} &= \lambda c_\Psi(\xi,\lambda\tau) + 2\lambda^2\left|\{p_\Psi,\Psi\}(x_0,\xi,\lambda\tau)\right|^2 + C_1\lambda\frac{|p_\Phi(x_0,\xi,\tau)|^2}{|\xi|^2+\tau^2} \\ &= \lambda\left(c_\Psi(\xi,\lambda\tau) + 2\lambda\left|\{p_\Psi,\Psi\}(x_0,\xi,\lambda\tau)\right|^2 + C_1\frac{|p_\Psi(x_0,\xi,\lambda\tau)|^2}{|\xi|^2+\tau^2}\right). \end{aligned}$$

Now taking $\lambda \geq \max\{C_1/2, 1\}$ and using (2.18) yields

$$\begin{aligned} c_\Phi(\xi,\tau) + C_1\lambda\frac{|p_\Phi(x_0,\xi,\tau)|^2}{|\xi|^2+\tau^2} &\geq \lambda\left(c_\Psi(\xi,\lambda\tau) + C_1\left|\{p_\Psi,\Psi\}(x_0,\xi,\lambda\tau)\right|^2 + C_1\frac{|p_\Psi(x_0,\xi,\lambda\tau)|^2}{|\xi|^2+\lambda^2\tau^2}\right) \\ &\geq C_2\lambda(|\xi|^2+\lambda^2\tau^2) \geq C_2\lambda(|\xi|^2+\tau^2). \end{aligned}$$

When recalling the definition of c_Φ, this readily implies (2.11), and also (2.10) in the limit $\tau \to 0^+$ (with Lemma 2.1.6). This concludes the proof that Φ is pseudoconvex for P at x_0 in the sense of Definition 2.1.5. □

It only remains to prove Lemma 2.2.6.

Proof of Lemma 2.2.6. We compute

$$\partial_j\Phi = \lambda\partial_j\Psi e^{\lambda\Psi}, \quad \partial_{j,k}\Phi = \lambda\partial_{j,k}\Psi e^{\lambda\Psi} + \lambda^2(\partial_j\Psi)(\partial_k\Psi)e^{\lambda\Psi},$$

which we write in a shorter way as

$$d\Phi = \lambda e^{\lambda\Psi}d\Psi, \quad \operatorname{Hess}(\Phi)(\xi,\widetilde{\xi}) = \lambda\operatorname{Hess}(\Psi)(\xi;\widetilde{\xi})e^{\lambda\Psi} + \lambda^2(\xi\cdot\partial_x\Psi)(\widetilde{\xi}\cdot\partial_x\Psi)e^{\lambda\Psi}.$$

Taken at the point x_0, and recalling (2.17), this implies

$$d\Phi(x_0) = \lambda d\Psi(x_0),$$
$$\mathrm{Hess}(\Phi)(x_0)(\xi, \widetilde{\xi}) = \lambda \, \mathrm{Hess}(\Psi)(x_0)(\xi; \widetilde{\xi}) + \lambda^2(\xi \cdot \partial_x\Psi(x_0))(\widetilde{\xi} \cdot \partial_x\Psi(x_0)).$$

Using Lemma 2.2.3, we now obtain (we drop the fact that Ψ and the different derivatives of Ψ are taken at x_0)

$$\begin{aligned}
c_\Phi(\xi,\tau) &= \frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\}(x_0,\xi,\tau) \\
&= \frac{2}{\tau}\,\mathrm{Im}\left[\partial_\xi p(x_0, \xi - i\tau\lambda d\Psi)\cdot(\partial_x p(x_0,\xi + i\tau\lambda d\Psi))\right] \\
&\quad +2\lambda\,\mathrm{Hess}(\Psi)\left[\partial_\xi p(x_0,\xi - i\tau\lambda d\Psi); \partial_\xi p(x_0, \xi + i\tau\lambda d\Psi)\right] \\
&\quad +2\lambda^2\left(\partial_\xi p(x_0,\xi - i\tau\lambda d\Psi)\cdot\partial_x\Psi\right)\left(\partial_\xi p(x_0,\xi + i\tau\lambda d\Psi)\cdot\partial_x\Psi\right) \\
&= \lambda c_\Psi(\xi,\lambda\tau) + 2\lambda^2\left|\{p,\Psi\}(x_0,\xi + i\tau\lambda d\Psi)\right|^2 \\
&= \lambda c_\Psi(\xi,\lambda\tau) + 2\lambda^2\left|\{p_\Psi,\Psi\}(x_0,\xi,\lambda\tau)\right|^2,
\end{aligned}$$

proving the lemma. □

2.2.2 Reducing the strong pseudoconvexity assumption to the condition on the real space

In the particular case of differential operators of order two, with real principal symbol, an additional simplification occurs. More precisely, Condition (2.13) on the real space implies Condition (2.14) in the complex space. This is a very particular situation.

Proposition 2.2.7. *Let $\Omega \ni x_0$ be an open set, $P \in \mathrm{Diff}^2(\Omega)$ with real-valued principal symbol p_2 and $\Psi \in C^\infty(\Omega)$ real-valued. Assume that the oriented surface $S = \{\Psi = \Psi(x_0)\}$ satisfies Condition (2.13) at x_0. Then $S = \{\Psi = \Psi(x_0)\}$ is strongly pseudoconvex with respect to P at x_0 (i.e. both conditions (2.13) and (2.14) are satisfied).*

This proposition states that in the case of *real symbols of order two*, we can get rid of Condition (2.14) on the complex domain (this is no longer the case if P is not of order two, or if its principal symbol is not real). Therefore the only remaining geometric assumption for the unique continuation theorem (to be stated in the next section) is (2.13). Its geometric content is commented in Section 2.3.2 below.

We split the proof of Proposition 2.2.7 into two lemmata, concerned with the non-characteristic case ($p_2(x_0, d\Psi(x_0)) \neq 0$) and the characteristic case ($p_2(x_0, d\Psi(x_0)) = 0$), respectively.

Lemma 2.2.8. *Assume p_2 is a real symbol of order two near x_0, and Ψ is such that $p_2(x_0, d\Psi(x_0)) \neq 0$. Then, for any $\xi \in \mathbb{R}^n$ we have*

$$p_\Psi(x_0,\xi,\tau) = \{p_\Psi,\Psi\}(x_0,\xi) = 0 \quad \Longrightarrow \tau = 0. \tag{2.19}$$

In this case, Assumption (2.14) is thus empty.

Lemma 2.2.9. *Assume p_2 is a real symbol of order two near x_0, and Ψ is such that $p_2(x_0, d\Psi(x_0)) = 0$. Assume also (2.13) for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Then we also have (2.14).*

Both proofs of Lemmata 2.2.8 and 2.2.9 rely on the fact that for fixed $\xi \in \mathbb{R}^n$,

$$f(z) = p_2(x_0, \xi + zd\Psi(x_0)) = p_2(x_0,\xi) + z^2 p_2(x_0, d\Psi(x_0)) + 2z\tilde{p}_2(x_0,\xi, d\Psi(x_0)),$$

is a *second order polynomial* in the variable z, with *real coefficients*. Moreover, the assumption of (2.19) (resp. of (2.14)) implies that

$$f(i\tau) = p_2(x_0, \xi + i\tau d\Psi(x_0)) = p_\Psi(x_0,\xi,\tau) = 0 \quad \text{and}$$
$$f'(i\tau) = \partial_\xi p_2(x_0,\xi + i\tau d\Psi(x_0))\cdot\partial_x\Psi(x_0) = \{p_2,\Psi\}(x_0,\xi + i\tau d\Psi(x_0)) = \{p_\Psi,\Psi\}(x_0,\xi) = 0,$$

that is to say, $z = i\tau$ ($\tau \in \mathbb{R}^+$) is a double root of the polynomial f.

Proof of Lemma 2.2.8. Since the coefficient in front of z^2, namely $p_2(x_0, d\Psi(x_0))$ is non-zero, the polynomial f has two complex roots which are either both in $\mathbb{R}$, or complex conjugate. That $z = i\tau$ ($\tau \in \mathbb{R}^+$) is a double root of the polynomial f implies $\tau = 0$. □

The proof of Lemma 2.2.9 relies on tedious computations, that are collected in Section 2.2.3 below.

Proof of Lemma 2.2.9. Using that $p_2(x_0, d\Psi(x_0)) = 0$ together with Lemma 2.2.11, we obtain that

$$f(z) = p_2(x_0, \xi) + z\{p_2, \Psi\}(x_0, \xi).$$

The fact that f has a double root (at $i\tau$) implies that actually, f is the zero polynomial, that is $p_2(x_0, \xi) = \{p_2, \Psi\}(x_0, \xi) = 0$. Hence, Assumption (2.13) taken at the point ξ implies that either $\{p_2, \{p_2, \Psi\}\}(x_0, \xi) > 0$ or $\xi = 0$. Moreover, the assumption $p_2(x_0, d\Psi(x_0)) = 0$ also yields that

$$\{p_2, \Psi\}(x_0, d\Psi(x_0)) = 2\tilde{p}_2(x_0, d\Psi(x_0), d\Psi(x_0)) = 2p_2(x_0, d\Psi(x_0)) = 0.$$

Hence, Assumption (2.13) taken at the point $\xi = d\Psi(x_0)$ implies $\{p_2, \{p_2, \Psi\}\}(x_0, d\Psi(x_0)) > 0$. Finally, according to Lemma 2.2.12 we have

$$\frac{1}{i\tau}\{\overline{p_\Psi}, p_\Psi\}(x_0, \xi, \tau) = 2\{p_2, \{p_2, \Psi\}\}(x_0, \xi) + 2\tau^2\{p_2, \{p_2, \Psi\}\}(x_0, d\Psi(x_0)),$$

and we have just proved that the first term in the right hand-side is nonnegative (even positive if $\xi \neq 0$) and the second term is positive, implying

$$\frac{1}{i\tau}\{\overline{p_\Psi}, p_\Psi\}(x_0, \xi, \tau) > 0 \quad \text{for } \tau > 0.$$

□

Remark 2.2.10. A simpler proof, could be made if we assume that we are in some coordinates so that $\Psi = x_1$. Actually, this is not a loss of generality since we could prove (but we did not do it yet) that the assumptions and conclusions of Proposition 2.2.7 are invariant by change of coordinates and of defining function for the surface. In that case, we can check that actually f can never be identically zero. Indeed, if it happens, we have $0 = f(s) = p_\Psi = p_2(x_0, \xi + se_1)$ and $\partial_{\xi_1} p = 0$. It gives

$$\{p, \{p, \Psi\}\} = \{p, \{p, x_1\}\} = \{p, \partial_{\xi_1} p\} = \nabla_\xi p \cdot \nabla_x \partial_{\xi_1} p - \nabla_x p \cdot \nabla_\xi \partial_{\xi_1} p = 0.$$

This is impossible for $\xi \neq 0$ since we have $p(x, \xi) = \{p, \Psi\} = 0$ in the considered points. It contradict the first assumption.

Note that the cancellation of $\{p, \{p, \Psi\}\}$ under these assumptions is specific to the chosen coordinates.

2.2.3 Computations for real symbols of order 2

This section contains many tedious computations that are used only in the previous section, where we remove the condition on the complex domain. These computations may/should be skipped at first (and second) reading. They all heavily rely on the assumption that the symbol $p(x, \xi)$ be *real and homogeneous of degree* 2, using the language of quadratic forms. We start with the following properties.

Lemma 2.2.11. *Let $p(x, \xi) = \sum_{1 \leq k,l \leq n} a^{kl}(x)\xi_k \xi_l$ be a homogeneous symbol of order 2, with real-valued coefficients. Denote by $\tilde{p}(x, \xi, \eta)$ its polar form, namely $\tilde{p}(x, \xi, \eta) = \sum_{k,l} \frac{a^{kl}(x) + a^{lk}(x)}{2}\xi_k \eta_l$. Then, we have*

$$\{p, \Psi\}(x, \xi) = 2\tilde{p}(x, \xi, d\Psi(x)) \tag{2.20}$$

$$p_\Psi(x, \xi, \tau) = p(x, \xi) - \tau^2 p(x, d\Psi(x)) + i\tau\{p, \Psi\}(x, \xi) \tag{2.21}$$

$$\{p_\Psi, \Psi\}(x, \xi, \tau) = 2\tilde{p}(x, \xi, d\Psi(x)) + 2i\tau p(x, d\Psi(x)). \tag{2.22}$$

Moreover, assume f only depends on x, then

$$\{f, \{p, \Psi\}\}(x) = -\{p, f\}(x, d\Psi(x)), \tag{2.23}$$

which, in particular, does not depend on ξ.

Recall that $p(x,\xi) = \tilde{p}(x,\xi,\xi)$ and $p(x,\xi+\eta) = p(x,\xi) + 2\tilde{p}(x,\xi,\eta) + p(x,\eta)$ for $\xi,\eta \in \mathbb{C}^n$.

Proof. First, by linearity, it is enough to prove (2.20) for $p(x,\xi) = a(x)\xi_k\xi_l$ and $\tilde{p}(x,\xi,\eta) = \frac{a(x)}{2}(\xi_k\eta_l + \xi_l\eta_k)$. This implies

$$\{p,\Psi\}(x,\xi) = \partial_\xi(a(x)\xi_k\xi_l)\cdot\partial_x\Psi(x) = a(x)\left[\xi_k(\partial_l\Psi) + \xi_l(\partial_k\Psi)\right] = 2\tilde{p}(x,\xi,d\Psi(x)), \tag{2.24}$$

which proves (2.20). Second, expanding the bilinear form, we have

$$p_\Psi(x,\xi,\tau) = p(x,\xi+i\tau d\Psi(x)) = p(x,\xi) - \tau^2 p(x,d\Psi(x)) + 2i\tau\tilde{p}(x,\xi,d\Psi(x)).$$

Recalling (2.20), this proves (2.21). Third, we have

$$\{p_\Psi,\Psi\}(x,\xi,\tau) = \partial_\xi\big(p(x,\xi+i\tau d\Psi)\big)\cdot\partial_x\Psi = (\partial_\xi p)(x,\xi+i\tau d\Psi)\cdot\partial_x\Psi = \{p,\Psi\}(x,\xi+i\tau d\Psi)$$

Using (2.20), we obtain

$$\{p_\Psi,\Psi\}(x,\xi,\tau) = 2\tilde{p}(x,\xi+i\tau d\Psi,d\Psi) = 2\tilde{p}(x,\xi,d\Psi) + 2i\tau\tilde{p}(x,d\Psi,d\Psi) = 2\tilde{p}(x,\xi,d\Psi) + 2i\tau p(x,d\Psi),$$

which yields (2.22). Finally, writing again $p(x,\xi) = a(x)\xi_k\xi_l$ and recalling (2.24), we obtain

$$\begin{aligned}\{f,\{p,\Psi\}\} &= \{f,a(x)\left[\xi_k(\partial_l\Psi) + \xi_l(\partial_k\Psi)\right]\} = -a(x)\left[(\partial_k f)(\partial_l\Psi) + (\partial_l f)(\partial_k\Psi)\right] \\ &= -2\tilde{p}(x,df,d\Psi) = -\{p,f\}(x,d\Psi),\end{aligned}$$

which proves (2.23). □

Lemma 2.2.12. *Let p be a real-valued symbol being homogeneous of degree 2. Then*

$$\frac{1}{i\tau}\left\{\overline{p_\Psi},p_\Psi\right\}(x,\xi) = 2\left\{p,\{p,\Psi\}\right\}(x,\xi) + 2\tau^2\left\{p,\{p,\Psi\}\right\}(x,d\Psi(x)).$$

Proof. Using the expression of p_Ψ in (2.21) together with $\{\bar{a},a\} = 2i\{\mathrm{Re}(a),\mathrm{Im}(a)\}$ (using that $\{a-b,a+b\} = 2\{a,b\}$), we have

$$\begin{aligned}\frac{1}{i\tau}\left\{\overline{p_\Psi},p_\Psi\right\} &= 2\left\{p-\tau^2 p(x,d\Psi),\{p,\Psi\}\right\} \\ &= 2\left\{p,\{p,\Psi\}\right\} - 2\tau^2\left\{p(x,d\Psi),\{p,\Psi\}\right\}.\end{aligned}$$

Finally, using Lemma 2.2.13 below, we obtain

$$\frac{1}{i\tau}\left\{\overline{p_\Psi},p_\Psi\right\} = 2\left\{p,\{p,\Psi\}\right\} + 2\tau^2\left\{p,\{p,\Psi\}\right\}(x,d\Psi),$$

which proves the lemma. □

For the above proof to be complete, it only remains to prove the following lemma.

Lemma 2.2.13. *Let $p : \Omega\times\mathbb{R}^n \to \mathbb{R}$ be a homogeneous symbol of degree 2, with real-valued coefficients and $\Psi \in C^\infty(\Omega)$. Then, we have*

$$\left\{p(\cdot,d\Psi),\{p,\Psi\}\right\}(x) = -\left\{p,\{p,\Psi\}\right\}(x,d\Psi(x)),$$

which, in particular, does not depend on ξ.

Proof. We start from the following general formula (for any symbols p,q), proved in Lemma 2.2.14 below,

$$\{p(\,\cdot,d\Psi),q\}(x,d\Psi(x)) + \{p,q(\cdot,d\Psi)\}(x,d\Psi(x)) = \{p,q\}(x,d\Psi(x)).$$

We write this identity with $q(x,\xi) = \{p,\Psi\}(x,\xi)$ (which is a homogeneous polynomial of degree one, so that its Poisson bracket with a function of x only is a function of x only), yielding

$$\left\{p(\cdot,d\Psi),\{p,\Psi\}\right\}(x) + \left\{p,\{p,\Psi\}(\cdot,d\Psi)\right\}(x,d\Psi(x)) = \left\{p,\{p,\Psi\}\right\}(x,d\Psi(x)). \tag{2.25}$$

But for real-valued homogeneous symbols of degree 2, Identity (2.20) gives $\{p, \Psi\}(x, d\Psi(x)) = 2\tilde{p}(x, d\Psi(x), d\Psi(x)) = 2p(x, d\Psi(x))$. Using this, together with (2.23), we have

$$\{p, \{p, \Psi\}(\cdot, d\Psi)\}\,(x, d\Psi(x)) = 2\{p, p(\cdot, d\Psi)\}(x, d\Psi(x)) = -2\{p(\cdot, d\Psi), \{p, \Psi\}\}(x).$$

Inserting this into (2.25), we obtain

$$\{p(\cdot, d\Psi), \{p, \Psi\}\}\,(x) - 2\{p(\cdot, d\Psi), \{p, \Psi\}\}(x) = \{p, \{p, \Psi\}\}\,(x, d\Psi(x)),$$

which is the sought result. ☐

Lemma 2.2.14. *For all symbols $p, q \in C^\infty(\Omega \times \mathbb{R}^n)$ and all function $\Psi \in C^\infty(\Omega)$, we have*

$$\{p(\cdot, d\Psi), q\}(x, d\Psi(x)) + \{p, q(\cdot, d\Psi)\}(x, d\Psi(x)) = \{p, q\}(x, d\Psi(x)). \tag{2.26}$$

Proof. On the one hand, the right hand-side of (2.26) writes

$$\{p, q\}(x, d\Psi(x)) = (\partial_\xi p)(x, d\Psi(x)) \cdot (\partial_x q)(x, d\Psi(x)) - (\partial_x p)(x, d\Psi(x)) \cdot (\partial_\xi q)(x, d\Psi(x)). \tag{2.27}$$

On the other hand, we have

$$\{p(\cdot, d\Psi), q\}(x, \xi) = -(\partial_x p)(x, d\Psi(x)) \cdot (\partial_\xi q)(x, \xi) - \operatorname{Hess}\Psi\left[(\partial_\xi p)(x, \partial\Psi(x)); (\partial_\xi q)(x, \xi)\right].$$

Applied to $\xi = d\Psi$, this gives an expression of the first term in the left hand-side of (2.26), namely

$$\begin{aligned}\{p(\cdot, d\Psi), q\}(x, d\Psi(x)) = &-(\partial_x p)(x, d\Psi(x)) \cdot (\partial_\xi q)(x, d\Psi(x)) \\ &- \operatorname{Hess}\Psi\left[(\partial_\xi p)(x, d\Psi(x)); (\partial_\xi q)(x, d\Psi(x))\right].\end{aligned}$$

By symmetry, we also have

$$\begin{aligned}\{p, q(\cdot, d\Psi)\}(x, d\Psi(x)) = &(\partial_\xi p)(x, d\Psi(x)) \cdot (\partial_x q)(x, d\Psi(x)) \\ &+ \operatorname{Hess}\Psi\left[(\partial_\xi p)(x, d\Psi(x)); (\partial_\xi q)(x, d\Psi(x))\right].\end{aligned}$$

The sum of the last two identities gives the right hand-side (2.27), which proves (2.26). ☐

2.3 The unique continuation theorem

In this chapter, collecting all results we proved so far, we are prepared to state and prove a very general result of unique continuation for operators of order 2, with real principal symbol.

2.3.1 Statement and examples

The geometric definition we need is the following.

Definition 2.3.1 (Strongly pseudoconvexity surface for operators of order two with real principal symbols)**.** Let $\Omega \ni x_0$ be an open set, $P \in \operatorname{Diff}^2(\Omega)$ with real-valued principal symbol p_2 and $\Psi \in C^\infty(\Omega)$ real-valued. We say that the oriented hypersurface $S = \{\Psi = \Psi(x_0)\}$ is strongly pseudoconvex with respect to P at x_0 if it satisfies

$$p_2(x_0, \xi) = \{p_2, \Psi\}(x_0, \xi) = 0 \Longrightarrow \{p_2, \{p_2, \Psi\}\}(x_0, \xi) > 0 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.28}$$

Note that $\{p, \Psi\}(x_0, \xi) = \partial_\xi p(x_0, \xi) \cdot \partial_x \Psi(x_0)$.

We can check that Definition 2.3.1 is invariant if we change the defining function Ψ (see Lemma 2.2.2). That is why this is a geometric property of the oriented surface solely. See Section 2.3.2 for an interpretation as convexity with respect to the bicharacteristic curves.

The geometric condition (2.28) has to be compared with that discussed for vector fields in Examples 3-4-5 in Section 1.2.1.

Theorem 2.3.2 (Real operators of order 2)**.** *Let $\Omega \ni x_0$ be an open set of $\mathbb{R}^n$, and let $P \in \mathrm{Diff}^2(\Omega)$ with* real *principal symbol p_2. Assume that the oriented hypersurface $S = \{\Psi = \Psi(x_0)\}$ is strongly pseudoconvex with respect to P at x_0. Then, there exists a neighborhood V of x_0 so that for all $u \in H^1(\Omega)$, we have*

$$\left\{ \begin{array}{l} Pu = 0 \text{ in } \Omega, \\ u = 0 \text{ in } \Omega \cap \{\Psi > \Psi(x_0)\} \end{array} \right\} \Longrightarrow u = 0 \text{ in } V. \tag{2.29}$$

Another (slightly weaker) way to formulate the conclusion of the theorem is to say that $x_0 \notin \mathrm{supp}(u)$. Here, we have assumed that all coefficients of P are smooth for simplicity. Finer regularity assumptions are discussed in Section 2.3.4 below.

Remark 2.3.3 (Elliptic case)**.** Note now that in the particular case where the operator P is elliptic at x_0, i.e. $p_2(x_0, \xi) \geq c|\xi|^2$, then the condition $p_2(x_0, \xi) = 0$ is never fulfilled when $\xi \neq 0$ and (2.28) is empty. This is the following corollary.

Corollary 2.3.4 (Real elliptic operators of order 2)**.** *Let Ω an open set of $\mathbb{R}^n$, $x_0 \in \Omega$ and $S \ni x_0$ be a local hypersurface at x_0. Let $P \in \mathrm{Diff}^2(\Omega)$ with* real *principal symbol p_2. Assume also that P is elliptic at x_0, that is, there exists $c > 0$ so that $p_2(x_0, \xi) \geq c|\xi|^2$ for all $\xi \in \mathbb{R}^n$. Then, there exists V a neighborhood of x_0 so that for any $u \in C^\infty(\Omega)$,*

$$\left\{ \begin{array}{rcl} Pu & = & 0 \text{ in } \Omega, \\ u & = & 0 \text{ in } \Omega \cap S^+ \end{array} \right. \Longrightarrow u = 0 \text{ on } V. \tag{2.30}$$

Here, S^+ denotes one (any) side of S.

This means that for elliptic operators of order 2, there is no geometric condition for unique continuation across a hypersurface. Again, regularity issues are discussed in Section 2.3.4 below.

Remark 2.3.5 (Operators with constant coefficients)**.** Consider here the simple case where $P = AD \cdot D$ where A is a constant real symmetric matrix. This is $p_2(x, \xi) = A\xi \cdot \xi$. We have $\{p_2, \Psi\}(x, \xi) = 2A\xi \cdot d\Psi(x)$ and $\{p_2, \{p_2, \Psi\}\}(x, \xi) = \{A\xi \cdot \xi, 2A\xi \cdot d\Psi(x)\} = 4 \,\mathrm{Hess}\, \Psi(x)(A\xi, A\xi)$. Condition (2.28) rewrites

$$A\xi \cdot \xi = 0 \quad \text{and} \quad A\xi \cdot d\Psi(x_0) = 0 \Longrightarrow \mathrm{Hess}\, \Psi(x_0)(A\xi, A\xi) > 0 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

Remark 2.3.6 (The wave operator)**.** We discuss here the case of the wave operator with constant coefficients, which is a particular case of the above examples with $A = \mathrm{diag}(-1, 1, \cdots, 1)$. In the case of the wave equation, $P = \partial_t^2 - \Delta$, $p = -\xi_t^2 + |\xi_x|^2$, we compute (using that Ψ does not depend on ξ)

$$\begin{aligned} \{p, \Psi\} &= \nabla_\xi p \cdot \nabla_{(t,x)} \Psi = -2\xi_t \partial_t \Psi + 2\xi_x \cdot \nabla_x \Psi \\ \{p, \{p, \Psi\}\} &= \nabla_\xi p \cdot \nabla_{(t,x)} \{p, \Psi\} - \nabla_{(t,x)} p \cdot \nabla_\xi \{p, \Psi\} \\ &= \nabla_\xi p \cdot \nabla_{(t,x)} \{p, \Psi\} \\ &= -2\xi_t \partial_t \left[-2\xi_t \partial_t \Psi + 2\xi_x \cdot \nabla_x \Psi\right] + 2\xi_x \cdot \nabla_x \left[-2\xi_t \partial_t \Psi + 2\xi_x \cdot \nabla_x \Psi\right] \\ &= 4 \left[\xi_t^2 \partial_t^2 \Psi - 2\xi_t \xi_x \cdot \nabla_x \partial_t \Psi + \mathrm{Hess}_x(\Psi)(\xi_x, \xi_x)\right] \end{aligned}$$

We now write the strong pseudoconvexity condition (2.28) specialized in the point $(t, x) = (0, 0)$ (the operator is translation invariant in (t, x)), in different situations.

- If $|\partial_t \Psi(0)| > |\nabla_x \Psi(0)|$: the surface $\{\Psi = \Psi(0)\}$ is called spacelike (its normal vector $\nabla_{t,x} \Psi$ is timelike). The first two conditions imply $|\xi_t \partial_t \Psi(0)| = |\xi_x \cdot \partial_x \Psi(0)| \leq |\xi_x||\partial_x \Psi(0)| = |\xi_t||\partial_x \Psi(0)| < |\xi_t||\partial_t \Psi(0)|$. This is a contradiction, and hence Condition (2.28) is empty.

  Any spacelike surface satisfies the unique continuation. This is very natural. Actually, the Cauchy problem is hyperbolic and indeed locally wellposed for any spacelike hypersurface (like for instance the wave equation posed with initial data at $t = 0$, see Theorem 1.2.1).

- In several applications, the typical unique continuation result we need is across hypersurfaces of the form $\Psi(t, x) = \varphi(x)$. The strong pseudoconvexity condition then writes

$$\xi_t^2 = |\xi_x|^2 \text{ and } \xi_x \cdot \nabla_x \varphi(0) = 0 \Longrightarrow \mathrm{Hess}_x(\varphi)(0)(\xi_x, \xi_x) > 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Typically, if $\Psi(t,x) = |x|^2 - 1$, the condition holds when we want to prove the unique continuation from the exterior of the ball to the interior and not in the other direction. There are actually counterexamples if we allow a potential smooth in t and x (see Alinhac-Baouendi [AB95]).

Note also that for the $1D$ wave equation, the constraint $\xi_x \cdot \nabla_x \varphi(0) = 0$ is much more demanding and implies $\xi_x = 0$ and $\xi = 0$ if $\xi_t^2 = |\xi_x|^2$. This is natural since we can exchange the time and space variable. So the finite speed of propagation (or a more refined version of it) implies the unique continuation across any non characteristic hypersurface.

Theorem 2.3.2 will be proved in Section 2.3.3. Before this, let us describe the underlying geometry of the condition (2.28).

2.3.2 Geometric interpretation of pseudoconvexity in the case of real symbol of order 2

In this section, we explain the geometric content of the condition of Definition 2.3.1. For this we need to introduce the Hamiltonian flow of the symbol p_2. Recall that $\{p_2, \cdot\}$ is a derivation on $C^\infty(\Omega \times \mathbb{R}^n)$ (see Definition 1.3.7 and the remarks thereafter) and can thus be identified with the vector field

$$H_{p_2}(x,\xi) = \partial_\xi p_2(x,\xi) \cdot \partial_x - \partial_x p_2(x,\xi) \cdot \partial_\xi,$$

on $\Omega \times \mathbb{R}^n$. We denote by χ_s the associated flow, defined by

$$\begin{cases} \frac{d}{ds}\chi_s(x_0,\xi_0) = H_{p_2}\big(\chi_s(x_0,\xi_0)\big), \\ \chi_0(x_0,\xi_0) = (x_0,\xi_0), \end{cases} \tag{2.31}$$

and called the Hamiltonian flow of p_2. Remark that $H_{p_2}(p_2) = \{p_2, p_2\} = 0$ so that p_2 is preserved along the flow: $p_2 \circ \chi_s(x_0,\xi_0) = p_2(x_0,\xi_0)$. Note also that the flow χ_s is (at least) locally defined in (s,x,ξ) a neighborhood of $(0,x_0,\xi_0)$ according to the Cauchy-Lipschitz theorem.

If we now denote by $(x_s,\xi_s) = \chi_s$, that is $\chi_s(x_0,\xi_0) = \big(x_s(x_0,\xi_0), \xi_s(x_0,\xi_0)\big)$ and recall the definition of the Poisson bracket $\{p_2,\cdot\} = \partial_\xi p_2 \cdot \partial_x - \partial_x p_2 \cdot \partial_\xi$, (2.31) now reads

$$\begin{cases} \dfrac{d}{ds}x_s(x_0,\xi_0) = \partial_\xi p_2\big(\chi_s(x_0,\xi_0)\big), \\ \dfrac{d}{ds}\xi_s(x_0,\xi_0) = -\partial_x p_2\big(\chi_s(x_0,\xi_0)\big), \\ \big(x_s(x_0,\xi_0), \xi_s(x_0,\xi_0)\big)|_{s=0} = (x_0,\xi_0). \end{cases} \tag{2.32}$$

With these definitions in hand, we can now reformulate the strong pseudoconvexity condition of Definition 2.3.1. Namely, note that we have

$$\{p_2, \Psi\}(x_0,\xi) = H_{p_2}(\Psi)(x_0,\xi) = \frac{d}{ds}\Psi \circ x_s(x_0,\xi)|_{s=0},$$
$$\{p_2, \{p_2, \Psi\}\}(x_0,\xi) = H_{p_2}\big(H_{p_2}(\Psi)\big)(x_0,\xi) = \frac{d^2}{ds^2}\Psi \circ x_s(x_0,\xi)|_{s=0}.$$

Now, if for $\xi \in \mathbb{R}^n$ we define $c_\xi(s) = \Psi \circ x_s(x_0,\xi)$ (2.28) is equivalent to:

$$\text{For all } \xi \in \mathbb{R}^n \setminus \{0\}, \text{ we have:} \quad p_2(x_0,\xi) = 0 \text{ and } \dot{c}_\xi(0) = 0 \implies \ddot{c}_\xi(0) > 0.$$

This means that for all $\xi \in \mathbb{R}^n \setminus \{0\}$,

- if ξ is noncharacteristic ($p_2(x_0,\xi) \neq 0$), we don't care;
- if (ξ is characteristic and) the (projected) Hamiltonian curve $x_s(x_0,\xi)$ is not tangent to $S = \{\Psi = \Psi(x_0)\}$ at $s = 0$, we don't care;
- if ξ is characteristic and the curve $x_s(x_0,\xi)$ is tangent to $S = \{\Psi = \Psi(x_0)\}$, then it should have non-vanishing second derivative (tangency at order 2) and the curve $(x_s(x_0,\xi))_{s\in(-\varepsilon,\varepsilon)}$ should stay in $\{\Psi \geq \Psi(x_0)\}$.

♣ `picture!`

This excludes the following situations

- tangent characteristic curves staying in $\{\Psi \leq \Psi(x_0)\}$;
- contacts of higher order with the tangent at x_0.

2.3.3 Unique continuation: Proof of Theorem 2.3.2

In this section, we give the final proof of Theorem 2.3.2. ♣ `explain geometric convexification`

After the geometric preliminaries, it consists essentially in using Lemma 1.2.8 (which we reprove here in a different way).

Proof of Theorem 2.3.2. ♣ `draw a picture` We first remark that we may assume that $\Psi(x_0) = 0$ (up to changing Ψ into $\Psi - \Psi(x_0)$, which does not change the assumption), so that $S = \{\Psi = 0\}$. Let u be a C^∞ solution of $Pu = 0$ in Ω so that $u = 0$ on $\Omega \cap \{\Psi > 0\}$. The surface $S = \{\Psi = 0\}$ being strongly pseudoconvex at x_0, Proposition 2.2.5 allows to produce a new function Φ pseudoconvex for functions so that (after having changed Φ into $\Phi - 1$) $\{\Phi = 0\} = \{\Psi = 0\}$, $\{\Phi > 0\} = \{\Psi > 0\}$ and $\{\Phi < 0\} = \{\Psi < 0\}$.

Proposition 2.1.9 yields the existence of $\varepsilon > 0$, such that $\Phi_\varepsilon = \Phi - \varepsilon|x-x_0|^2$ satisfies the pseudoconvexity for functions. As a consequence of Proposition 2.1.7 and Theorem 2.1.1, it therefore satisfies the following properties

1. there exist $R > 0$, $C > 0$ and $\tau_0 > 0$ so that we have the following estimate
$$\tau^3 \left\|e^{\tau\Phi_\varepsilon} w\right\|_{L^2}^2 + \tau \left\|e^{\tau\Phi_\varepsilon}\nabla w\right\|_{L^2}^2 \leq C \left\|e^{\tau\Phi_\varepsilon} Pw\right\|_{L^2}^2, \tag{2.33}$$
for any $w \in C^\infty(B(x_0, R))$ and $\tau \geq \tau_0$.
2. there exists $\eta > 0$ so that $\Phi_\varepsilon(x) \leq -\eta$ for $x \in \{\Phi \leq 0\} \cap \{|x - x_0| \geq R/2\}$,
3. there exists a neighborhood $V \subset B(x_0, R/2)$ of x_0 so that $\Phi_\varepsilon(x) \geq -\eta/2$ for $x \in V$.

Property 1 is a consequence of Theorem 2.1.1, and R is fixed by that theorem.

Property 2 is true thanks to the parameter ε in the geometric convexification. Indeed, for $|x-x_0| \geq R/2$, we have $\Phi_\varepsilon(x) \leq \Phi(x) - \varepsilon R^2/4$. If $\Phi(x) \leq 0$, this implies $\Phi_\varepsilon(x) \leq -\varepsilon R^2/4$, so that we can take $\eta = -\varepsilon R^2/4$.

Property 3 is only a continuity argument since $\Phi_\varepsilon(x_0) = 0$.

Pick $\chi \in C_c^\infty(B(x_0, R))$ so that $\chi = 1$ on $B(x_0, R/2)$. We want to apply the Carleman estimate to $w = \chi u$ solution of $Pw = \chi Pu + [P, \chi]u = [P, \chi]u$. Notice that $[P, \chi]$ is a classical differential operator of order 1 with coefficients supported in the set $\{\frac{R}{2} \leq |x - x_0| \leq R\}$. As a consequence, using that $\operatorname{supp}(u) \subset \{\Phi \leq 0\}$, this implies
$$\operatorname{supp}([P, \chi]u) \subset \{\Phi \leq 0\} \cap \{\frac{R}{2} \leq |x - x_0| \leq R\},$$
where we have $\Phi_\varepsilon(x) \leq -\eta$ (according to Property 2). In particular, this implies $\left\|e^{\tau\Phi_\varepsilon} Pw\right\|_{L^2} \leq Ce^{-\tau\eta} \|u\|_{H^1(B(x_0,R))}$.

Moreover, since $\Phi_\varepsilon(x) \geq -\eta/2$ and $\chi = 1$ on V, we have
$$\tau^{3/2} \left\|e^{\tau\Phi_\varepsilon} w\right\|_{L^2} \geq \tau_0^{3/2} \left\|e^{\tau\Phi_\varepsilon} \chi u\right\|_{L^2(V)} \geq \tau_0^{3/2} \left\|e^{-\tau\eta/2} u\right\|_{L^2(V)} = \tau_0^{3/2} e^{-\tau\eta/2} \|u\|_{L^2(V)}.$$

So the Carleman estimate (2.33) implies
$$e^{-\tau\eta/2} \|u\|_{L^2(V)} \leq C \left\|e^{\tau\Phi_\varepsilon} Pw\right\|_{L^2} \leq Ce^{-\tau\eta} \|u\|_{H^1(B(x_0,R))}.$$

This gives $\|u\|_{L^2(V)} \leq Ce^{-\tau\eta/2} \|u\|_{H^1(B(x_0,R))}$ and $u = 0$ on V by letting τ tend to infinity.

Note finally that, in order for the result to hold for $u \in H^1(\Omega)$, we need to remark that a density argument shows that the Carleman estimate is still valid for all $w \in H^1(\Omega)$ such that $\operatorname{supp}(w) \subset B(x_0, R)$ and $Pw \in L^2(\Omega)$. Here, in case $u \in H^1(\Omega)$, we have $w = \chi u \in H^1(\Omega)$ with $\operatorname{supp}(w) \subset \operatorname{supp}(\chi) \subset B(x_0, R)$ and $Pw = 0 + [P, \chi]u \in L^2(\Omega)$ since $[P, \chi] \in \mathrm{Diff}^1(\Omega)$ and $u \in H^1(\Omega)$. Hence, the Carleman estimate applies and the remainder of the proof remains unchanged. ☐

2.3.4 Lowering regularity requirements

In this section, we explain how the regularity of the coefficients of P or the solution u can be lowered in different contexts.

Lowering regularity of the coefficients of P

Theorem 2.3.7. *The conclusions of Theorem 2.3.2 and Corollary 2.3.4 hold as well if the Assumption $u \in H^1_{\mathrm{loc}}(\Omega)$ and $Pu = 0$ in (2.29)-(2.30) is replaced by the following assumption: $u \in H^1_{\mathrm{loc}}(\Omega)$ is such that $Pu \in L^2_{\mathrm{loc}}(\Omega)$ and there is Ω' a neighborhood of x_0 and $C > 0$ such that*

$$|Pu|(x) \leq C \sum_{k=1}^{n} |D_k u|(x) + C|u|(x), \qquad \text{for almost every } x \in \Omega'. \tag{2.34}$$

In particular, the conclusions of Theorem 2.3.2 and Corollary 2.3.4 hold if the operator $P \in \mathrm{Diff}^2(\Omega)$ is replaced by

$$P = \sum_{i,j=1}^{n} a^{ij}(x) D_i D_j + \sum_{k=1}^{n} b_k(x) D_k + c(x), \tag{2.35}$$

with $a^{ij} \in C^\infty(\Omega)$ real-valued, $b_k, c \in L^\infty_{\mathrm{loc}}(\Omega)$. In this case, p_2 denotes $p_2(x,\xi) = \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j$.

Note that we have to re-define p_2 in the latter situation since $P \notin \mathrm{Diff}^2(\Omega)$. ♣ `write a proof if` $a^{ij} \in C^1(\Omega)$ `?` Note that (2.34) is no longer a PDE, but a slightly weaker "differential inequality". In the situation of Corollary 2.3.4, the ellipticity assumption at the point x_0 writes: there exists $c > 0$ so that

$$\sum_{i,j=1}^{n} a^{ij}(x_0)\xi_i\xi_j \geq c|\xi|^2, \text{ for all } \xi \in \mathbb{R}^n.$$

Proof of Theorem 2.3.7. The proof follows essentially the same as that of Theorem 2.3.2. The geometry is the same and the only point is how to get rid of the terms in the right hand-side of (2.34). We start from the same Carleman inequality (2.33), namely

$$\tau^3 \left\|e^{\tau\Phi_\varepsilon} w\right\|_{L^2}^2 + \tau \left\|e^{\tau\Phi_\varepsilon} \nabla w\right\|_{L^2}^2 \leq C \left\|e^{\tau\Phi_\varepsilon} P w\right\|_{L^2}^2, \tag{2.36}$$

and apply it again to $w = \chi u$. On the right hand-side, writing $Pw = P\chi u = [P,\chi]u + \chi P u$ and using (2.34), we obtain,

$$\begin{aligned}
\left\|e^{\tau\Phi_\varepsilon} P w\right\|_{L^2} &\leq \left\|e^{\tau\Phi_\varepsilon}[P,\chi]u\right\|_{L^2} + C\sum_{k=1}^{n} \left\|e^{\tau\Phi_\varepsilon}\chi D_k u\right\|_{L^2} + C\left\|e^{\tau\Phi_\varepsilon}\chi u\right\|_{L^2} \\
&\leq \left\|e^{\tau\Phi_\varepsilon}[P,\chi]u\right\|_{L^2} + C\sum_{k=1}^{n} \left(\left\|e^{\tau\Phi_\varepsilon} D_k \chi u\right\|_{L^2} + \left\|e^{\tau\Phi_\varepsilon}[D_k,\chi]u\right\|_{L^2}\right) + C\left\|e^{\tau\Phi_\varepsilon}\chi u\right\|_{L^2} \\
&\leq \left\|e^{\tau\Phi_\varepsilon}[P,\chi]u\right\|_{L^2} + C\sum_{k=1}^{n} \left\|e^{\tau\Phi_\varepsilon}[D_k,\chi]u\right\|_{L^2} + C\left\|e^{\tau\Phi_\varepsilon}\nabla w\right\|_{L^2} + +C\left\|e^{\tau\Phi_\varepsilon} w\right\|_{L^2}.
\end{aligned}$$

Now, the last two terms can be absorbed in the left hand-side of the Carleman estimate (2.36) for τ large enough, and we obtain

$$\tau^3 \left\|e^{\tau\Phi_\varepsilon} w\right\|_{L^2}^2 + \tau \left\|e^{\tau\Phi_\varepsilon}\nabla w\right\|_{L^2}^2 \leq C\left\|e^{\tau\Phi_\varepsilon}[P,\chi]u\right\|_{L^2}^2 + C\sum_{k=1}^{n}\left\|e^{\tau\Phi_\varepsilon}[D_k,\chi]u\right\|_{L^2}^2.$$

The important point here is that on the right hand-side, only derivatives of χ appear: the term $[D_k,\chi]u$ enjoys the same support properties as $[P,\chi]u$. Hence, from this point forward, we can follow the proof of Theorem 2.3.2 line by line. □

Lowering regularity of the solution u

♣ `to be written via propagation of singularities as a blackbox`

2.3.5 From local to global uniqueness (elliptic operators)

The unique continuation result that we obtain is only local. Yet, we could expect to iterate this result with a well chosen sequence of hypersurfaces. It turns out that it is not easy to do in the general case. Take for instance the wave equation. Suppose that you are given an open set ω and $T > 0$ so that a solution u satisfies

$$\begin{cases} (\partial_t^2 - \Delta)u &= 0 \text{ on } [0,T] \times \Omega \\ u &= 0 \text{ on } [0,T] \times \omega \end{cases}$$

The question of defining what is the "domain of dependence" for the unique continuation that can be obtained with our unique continuation using iterated pseudoconvex surfaces (as described in Remark 2.3.6) is not clear.

An easier situation is the elliptic case where the strong pseudoconvexity condition for the surface S is empty. This allows to obtain the following global result.

Theorem 2.3.8 (Global result in the elliptic case)**.** *Let Ω be a connected open set and P satisfying the assumptions of Corollary 2.3.4. Let u be a $H^1(\Omega)$ solution to $Pu = 0$ on Ω, that satisfies $u = 0$ on an arbitrary nonempty open set $\omega \subset \Omega$. Then, $u = 0$ on Ω.*

The proof uses the local result together with a connectedness argument.

Proof. We define $F = \text{supp}(u)$, which is a closed subset of Ω, and $\partial F = F \setminus \text{Int}(F) \subset \Omega$ the boundary of F. The proof proceeds in two steps: first proving that $\partial F = \emptyset$ by contradiction, and then concluding with a connectedness argument.

Let us first prove that $\partial F = \emptyset$. Assuming $\partial F \neq \emptyset$, there exists $x \in \partial F \subset \Omega$. Define R so that $B(x, R) \subset \Omega$. Take $x_1 \in \Omega \setminus F$ with $|x - x_1| < R/2$ (it exists since $x \in \partial F$). So, we have $u(y) = 0$ in a neighborhood of x_1 by definition of the support. Next define $r_1 = \sup\{r \in [0, R/2]; u(x) = 0 \text{ in } B(x_1, r)\}$. We know that $r_1 > 0$. So, we have obtained $u = 0$ in $B(x_1, r_1)$.

Assume $r_1 < R/2$. Since $|x - x_1| < R/2$ and $B(x, R) \subset \Omega$, $B(x_1, R/2) \subset \Omega$. So, we can apply Theorem 2.3.2 to any point $x_0 \in S(x_1, r_1)$ the sphere of radius r_1 and of center x_1 to get that for any $x_0 \in S(x_1, r_1)$, there exists r_{x_0} so that $u(y) = 0$ in $B(x_0, r_{x_0})$. Covering $S(x_1, r_1)$ by a finite number of such balls using the compactness of $S(x_1, r_1)$ we get one ε so that $u(y) = 0$ on $B(x_1, r_1 + \varepsilon)$ contradicting the definition of r_1. So, we have $r_1 = R/2$.

But since $|x - x_1| < R/2$, there exists a neighborhood of x included in $B(x_1, R/2)$. In particular, $u = 0$ in this neighborhood. This contradicts the assumption that $x \in \partial F$. As a consequence, we have obtained $\partial F = \emptyset$.

Now, since $\partial F = F \setminus \text{Int}(F) = \emptyset$, we have $F = \text{Int}(F)$ and is thus closed and open. Moreover, we have $F \neq \Omega$ since $\omega \cap F = \emptyset$. The connectedness of Ω then yields $F = \emptyset$, i.e. $u = 0$ on Ω. □

♣ `faire un dessin`

A first useful application of this result is to eigenfunctions.

Corollary 2.3.9. *Let $\Omega \subset \mathbb{R}^n$ be a connected open set. Denote by $-\Delta$ the Laplace operator. Assume $\psi \in H^1(\Omega)$ satisfies $-\Delta\psi = \lambda\psi$ on Ω and $\psi = 0$ on a nonempty open set $\omega \subset \Omega$. Then we have $\psi = 0$ on Ω.*

This means that eigenfunctions of the Laplace operator never vanish on a nonempty open set. The same result also holds for the Laplace-Beltrami operator $-\Delta_g$ on a Riemannian manifold $(\mathcal{M}, g)$.

2.4 Quantitative estimates and application to eigenfunctions

In this section, we want to give some estimates that quantify the unique continuation, that is some inequality proving in some sense the implication

$$\begin{cases} Pu & \text{small in} & \Omega, \\ u & \text{small in} & U \end{cases} \Longrightarrow u \text{ small in } \widetilde{U}.$$

described in the introduction. More precisely, we would like to have some estimates of the kind $\|u\|_{\widetilde{U}} \leq \varphi(\|u\|_U + \|Pu\|_\Omega, \|u\|_\Omega)$, with $\varphi(a,b) \to 0$ as $a \to 0$ when b is bounded. The ideal situation would be some linear estimate in a, independent on b. This is the case when the Cauchy problem is wellposed. For instance the wave operator across the surface $\{t = 0\}$. Yet, in those cases, Carleman estimates are generally not the best way to get uniqueness and good estimates. We will be interested in some case where the Cauchy problem is ill-posed and linear estimates are not expected to occur.

In our situation, the estimates that we can expect are more of Hölder type, that is $\varphi(a,b) = a^\theta b^{1-\theta}$. We could derive such local estimates in the general context of Theorem 2.3.2. We shall however restrict our attention to elliptic operators, for which the geometric setting is simpler, the globalization is possible, and which have several interesting applications.

2.4.1 Interlude: a semiglobal Carleman estimate

In Section 2.1 above, we proved:

- Φ is a pseudoconvex function for P at x_0 $\implies$ the Carleman estimate holds for all $u \in C_c^\infty(B(x_0, r))$ for some $r > 0$
- $S = \{\Psi = \Psi(x_0)\}$ is a strictly pseudoconvex surface for P at x_0 $\implies$ $\Phi := e^{\lambda\Psi}$ is a pseudoconvex function for P at x_0 for λ large.

These are local result at/near the point x_0. Here, we shall need similar results on a whole (relatively compact) open set Ω. We state the results without proofs.

We first state the analogue of the Carleman estimate of Theorem 2.1.1. Its proof is exactly the same as that of Theorem 2.1.1, except that we use the "semiglobal" Gårding inequality of Proposition 1.3.18 instead of the local one (Proposition 1.3.14).

Theorem 2.4.1 (Semiglobal Carleman estimate). *Let Ω be an open subset of $\mathbb{R}^n$ such that $\overline{\Omega}$ is compact. Let $P \in \mathrm{Diff}^2(\Omega)$ be a (classical) differential operator with* real-valued principal symbol *p_2 and $\Phi \in C^\infty(\overline{\Omega}; \mathbb{R})$.*

Then, the following statements are equivalent:

1. *There exist $C, \tau_0 > 0$ so that we have the following estimate*
$$\tau^3 \left\|e^{\tau\Phi}u\right\|_{L^2}^2 + \tau \left\|e^{\tau\Phi}\nabla u\right\|_{L^2}^2 \leq C \left\|e^{\tau\Phi}Pu\right\|_{L^2}^2, \quad \text{for all } u \in C_c^\infty(\Omega), \tau \geq \tau_0; \tag{2.37}$$
2. *There exist $C, \tau_0 > 0$ so that we have the following estimate*
$$\tau \|v\|_{H_\tau^1}^2 \leq C \|P_\Phi v\|_{L^2}^2, \quad \text{for all } v \in C_c^\infty(\Omega), \tau \geq \tau_0; \tag{2.38}$$
3. *There exist $C_1, C_2 > 0$ such that for all $(x, \xi, \tau) \in \overline{\Omega} \times \mathbb{R}^n \times \mathbb{R}_+^*$,*
$$\frac{C_1}{|\xi|^2 + \tau^2}\left[(\operatorname{Re} p_\Phi)^2 + (\operatorname{Im} p_\Phi)^2\right](x,\xi,\tau) + \frac{1}{\tau}\left\{\operatorname{Re} p_\Phi, \operatorname{Im} p_\Phi\right\}(x,\xi,\tau) \geq C_2\left(|\xi|^2 + \tau^2\right). \tag{2.39}$$
4. *The function Φ is pseudoconvex with respect to P on $\overline{\Omega}$, i.e. it satisfies*
$$\{p_2, \{p_2, \Phi\}\}(x,\xi) > 0, \qquad \text{if } p_2(x,\xi) = 0 \text{ and } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n \setminus 0; \tag{2.40}$$
$$\frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\}(x,\xi,\tau) > 0, \quad \text{if } p_\Phi(x,\xi,\tau) = 0 \text{ and } (x,\xi,\tau) \in \overline{\Omega} \times \mathbb{R}^n \times \mathbb{R}_*^+, \tag{2.41}$$

   where $p_\Phi(x,\xi,\tau) = p_2(x, \xi + i\tau d\Phi(x))$.

Proposition 2.4.2 (Analytic convexification). *Let Ω be an open set such that $\overline{\Omega}$ is compact, $P \in \mathrm{Diff}^2(\Omega)$ with real-valued principal symbol p_2 and $\Psi \in C^\infty(\overline{\Omega})$. Assume that for all $x \in \overline{\Omega}$, the oriented hypersurface $S_x = \{\Psi = \Psi(x)\}$ is strongly pseudoconvex with respect to P at x (Definition 2.2.1). Then there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, the function $\Phi = e^{\lambda\Psi}$ is pseudoconvex with respect to P on $\overline{\Omega}$ (in the sense of (2.40)-(2.41)).*

And thus, the Carleman estimate of Theorem 2.4.1 holds with weight Φ. The proof of this proposition is exactly the same as that of Proposition 2.2.5.

2.4.2 Local interpolation estimates for elliptic operators

In what follows, we will remain in the elliptic framework where any smooth surface is strongly pseudoconvex. It allows to simplify 2 important facts:

- we can choose compact surfaces, like spheres. One advantage is that we can skip the geometric convexification;
- the globalization is much easier, as we saw in Theorem 2.3.8 for the (qualitative) unique continuation result.

We first state the local result.

Theorem 2.4.3 (Local quantitative estimates for real elliptic operators of order 2)**.** *Let Ω, P be as in Corollary 2.3.4, $x_0 \in \Omega$. Let $r > 0$ so that $B(x_0, 3r) \subset \Omega$.*

Then, there exists $C > 0$, $0 < \delta < 1$ so that

$$\|u\|_{H^1(B(x_0,2r))} \leq C \left[\|u\|_{H^1(B(x_0,r))} + \|Pu\|_{L^2(B(x_0,3r))}\right]^\delta \|u\|^{1-\delta}_{H^1(B(x_0,3r))}$$

for any $u \in C^\infty(\Omega)$.

Note that in the context of complex analysis (for holomorphic functions on $\mathbb{C}$), this inequality is usually called the "Hadamard three spheres inequality" (sometimes also used for three lines) and can be proved with different methods, using analyticity. A remarkable fact here is that the regularity requirements on the coefficients are relatively low (and could be lowered with the same techniques).

Proof. We consider the relatively compact open set $U = B(x_0, 3r) \setminus \overline{B}(x_0, r/2)$. Denote $\Psi = -|x - x_0| \in C^\infty(\overline{U}; \mathbb{R})$. Moreover, we have $d\Psi \neq 0$ on $\overline{U}$, and P elliptic, so that for all $x \in \overline{U}$, the set $\{\Psi = \Psi(x)\}$ is a smooth strongly pseudoconvex hypersurface with respect to P at x. According to Proposition 2.4.2, the function $\Phi = e^{\lambda\Psi}$ is thus pseudoconvex on $\overline{U}$ for λ large enough, fixed from now on.

Theorem 2.4.1 applies and yields the existence of $C, \tau_0 > 0$ so that we have the following estimate

$$\tau^3 \left\|e^{\tau\Phi} w\right\|^2_{L^2} + \tau \left\|e^{\tau\Phi} \nabla w\right\|^2_{L^2} \leq C \left\|e^{\tau\Phi} P w\right\|^2_{L^2}, \quad \text{for all } w \in C^\infty_c(U), \tau \geq \tau_0. \tag{2.42}$$

Taking now $\chi \in C^\infty_c(U)$ so that $\chi = 1$ on $\overline{B}(x_0, 5r/2) \setminus B(x_0, r)$, we want to apply the estimate (2.42) to the function $w = \chi u \in C^\infty_c(U)$.

Concerning the right hand-side, we have $Pw = \chi P u + [P, \chi]u$ where $[P, \chi]$ is of order 1 supported in two different connected subsets of U:

- $|x - x_0| \in [r/2, r]$, where $\Phi \leq e^{-\lambda r/2} := \rho_3$. The corresponding term is bounded by
$$\left\|e^{\tau\Phi}[P, \chi]u\right\|_{L^2(|x-x_0|\in[r/2,r])} \leq Ce^{\tau\rho_3} \|u\|_{H^1(B(x_0,r))}$$
- $|x - x_0| \in [5r/2, 3r]$, where $\Phi \leq e^{-\lambda 5r/2} := \rho_1$. The corresponding term is bounded by
$$\left\|e^{\tau\Phi}[P, \chi]u\right\|_{L^2(|x-x_0|\in[5r/2,3r])} \leq Ce^{\tau\rho_1} \|u\|_{H^1(B(x_0,3r))}$$

The term corresponding to $\chi P u$ is bounded by $Ce^{\tau\rho_3} \|Pu\|_{L^2(B(0,3r))}$ since $\Phi \leq \rho_3$ on $\text{supp}(\chi)$.

The (square root of the) right hand-side of (2.42) is estimated from below by

$$\tau^{3/2} \left\|e^{\tau\Phi} w\right\|_{L^2} + \tau^{1/2} \left\|e^{\tau\Phi} \nabla w\right\|_{L^2} \geq c_0 \left\|e^{\tau\Phi}(\chi u)\right\|_{L^2(|x-x_0|\in[r,2r])} + c_0 \left\|e^{\tau\Phi} \nabla(\chi u)\right\|_{L^2(|x-x_0|\in[r,2r])}.$$

But since $\chi = 1$ on $|x - x_0| \in [r, 2r]$, we have, for a different constant C

$$\begin{aligned}
\tau^{3/2} \left\|e^{\tau\Phi} w\right\|_{L^2} + \tau^{1/2} \left\|e^{\tau\Phi} \nabla w\right\|_{L^2} &\geq C \left\|e^{\tau\Phi} \nabla u\right\|_{L^2(|x-x_0|\in[r,2r])} + C \left\|e^{\tau\Phi} u\right\|_{L^2(|x-x_0|\in[r,2r])} \\
&\geq Ce^{\tau\rho_2} \left[\|\nabla u\|_{L^2(|x-x_0|\in[r,2r])} + \|u\|_{L^2(|x-x_0|\in[r,2r])}\right]
\end{aligned}$$

where $\rho_2 := e^{-2\lambda r}$ is chosen so that $\Phi \geq \rho_2$ on the set $\{|x - x_0| \in [r, 2r]\}$.

Combining these estimates in (2.42), we finally obtain

$$e^{\tau\rho_2}\|u\|_{H^1(|x-x_0|\in[r,2r])}\leq Ce^{\tau\rho_1}\|u\|_{H^1(B(x_0,3r))}+Ce^{\tau\rho_3}\left[\|u\|_{H^1(B(x_0,r))}+\|Pu\|_{L^2(B(x_0,3r))}\right].$$

with $\rho_1<\rho_2<\rho_3$. This gives

$$\|u\|_{H^1(|x-x_0|\in[r,2r])}\leq Ce^{-C_1\tau}\|u\|_{H^1(B(x_0,3r))}+Ce^{C_2\tau}\left[\|u\|_{H^1(B(x_0,r))}+\|Pu\|_{L^2(B(x_0,3r))}\right].$$

with $C_1=\rho_2-\rho_1>0$ and $C_2=\rho_3-\rho_2>0$. Next, we apply the following Lemma of interpolation type, for which we postpone the proof.

Lemma 2.4.4. *Given $C_1,C_2,C_3,\tau_0>0$, there exists $C>0$ such that for all $a,b,c\geq 0$, we have*

$$\left.\begin{array}{l} a\leq e^{-C_1\tau}b+e^{C_2\tau}c, \quad \text{for all } \tau\geq\tau_0 \\ a\leq C_3 b.\end{array}\right\}\Longrightarrow a\leq Cb^{1-\delta}c^{\delta},\quad \delta=\frac{C_1}{C_1+C_2}.$$

Applying this Lemma with

$$a=\|u\|_{H^1(|x-x_0|\in[r,2r])}/C,\quad b=\|u\|_{H^1(B(x_0,3r))},\quad c=\left[\|u\|_{H^1(B(x_0,r))}+\|Pu\|_{L^2(B(x_0,3r))}\right],$$

and noticing that $a\leq b/C$, we obtain, with a different constant $C>0$,

$$\|u\|_{H^1(|x-x_0|\in[r,2r])}\leq C\|u\|^{1-\delta}_{H^1(B(x_0,3r))}\left[\|u\|_{H^1(B(x_0,r))}+\|Pu\|_{L^2(B(0,3r))}\right]^{\delta}.$$

Moreover, we have, if $C\geq 1$,

$$\|u\|_{H^1(B(x_0,r))}\leq\|u\|^{1-\delta}_{H^1(B(x_0,3r))}\|u\|^{\delta}_{H^1(B(x_0,r))}\leq C\|u\|^{1-\delta}_{H^1(B(x_0,3r))}\left[\|u\|_{H^1(B(x_0,r))}+\|Pu\|_{L^2(B(0,3r))}\right]^{\delta}.$$

This gives the expected result by summing up. ☐

Proof of the Lemma 2.4.4. We minimize in τ. The minimum is reached for $\tau=\frac{\ln\left(\frac{bC_1}{cC_2}\right)}{C_1+C_2}$. To simplify (actually, it is just changing b by bC_2/C_1), we apply the formula for $\tau_1=\frac{\ln\left(\frac{b}{c}\right)}{C_1+C_2}$. It gives, if $\tau_1\geq\tau_0$,

$$\begin{aligned} a &\leq e^{-\frac{C_1}{C_1+C_2}\ln\left(\frac{b}{c}\right)}b+e^{\frac{C_2}{C_1+C_2}\ln\left(\frac{b}{c}\right)}c \\ &\leq \left(\frac{b}{c}\right)^{-\delta}b+\left(\frac{b}{c}\right)^{1-\delta}c=2b^{1-\delta}c^{\delta}.\end{aligned}$$

where $\delta=\frac{C_1}{C_1+C_2}$.

In the case $\tau_1\leq\tau_0$, this means $\frac{b}{c}\leq e^{\tau_0(C_1+C_2)}$, so $b\leq C(\tau_0,C_1,C_2)c$. So, the assumption $a\leq C_3b$ gives $a\leq C_3b^{1-\delta}b^{\delta}\leq Cb^{1-\delta}c^{\delta}$ with a new constant depending on τ_0,C_1,C_2,C_3.

This gives the expected estimate in both cases with an appropriate constant $C>0$. ☐

2.4.3 Semi-global interpolation estimates for elliptic operators

Now, we want to obtain similar global estimates. This works since, as we shall see, interpolation estimates like that of Theorem 2.4.3 "propagate well".

Theorem 2.4.5 (Global quantitative estimates for real elliptic operators of order 2)**.** *Let $\Omega\subset\mathbb{R}^n$ be a connected open set. Let P be as in (2.35) with $a^{ij}\in C^\infty(\Omega)$ real-valued* elliptic *at all points of Ω, b_k, $c\in L^\infty_{\mathrm{loc}}(\Omega)$. Let K be a compact subset of Ω and ω be a non-empty open subset of Ω.*

Then, there exists $C>0$, $\delta\in(0,1)$ so that

$$\|u\|_{H^1(K)}\leq C\left[\|u\|_{H^1(\omega)}+\|Pu\|_{L^2(\Omega)}\right]^{\delta}\|u\|^{1-\delta}_{H^1(\Omega)} \tag{2.43}$$

for any $u\in C^\infty(\Omega)$.

Note that, at this point, we are not able to dominate the full $\|u\|_{H^1(\Omega)}$ norm. Indeed, we did not prove anything near the boundary $\partial\Omega$. This requires additional work, see Section 2.5 below.

Remark 2.4.6. Note that interpolation inequalities like (2.43) are trivial for $\delta = 0$, and would be very strong (but false!) for $\delta = 1$. Actually, the larger δ is, the stronger Inequality (2.43) is. More precisely, Inequality (2.43) for some δ_0 implies the same inequality for all $\delta \in (0, \delta_0]$ (with the same constant C). Indeed, since $\delta \leq \delta_0$, we can decompose $\|u\|_{H^1(K)} = \|u\|_{H^1(K)}^{\frac{\delta}{\delta_0}} \|u\|_{H^1(B(x_1,r_x))}^{1-\frac{\delta}{\delta_0}}$ where both exponent are nonnegative. Using (2.43) with δ_0 for the first term and $\|u\|_{H^1(K)} \leq \|u\|_{H^1(\Omega)}$ for the second, we obtain,

$$\begin{aligned}\|u\|_{H^1(K)} &\leq C\left(\left[\|u\|_{H^1(\omega)} + \|Pu\|_{L^2(\Omega)}\right]^{\delta_0} \|u\|_{H^1(\Omega)}^{1-\delta_0}\right)^{\frac{\delta}{\delta_0}} \|u\|_{H^1(\Omega)}^{1-\frac{\delta}{\delta_0}} \\ &\leq C\left[\|u\|_{H^1(\omega)} + \|Pu\|_{L^2(\Omega)}\right]^{\delta} \|u\|_{H^1(\Omega)}^{1-\delta},\end{aligned}$$

and (2.43) is valid for $\delta \leq \delta_0$. This will be used in the proof below.

Proof. Fix first $x_0 \in \omega$ and $r_0 > 0$ such that $B(x_0, r_0) \subset \omega$. By compactness, it is enough to prove the following statement: for any $x \in K$, there exist $0 < r_x < r_0$, $C_x > 0$, $\delta_x \in (0,1)$ so that $B(x, r_x) \subset \Omega$ and

$$\|u\|_{H^1(B(x,r_x))} \leq C\left[\|u\|_{H^1(B(x_0,r_0))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta_x} \|u\|_{H^1(\Omega)}^{1-\delta_x}. \tag{2.44}$$

Indeed, we recover K by a finite number of such balls $K \subset \cup_{i\in I} B(x_i, r_{x_i})$. Take $C = \max_{i\in I} C_{x_i}$ and $\delta = \min_{i\in I} \delta_{x_i}$. According to Remark 2.4.6, Inequality (2.44) is still true with δ_{x_i} replaced by δ (and C_{x_i} replaced by C), that is, for all $i \in I$,

$$\|u\|_{H^1(B(x_i,r_{x_i}))} \leq C\left[\|u\|_{H^1(B(x_0,r_0))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta} \|u\|_{H^1(\Omega)}^{1-\delta}.$$

By summing up over $i \in I$ and using the covering property, we would obtain

$$\|u\|_{H^1(K)} \leq C\left[\|u\|_{H^1(B(x_0,r_0))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta} \|u\|_{H^1(\Omega)}^{1-\delta}.$$

We are thus left to prove (2.44) for any $x \in K$.

We may also assume $\|Pu\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)}$. Indeed, if not, the result is straightforward since the right hand side is larger than $\|u\|_{H^1(\Omega)}$.

We will need the following geometric Lemma that will be prove later on.

Lemma 2.4.7. *Under the previous assumptions, let x_0 and $x_1 \in \Omega$ and $r_0 > 0$. Then, there exist $r \in (0, r_0]$, $N \in \mathbb{N}$ and a sequence of points y_k, $k = 0, \cdots, N$ so that*

- *$y_0 = x_0$, $y_N = x_1$.*
- *$\overline{B}(y_{k+1}, r) \subset B(y_k, 2r)$.*
- *$\overline{B}(y_k, 3r) \subset \Omega$.*

Assuming this Lemma, we prove recursively the following property: there exist C_k and $\delta_k \in (0,1)$ so that

$$\|u\|_{H^1(B(y_k,r))} \leq C_k\left[\|u\|_{H^1(B(x_0,r))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta_k} \|u\|_{H^1(\Omega)}^{1-\delta_k}. \tag{2.45}$$

- The property is true for $k = 0$ for $C = 1$ and any $\delta_k \in [0,1]$ since $\|u\|_{H^1(B(x_0,r))} \leq \|u\|_{H^1(\Omega)}$.
- Assume the property true for $k < N$. Theorem 2.4.3 applied at the point y_k (which can be applied since $\overline{B}(y_k, 3r) \subset \Omega$) gives $C > 0$, $0 < \delta < 1$ so that

$$\|u\|_{H^1(B(y_k,2r))} \leq C\left[\|u\|_{H^1(B(y_k,r))} + \|Pu\|_{L^2(B(y_k,3r))}\right]^{\delta} \|u\|_{H^1(B(y_k,3r))}^{1-\delta}.$$

Since $B(y_{k+1}, r) \subset B(y_k, 2r)$ and $B(y_k, 3r) \subset \Omega$, it gives

$$\|u\|_{H^1(B(y_{k+1},r))} \leq C \left[\|u\|_{H^1(B(y_k,r))} + \|Pu\|_{L^2(\Omega)}\right]^\delta \|u\|_{H^1(\Omega)}^{1-\delta}.$$

The assumption at step k and the fact that $\delta > 0$ gives

$$\|u\|_{H^1(B(y_{k+1},r))} \leq C \left[C_k \left[\|u\|_{H^1(B(x_0,r))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta_k} \|u\|_{H^1(\Omega)}^{1-\delta_k} + \|Pu\|_{L^2(\Omega)}\right]^\delta \|u\|_{H^1(\Omega)}^{1-\delta}.$$

Since we have assumed $\|Pu\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)}$, we have

$$\|Pu\|_{L^2(\Omega)} \leq \left[\|u\|_{H^1(B(x_0,r))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta_k} \|u\|_{H^1(\Omega)}^{1-\delta_k}.$$

So, we are left with some different constant C_{k+1}

$$\begin{aligned}\|u\|_{H^1(B(y_{k+1},r))} &\leq C_{k+1} \left[\left[\|u\|_{H^1(B(x_0,r))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta_k} \|u\|_{H^1(\Omega)}^{1-\delta_k}\right]^\delta \|u\|_{H^1(\Omega)}^{1-\delta} \\ &\leq C_{k+1} \left[\|u\|_{H^1(B(x_0,r))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta_k \delta} \|u\|_{H^1(\Omega)}^{1-\delta_k\delta}.\end{aligned}$$

So, it gives the result with $\delta_{k+1} = \delta_k \delta$.

□

Proof of Lemma 2.4.7. Since Ω is an open connected set of $\mathbb{R}^n$, it is connected by arc and we can find $\gamma : [0, 1] \to \Omega$ a continuous path in Ω so that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

The interval $[0, 1]$ is a compact set. Denote $d = \max_{t\in[0,1]}(\operatorname{dist}(\gamma(t), \Omega^c)$. We fix $r = \min\{d/4, r_0\}$. By compactness, γ is also uniformly continuous on $[0, 1]$. So, there exists $\varepsilon > 0$ so that $|t - t'| \leq \varepsilon$ implies $|\gamma(t) - \gamma(t')| \leq r/2$. We take $N = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1$ and define

$$\begin{aligned} y_k &= \gamma(k\varepsilon) \text{ for } k = 0, \cdots, N-1 \\ y_N &= x_1 = \gamma(1). \end{aligned}$$

This fulfills the expected criterium. For instance, $\overline{B}(y_{k+1}, r) \subset B(y_k, 2r)$ is fulfilled if $|y_{k+1} - y_k| < r$. This works since for $k \leq N-2$, $|y_{k+1} - y_k| = |\gamma((k+1)\varepsilon) - \gamma(k\varepsilon)| \leq r/2$ by the uniform continuity assumption. For the last step, $k = N - 1$, the same argument applies since $y_N = \gamma(1)$ and $y_{N-1} = \gamma(\lfloor 1/\varepsilon \rfloor \varepsilon)$. We observe that $|1 - \left\lfloor \frac{1}{\varepsilon} \right\rfloor \varepsilon| \leq \varepsilon$ because $\left|\frac{1}{\varepsilon} - \left\lfloor \frac{1}{\varepsilon} \right\rfloor\right| \leq 1$ by definition. □

Below, when using estimates like those of Theorem 2.4.5, we shall need to replace local H^1 norms of u, by local L^2 norms. This is possible at the cost of an additional L^2 estimate of Pu. This uses the ellipticity of P.

Lemma 2.4.8 (Local elliptic estimates)**.** *Let $\Omega \subset \mathbb{R}^n$ and $P \in \operatorname{Diff}^2(\Omega)$ be elliptic with real principal symbol. Then, for all $U, \tilde{U} \subset \Omega$ open sets with $\overline{U}$ compact and $\overline{U} \subset \tilde{U}$, there exists $C > 0$ such that for all $u \in C^\infty(\Omega)$, we have*

$$\|u\|_{H^1(U)} \leq C \|u\|_{L^2(\tilde{U})} + C \|Pu\|_{L^2(\tilde{U})}.$$

Proof. First recall (see Example 1.3.12) that P can be rewritten as

$$P = \sum_{i,j=1}^n D_i a^{ij}(x) D_j + R_1, \quad R_1 \in \operatorname{Diff}^1(\Omega),$$

where $a^{ij} = a^{ji}$. Denoting $A = (a^{ij})_{i,j}$, this may be rewritten, with $b \in C^\infty(\overline{\Omega}; \mathbb{C}^n)$ and $c \in C^\infty(\overline{\Omega}; \mathbb{C})$, as

$$P = -\operatorname{div}(A(x)\nabla\cdot) + b(x) \cdot \nabla + c(x).$$

Now, we let $\chi \in C_c^\infty(\tilde{U})$ have $\chi = 1$ on $\overline{U}$ (supp(χ) may be taken compact since $\overline{U}$ is), and remark that

$$\|u\|_{H^1(U)}^2 \leq \|\chi\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(U)}^2. \tag{2.46}$$

Hence, it suffices to estimate $\|\chi\nabla u\|_{L^2(\Omega)}$. Using the uniform ellipticity of the matrix $A(x)$, and integrating by parts, we have

$$\|\chi\nabla u\|_{L^2(\Omega)}^2 = \int_\Omega \chi^2|\nabla u|^2 \leq C\operatorname{Re}\int_\Omega \chi^2 A\nabla u\cdot\overline{\nabla u} = -C\operatorname{Re}\int_\Omega \operatorname{div}(\chi^2 A\nabla u)\overline{u}.$$

We next write

$$-\operatorname{div}(\chi^2 A\nabla u) = -\chi^2\operatorname{div}(A\nabla u) - 2\chi\nabla\chi\cdot A\nabla u = \chi^2 Pu - \chi^2(b(x)\cdot\nabla u + c(x)u) - 2\chi\nabla\chi\cdot A\nabla u$$

and deduce, using the Cauchy-Schwarz inequality,

$$\begin{aligned}\|\chi\nabla u\|_{L^2(\Omega)}^2 &\leq C\int_\Omega \chi^2|Pu||u| + C\int_\Omega \left|\chi^2(b(x)\cdot\nabla u + c(x)u)\overline{u}\right| + C\int_\Omega \left|u\chi\nabla\chi\cdot A\overline{\nabla u}\right| \\ &\leq C\Big(\|\chi Pu\|_{L^2(\Omega)}\|\chi u\|_{L^2(\Omega)} + \|\chi\nabla u\|_{L^2(\Omega)}\|\chi u\|_{L^2(\Omega)} \\ &\quad + \|\chi u\|_{L^2(\Omega)}^2 + \|\chi\nabla u\|_{L^2(\Omega)}\|u\nabla\chi\|_{L^2(\Omega)}\Big).\end{aligned}$$

Recalling that $\operatorname{supp}(\chi)\subset\tilde{U}$ and $\operatorname{supp}\chi$ is compact, we have obtained, for all $\varepsilon>0$, the estimate

$$\|\chi\nabla u\|_{L^2(\Omega)}^2 \leq C\|Pu\|_{L^2(\tilde{U})}^2 + \varepsilon\|\chi\nabla u\|_{L^2(\Omega)}^2 + \frac{C}{\varepsilon}\|u\|_{L^2(\tilde{U})}^2,$$

which yields the sought result when taking $\varepsilon = 1/2$ and recalling (2.46). □

2.4.4 Application to tunneling estimates for eigenfunctions

From the quantitative estimate of Theorem 2.4.5, we can already get some applications about spectral estimates of eigenfunctions of second order elliptic operators. We first describe the context.

We will denote $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ the n-dimensional torus. This can be seen as $[0,1]^n$ with the necessary identification of points. Functions on $\mathbb{T}^n$ can be seen as functions on $\mathbb{R}^n$ with periodic boundary conditions. Let $A = (a^{ij})_{i,j=1}^n$ a symmetric matrix with $a^{ij}\in C^\infty(\mathbb{T}^n)$ real-valued. We define the operator $-\Delta_A u = -\operatorname{div}(A\nabla u) = -\sum_{i,j}\partial_i\left(a^{ij}\partial_j u\right)$. Assume also that Δ_A is elliptic, that is there exists C so that

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq C|\xi|^2, \text{ for all } (x,\xi)\in\Omega\times\mathbb{R}^n.$$

The operator Δ_A is also symmetric, that is $(\Delta_A u, v)_{L^2(\mathbb{T}^n)} = (u,\Delta_A v)_{L^2(\mathbb{T}^n)}$ for $u,v\in C^\infty(\mathbb{T}^n)$.

We can check that it can be extended to a positive self-adjoint operator on $L^2(\mathbb{T}^n, dx)$, with domain $H^2(\mathbb{T}^n)$. Therefore, since the embedding of $H^2(\mathbb{T}^n)$ into $L^2(\mathbb{T}^n)$ is compact, the resolvent $(-\Delta_A + Id)^{-1}$ is well defined and compact on L^2.

All this allows to define an orthonormal basis of $L^2(\mathbb{T}^n)$. There exist some functions $\psi_i\in C^\infty(\mathbb{T}^n)$, $\lambda_i\in\mathbb{R}$ (actually $\lambda_i\geq 0$ since $-\Delta_A$ is positive) so that

- $(\psi_i)_{i\in\mathbb{N}}$ is an orthonormal basis of the Hilbert space $L^2(\mathbb{T}^n)$
- $-\Delta_A\psi_i = \lambda_i\psi_i$.

We refer e.g. to [Bre83] Chapters VI and IX for more details about this construction.

Remark 2.4.9. The same construction holds for a general compact Riemannian manifold $(\mathcal{M}, g)$. Let us recall briefly objects and notations from Riemannian geometry. We denote by $\langle\cdot,\cdot\rangle_g = g(\cdot,\cdot)$ the inner product in $T\mathcal{M}$. Remark that this notation omits to mention the point $x\in\mathcal{M}$ at which the inner product takes place: this allows to write $\langle X, Y\rangle_g$ as a function on $\mathcal{M}$ (the dependence on x is omitted here as well) when X and Y are two vector fields on $\mathcal{M}$. We also denote for a vector field X, $|X|_g^2 = \langle X, X\rangle_g$.

We recall that the Riemannian gradient ∇_g of a function f is defined by

$$\langle\nabla_g f, X\rangle_g = df(X), \quad \text{for any vector field } X.$$

For a function f on $\mathcal{M}$, we denote by $\int f = \int_{\mathcal{M}} f(x) d\operatorname{Vol}_g(x)$ its integral on $\mathcal{M}$, where $d\operatorname{Vol}_g(x)$ is the Riemannian density. We denote by div_g the associated divergence, defined on a vector field X by

$$\int u \operatorname{div}_g X = -\int \langle \nabla_g u, X\rangle_g, \quad \text{for all } u \in C_c^\infty(\operatorname{Int}(\mathcal{M})).$$

We denote by $\Delta_g = \operatorname{div}_g \nabla_g$ the associated (nonpositive) Laplace-Beltrami operator.

Let us now recall how these objects write in local coordinates. In coordinates, for f a smooth function and $X = \sum_i X^i \frac{\partial}{\partial x_i}$, $Y = \sum_i Y^i \frac{\partial}{\partial x_i}$ smooth vector fields on $\mathcal{M}$, we have

$$\begin{aligned}
\langle X, Y\rangle_g &= \sum_{i=1}^n g_{ij} X^i Y^j, \\
\nabla_g f &= \sum_{i,j=1}^n g^{ij} (\partial_j f) \frac{\partial}{\partial x_i}, \\
\int f &= \int f(x) \sqrt{\det g(x)} dx, \\
\operatorname{div}_g(X) &= \sum_{i=1}^n \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g} X_i \right), \\
\Delta_g f &= \sum_{i,j=1}^n \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g} g^{ij} \partial_j f \right).
\end{aligned}$$

where $(g^{-1})_{ij} = g^{ij}$. Note that in particular, we have in any local chart $\Delta_g \in \operatorname{Diff}^2$ with principal symbol $\sum_{i,j=1}^n g^{ij} \xi_i \xi_j$, which is real and elliptic.

Recall also that for f, h smooth functions and $X = \sum_i X^i \frac{\partial}{\partial x_i}$ a smooth vector field on $\mathcal{M}$, we have

$$\begin{aligned}
\nabla_g (fh) &= (\nabla_g f) h + f (\nabla_g h), \\
\operatorname{div}_g (fX) &= \langle \nabla_g f, X\rangle_g + f \operatorname{div}_g(X),
\end{aligned}$$

The following result states a weak delocalization property of eigenfunctions of $-\Delta_g$.

Theorem 2.4.10 (Tunneling estimates for eigenfunctions)**.** *Assume $(\mathcal{M}, g)$ is a compact connected Riemannian manifold, and let $\omega \subset \mathcal{M}$ be a nonempty open subset. Then, there exist C and $\kappa > 0$ such that for all $(\lambda, \psi_\lambda) \in \mathbb{R}^+ \times H^2(\mathcal{M})$ with*

$$-\Delta_g \psi_\lambda = \lambda \psi_\lambda,$$

we have

$$\|\psi_\lambda\|^2_{L^2(\mathcal{M})} \leq C e^{\kappa\sqrt{\lambda}} \|\psi_\lambda\|^2_{L^2(\omega)}.$$

This is a kind of observability estimate for eigenfunctions: the partial observation of eigenfunctions on the small set ω allows one to recover *at least* an $C^{-1}e^{-\kappa\sqrt{\lambda}}$ proportion of its total energy. Another formulation is to say that eigenfunctions leave at least an exponentially small mass on any nonempty open set. This theorem shall be generalized later on to linear combinations of eigenfunctions.

Proof. Let us consider the manifold $\mathbb{R} \times \mathcal{M}$, in which we denote the variable (x_0, x). The operator $P = -\partial_{x_0}^2 - \Delta_g$ is a second order differential operator with real principal symbol, which reads $\xi_0^2 + \sum_{i,j=1}^n g^{ij} \xi_i \xi_j$, hence is elliptic.

Define $u_\lambda(x_0, x) = e^{x_0\sqrt{\lambda}} \psi_\lambda(x)$. We verify that $P u_\lambda = -\lambda u_\lambda - e^{x_0\sqrt{\lambda}} \Delta_g \psi_\lambda = 0$. We want to apply Theorem 2.4.5 to P and u_λ on an open set $\Omega = (1/2, 5/2) \times \mathcal{M}$ and $K = [1, 2] \times \mathcal{M}$. Note that we are not exactly in the configuration of the Theorem since $\mathbb{R} \times \mathcal{M}$ is not an open set of $\mathbb{R}^n$. But it can be checked that Theorem 2.4.5 holds equally well on a manifold (being a consequence of a local result, proved in local charts♣).

A combination of Theorem 2.4.5 (with $K = [1,2] \times \mathcal{M}$, $\Omega = (1/2, 5/2) \times \mathcal{M}$) with Lemma 2.4.8 yields the existence of $C > 0$, $\delta \in (0,1)$ such that

$$\|u_\lambda\|_{H^1([1,2]\times\mathcal{M})} \leq C \left[\|u_\lambda\|_{L^2((1,2)\times\omega)} + \|Pu_\lambda\|_{L^2((0,3)\times\mathcal{M})}\right]^\delta \|u_\lambda\|^{1-\delta}_{H^1((1/2,5/2)\times\mathcal{M})}.$$

Using again Lemma 2.4.8 for the last term in the right handside, we obtain

$$\|u_\lambda\|_{H^1([1,2]\times\mathcal{M})} \leq C \left[\|u_\lambda\|_{L^2((1,2)\times\omega)} + \|Pu_\lambda\|_{L^2((0,3)\times\mathcal{M})}\right]^\delta \left(\|u_\lambda\|_{L^2((0,3)\times\mathcal{M})} + \|Pu_\lambda\|_{L^2((0,3)\times\mathcal{M})}\right)^{1-\delta}$$

and, since $Pu_\lambda = 0$, this yields

$$\|u_\lambda\|_{L^2([1,2]\times\mathcal{M})} \leq C \|u_\lambda\|^\delta_{L^2((1,2)\times\omega)} \|u_\lambda\|^{1-\delta}_{L^2((0,3)\times\mathcal{M})} \tag{2.47}$$

But on $[1,2]$, we have $e^{\sqrt{\lambda}s} \geq e^{\sqrt{\lambda}}$. So $\|u_\lambda\|_{L^2([1,2]\times\mathcal{M})} \geq e^{\sqrt{\lambda}} \|\psi_\lambda\|_{L^2(\mathcal{M})}$. Similarly, we have

$$\|u_\lambda\|_{L^2((1,2)\times\omega)} \leq e^{2\sqrt{\lambda}} \|\psi_\lambda\|_{L^2(\omega)}$$

and

$$\|u_\lambda\|_{L^2((0,3)\times\mathcal{M})} \leq \sqrt{3} e^{3\sqrt{\lambda}} \|\psi_\lambda\|_{L^2(\mathcal{M})}$$

Combining the above three estimates in (2.47), we finally obtain

$$e^{\sqrt{\lambda}} \|\psi_\lambda\|_{L^2(\mathcal{M})} \leq C \left(e^{2\sqrt{\lambda}} \|\psi_\lambda\|_{L^2(\omega)}\right)^\delta \left(e^{3\sqrt{\lambda}} \|\psi_\lambda\|_{L^2(\mathcal{M})}\right)^{1-\delta}.$$

This rewrites as

$$\|\psi_\lambda\|^\delta_{L^2(\mathcal{M})} \leq C e^{(2\delta + 3(1-\delta) - 1)\sqrt{\lambda}} \|\psi_\lambda\|^\delta_{L^2(\omega)},$$

which gives the expected result. □

To conclude this section, let us discuss briefly the optimality of this lower bound. The rate $e^{c\sqrt{\lambda}}$ is not always optimal. This can be seen in dimension one: on $\mathbb{T}^1$ with the flat metric, the L^2 norm of eigenfunctions (namely, $\psi_k(x) = e^{\pm ikx}$ and combinations of $\pm$) are uniformly bounded from below on any nonempty open set.

However, there are some particular geometric situations $(\mathcal{M}, g, \omega)$ where it is optimal. The next proposition provides with such an example.

There are also geometries in which $c(\lambda_j) = e^{c\sqrt{\lambda_j}}$ can be replaced by a uniform constant, or sometimes a power of λ_j or $\log(\lambda_j)$. The general question of making the link between the geometric properties of $(\mathcal{M}, g, \omega)$ and the appropriate $c(\lambda)$ is a widely open problem in spectral geometry.

Proposition 2.4.11. *Consider $\mathcal{M} = \mathbb{S}^2$ with*

$$\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3, x_1^2 + x_2^2 + x_3^2 = 1\} = \{x \in \mathbb{R}^3, |x| = 1\},$$

endowed with the metric g inherited from the Euclidean metric on $\mathbb{R}^3$. Assume $\omega \subset \mathbb{S}^2$ is such that $\overline{\omega} \cap \{x_3 = 0\} = \emptyset$. Then, there are constants $C, c > 0$ and a sequence of functions $(\psi_k)_{k\in\mathbb{N}}$ such that

$$-\Delta_g \psi_k = k(k+1)\psi_k, \quad \|\psi_k\|_{L^2(\mathbb{S}^2)} = 1,$$

and

$$\|\psi_k\|_{L^2(\omega)} \leq C e^{-ck}.$$

The eigenfunctions ψ_k constructed below are called equatorial spherical harmonics and are known to concentrate exponentially on the equator (which is a geodesic curve) given by $x_3 = 0$.

Proof. We set
$$u_k = P_k|_{\mathbb{S}^2}, \quad P_k(x_1, x_2, x_3) = (x_1 + ix_2)^k,$$
and first remark that we have $u_k \in C^\infty(\mathbb{S}^2)$. Next, we work in a particular coordinate set. We denote by $N = (0, 0, 1)$ and $S = (0, 0, -1)$, the north and south poles, and have coordinates :

$$\begin{array}{ccc} (0,\pi)\times\mathbb{S}^1 & \to & \mathbb{S}^2 \setminus \{N, S\} \\ (s,\theta) & \mapsto & (\sin s\cos\theta, \sin s\sin\theta, \cos s) \end{array}$$

Remark that $s(x) = \mathrm{dist}_g(x, N)$, for $x \in \mathbb{S}^2$. In these coordinates, the metric g is given by $ds^2 + (\sin s)^2 d\theta^2$, the Riemannian volume element is $d\,\mathrm{Vol}_g = \sin s ds d\theta$, the Laplace-Beltrami operator is given by

$$\Delta_g = \frac{1}{\sin(s)}\partial_s \sin(s)\partial_s + \frac{1}{\sin^2(s)}\partial_\theta^2,$$

and the sequence u_k is defined by

$$u_k(s,\theta) = (\sin s\cos\theta + i\sin s\sin\theta)^k = \sin(s)^k e^{ik\theta}.$$

Let us check by a direct computation that this is an eigenfunction. We have

$$\frac{1}{\sin^2(s)}\partial_\theta^2 u_k = -k^2 e^{ik\theta}\sin(s)^{k-2}$$

and

$$\begin{aligned}\frac{1}{\sin(s)}\partial_s \sin(s)\partial_s u_k &= \frac{e^{ik\theta}}{\sin(s)}\partial_s\left(k\cos(s)\sin(s)^k\right) = \frac{ke^{ik\theta}}{\sin(s)}\left(k\cos(s)^2\sin(s)^{k-1} - \sin(s)^{k+1}\right) \\ &= e^{ik\theta}\left(k^2(1-\sin(s)^2)\sin(s)^{k-2} - k\sin(s)^k\right).\end{aligned}$$

Adding these two identities yields

$$\Delta_g u_k = -k^2 e^{ik\theta}\sin(s)^k - ke^{ik\theta}\sin(s)^k = -k(k+1)u_k, \tag{2.48}$$

and u_k indeed satisfies the eigenfunction equation for the eigenvalue $k(k+1)$ on $\mathbb{S}^2\setminus\{N, S\}$, that is, almost everywhere on $\mathbb{S}^2$. Now compute

$$\begin{aligned}\frac{1}{2\pi}\|u_k\|^2_{L^2(\mathbb{S}^2)} &= \frac{1}{2\pi}\int_{\mathbb{S}^2}|u_k(x)|^2 d\,\mathrm{Vol}_g(x) = \frac{1}{2\pi}\int_{(0,\pi)\times\mathbb{S}^1}(\sin s)^{2k+1}ds d\theta \\ &= \int_0^\pi (\sin s)^{2k+1}ds = \int_{-1}^1 (1-x_3^2)^k dx_3 = \int_{-1}^1 e^{k\log(1-x_3^2)}dx_3 \\ &= (1+O(\frac{1}{k}))\int_{\mathbb{R}} e^{-kx_3^2}dx_3 = \sqrt{\frac{\pi}{k}}(1+O(\frac{1}{k})),\end{aligned}$$

and hence

$$c_k := \|u_k\|_{L^2(\mathbb{S}^2)} \sim 2^{1/2}\pi^{3/4}k^{-1/4}, \quad \text{as } k \to +\infty. \tag{2.49}$$

Finally, we have

$$\|u_k\|^2_{L^2(B(S,r))} = \|u_k\|^2_{L^2(B(N,r))} = 2\pi\int_0^r (\sin s)^{2k+1}ds \le \frac{\pi}{k+1}r^{2k+2},$$

which proves that $\|u_k\|^2_{L^2(\omega)} \le Ce^{-\kappa k}$ as soon as $\omega \subset B(N,r) \cup B(S,r)$ with $r < 1$, which is the case if $\overline{\omega}\cap\{x_3 = 0\} = \emptyset$. Combined with (2.48), (2.49), and the fact that $u_k \in C^\infty(\mathbb{S}^2)$, this proves the sought result for $\psi_k := c_k^{-1}u_k$. □

2.5 Estimates at the boundary and applications

♣ Beware! this section needs a strong lifting (and has not been taught in class). En particulier, il faut changer Ψ (surfaces) et Φ (carleman)

In this section, we prove quantitative unique continuation estimates (namely interpolation inequalities) for elliptic operators near a boundary (assuming e.g. Dirichlet boundary conditions). This follows from Carleman estimates near the boundary. This allows to generalize the tunneling estimate of Theorem 2.4.10 to linear combinations of eigenfunctions, and to prove the observability/controllability of the heat equation.

2.5.1 Estimates at the boundary for elliptic operators with Dirichlet conditions

Theorem 2.5.1 (Global quantitative estimates for real elliptic operator of order 2)**.** *Let Ω connected with smooth boundary, and P be as in Theorem 2.3.2. Let $\Gamma \subset \partial\Omega$ a non empty open subset of the boundary. Let K be a compact subset of $\overline{\Omega}$.*

Then, there exists $C > 0$, $0 < \delta < 1$ so that

$$\|u\|_{H^1(K)} \leq C \left[\|\partial_\nu u\|_{L^2(\Gamma)} + \|Pu\|_{L^2(\Omega)}\right]^{\delta} \|u\|_{H^1(\Omega)}^{1-\delta}$$

for any $u \in C^\infty(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$.

Obtaining the previous estimates follows a similar path as previously, except that we need to prove some Carleman estimates until the boundary. By some change of variables, it is always possible (see Lemma **??**) to get to the following situation.

We decompose $x \in \mathbb{R}^n$ with $x = (x', x_n)$ $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. The boundary $\partial\Omega$ becomes the set $\{x_n = 0\}$ and P is of the form $D_{x_n}^2 + r(x, D_{x'})$ where $r(x, D_{x'})$ is a family of operator depending on $x = (x', x_n)$, but with derivatives only in x'.

We denote $K_{r_0} = \mathbb{R}_+^n \cap B(x_0, r_0)$ and $C_c^\infty(K_{r_0})$ is the set of functions in $C^\infty(\overline{\mathbb{R}_+^n})$ supported in $B(x_0, r_0)$. The index $+$ in the norms means that it is taken on $\mathbb{R}_+^n$.

Theorem 2.5.2 (Local Carleman estimate)**.** *Let $r_0 > 0$ and $P = D_{x_n}^2 + r(x, D_{x'})$ be a differential operator of order two on a neighborhood of K_{r_0}, with real principal part, where $r(x, D_{x'})$ is a smooth x_n family of second order operators in the (tangential) variable x'.*

Let ψ be quadratic polynomial such that $\psi'_{x_n} \neq 0$ on K_{r_0} and

$$\{p, \{p, \psi\}\}(x, \xi) > 0, \quad \text{if } p(x, \xi) = 0, \quad x \in K_{r_0}, \quad \xi \neq 0; \tag{2.50}$$

$$\frac{1}{i\tau}\{\overline{p}_\psi, p_\psi\}(x, \xi) > 0, \quad \text{if } p_\psi(x, \xi) = 0, \quad x \in K_{r_0}, \quad \tau > 0, \tag{2.51}$$

where $p_\psi(x, \xi) = p(x, \xi + i\tau\nabla\psi)$.

Then, there exist $C > 0$, $\tau_0 > 0$ such that for any $\tau > \tau_0$, we have for all $u \in C_c^\infty(K_{r_0/4})$

$$\begin{aligned} \tau\|e^{\tau\psi}u\|_{1,+,\tau}^2 \quad &\leq \quad C\left(\left\|e^{\tau\Psi}Pu\right\|_{0,+}^2 + \tau^3|(e^{\tau\Psi}u)_{|x_n=0}|_0^2\right. \\ &\qquad \left. + \tau\left|\left(D(e^{\tau\Psi}u)\right)_{|x_n=0}\right|_0^2\right). \end{aligned} \tag{2.52}$$

If moreover $\partial_{x_n}\psi > 0$ for $(x', x_n = 0) \in K_{r_0}$, then we have for all $u \in C_c^\infty(K_{r_0/4})$ such that $u_{|x_n=0} = 0$,

$$\tau\|e^{\tau\Psi}u\|_{1,+,\tau}^2 \leq C\left\|e^{\tau\Psi}Pu\right\|_{0,+}^2. \tag{2.53}$$

Note that the Theorem applies to real elliptic operators, but also to wave type operators with the associated pseudoconvexity condition.

We give a proof of this theorem in the appendix. The general idea is the following.

We would like to apply the same reasoning as before. Yet, we have to be more careful about the Gårding inequality in the case of boundary. One possibility is to use symbolic calculus only in the tangential variable x' where integration by parts are allowed without boundary terms. But the integration by parts for D_n^2 and its conjugated operator produce some boundary terms that we need to take into account.

How to deal with boundary terms?

In the variable x_n, the operator is D_n^2 and the conjugated operator is explicit $(D_n + i\tau(\partial_n\psi))^2 = D_n^2 - \tau^2(\partial_n\psi)^2 + 2i\tau(\partial_n\psi)D_n - (\partial_n^2\psi)$. The integration by parts can be explicitely computed. What is not a priori obvious is that there is no term of order 2 and 3.

What will save us is that fact that the real part and the imaginary part don't have the same number of derivative in x_n. Decomposing $P_\psi = Q^r + iQ^i$ as before, we check that Q^r has 2 derivatives in x_n (D_n^2) while Q^i only has one $(2i\tau(\partial_n\psi)D_n)$.

Let us look at each term, using the integration by part formula $(f, D_n g) = (D_n f, g) + i(f, g)_{x_n=0}$

- integrating by part of $(Q^r u, Q^i u) = (Q^i Q^r u, u) + boundaryterms$: the worst terms should come from the integration by part of $(D_n^2 u, (\partial_n\psi)D_n u)$ and we expect the boundary term to be of the form $i(D_n^2 u, (\partial_n\psi)u)_{x_n=0}$. For instance, the boundary term corresponding to the to this should be of the form
- integrating by part of $(Q^i u, Q^r u) = (Q^r Q^i u, u) + boundaryterms$: the worst terms should come from the integration by part of

$$
\begin{aligned}
((\partial_n\psi)D_n u, D_n^2 u) &= (D_n\left[(\partial_n\psi)D_n u\right], D_n u) + i((\partial_n\psi)D_n u, D_n u)_{x_n=0} \\
&= (D_n^2\left[(\partial_n\psi)u\right], D_n u) + i(D_n\left[(\partial_n\psi)D_n u\right], u)_{x_n=0} + i((\partial_n\psi)D_n u, D_n u)_{x_n=0}
\end{aligned}
$$

This gives the boundary terms

$$
i((\partial_n\psi)D_n^2 u, u)_{x_n=0} + i((D_n\partial_n\psi)D_n u, u)_{x_n=0} + i((\partial_n\psi)D_n u, D_n u)_{x_n=0}.
$$

The two terms of order 2 cancel. So we are left with some terms of order 1 that come into the boundary terms that can be handled by the Carleman method. The true computation contains some more terms, but with less derivative in x_1.

How to deal with interior terms?

The interior terms are more or less the same as in the boundaryless case. So, we could expect that their symbol satisfy the same positivity condition. Yet, we would like to use only a tangential Gårding inequality, that is only in the derivatives in the variable x' (with symbol only depending on the cotangent variable ξ'.)

The idea is to perform a kind of euclidian division of the commutator $i[Q^r, Q^i]$ by D_n. Indeed, we can factorize $i[Q^r, Q^i] = \tau\left[C_0 D_n^2 + C_1 D_n + C_2\right]$ where C_i are tangential operators (we have also used that Q^i can be written $\tau\widetilde{Q^i}$). Moreover, since Q^r contains some derivative in x_n with main coefficient D_n^2 while the main derivative of Q^i in x_n is $2i\tau(\partial_n\psi)D_n$ where $(\partial_n\psi) \neq 0$. This allows to perform a similar "euclidian division" with Q^r, Q^i which allows to write

$$
i[Q^r, Q^i] = \tau D_0 Q^r + D_1 Q^i + \tau D_2.
$$

Since the terms $\tau D_0 Q^r$ are in some sense weaker than $\|Q^r\|_{L^2}$ (and the same for Q^i), we are left with some tangential operator. D_2 is not always positive, but the final task is to transfer the information we have on p_ψ to this tangential operator.

2.5.2 Application to spectral estimates II: linear combinations of eigenfunctions

Using the boundary estimates of Theorem 2.5.1, it is possible to get a more precise result. Actually, the previous result remain true not only for eigenfunctions, but also for finite sum of eigenfunction. Since now, the stability estimate is still true for an open set with boundary, we state the result for an elliptic operator P with the Dirichlet boundary conditions. The framework will be quite similar to the previous one

Let Ω be a smooth compact open set with boundary. Let $A = (a^{ij})_{i,j=1}^n$ a symmetric matrix with $a^{ij} \in C^\infty(\Omega)$ real-valued. We define the operator $Pu = -div(A\nabla u) = -\sum_{i,j}\partial_i\left(a^{ij}\partial_j u\right)$. We consider (and we will still denote it P the selfadjoint extension of P associated to the Drichlet boundary condition, that is $u = 0$ on $\partial\Omega$. We use the same notation ψ_j and λ_j the eigenfunctions and eigenvalues.

Theorem 2.5.3. *Under the previous assumptions on P and the related ψ_i, λ_i. Let ω be an open subset of Ω. There exists C and $c>0$ so that we have the estimate uniform in λ*

$$\|u\|^2_{L^2(\Omega)} \leq Ce^{c\sqrt{\lambda}} \|u\|^2_{L^2(\omega)}.$$

for any $u=\sum_{\lambda_j\leq\lambda} u_j\psi_j$.

Proof. As before, we consider the elliptic operator $Q=-\partial_s^2+P$ the operator defined on $\mathbb{R}_s^+\times\Omega$.

Define $f(s,x)=\sum_{i,\lambda_j\leq\lambda} u_j \frac{\sinh(\sqrt{\lambda_j}s)}{\sqrt{\lambda_j}}\psi_j(x)$ and we easily verify that it satisfies $Qf=0$ and $f=0$ on $\mathbb{R}\times\partial\Omega$ and $\{0\}\times\Omega$. Theorem 2.5.1 gives, if we take $\Gamma=\{0\}\times\omega$. Note that we can assume without loss of generality that ω is far from $\partial\Omega$, in order to avoid problems with "corners" at the points $\{0\}\times\partial\Omega$. A weak form of the inequality is then

$$\|f\|_{L^2([0,1]\times\Omega)} \leq C\|\partial_s f\|^{\delta}_{L^2(\{0\}\times\omega)} \|f\|^{1-\delta}_{H^1([0,2[\times\Omega)}$$

We have $\partial_s f(0,x)=u(x)$, so $\|\partial_s f\|_{L^2(\{0\}\times\omega)}=\|u\|_{L^2(\omega)}$.

As before, using Parseval identity

$$\begin{aligned}
\|f\|^2_{H^1([0,2[\times\Omega)} &\leq \|f\|^2_{L^2([0,2[\times\Omega)}+\|\partial_s f\|^2_{L^2([0,2[\times\Omega)}+\int_0^2(-\Delta f,f)_{L^2(\Omega)}\\
&\leq C\int_0^2\sum_{\lambda_j\leq\lambda}|u_j|^2\left(\cosh(\sqrt{\lambda_j}s)^2+\sinh(\sqrt{\lambda_j}s)^2\right)\,ds\\
&\leq Ce^{c\sqrt{\lambda}}\sum_{\lambda_j\leq\lambda}|u_j|^2\leq Ce^{c\sqrt{\lambda}}\|u\|_{L^2(\Omega)}.
\end{aligned}$$

And similarly

$$\begin{aligned}
\|f\|^2_{L^2([0,1[\times\Omega)} &\geq \|f\|^2_{L^2([0,1[\times\Omega)}\\
&\geq C\int_0^1\sum_{\lambda_j\leq\lambda}|u_j|^2\frac{\sinh(\sqrt{\lambda_j}s)}{\lambda_j}^2\,ds\\
&\geq C\sum_{\lambda_j\leq\lambda}|u_j|^2\int_0^{\sqrt{\lambda_j}}\frac{\sinh(y)^2}{\lambda_j^{3/2}}dy\geq C'\|u\|^2_{L^2(\Omega)}.
\end{aligned}$$

So, we obtain, with some different constants C, c

$$\|u\|_{L^2(\Omega)}\leq Ce^{c\sqrt{\lambda}}\|u\|^{\delta}_{L^2(\omega)}\|u\|^{1-\delta}_{L^2(\Omega)}.$$

This gives the result. □

This type growth of the type $e^{\sqrt{\lambda}}$ is optimal whatever the geometry if $\overline{\omega}\neq\Omega$. See [LRL12].

2.5.3 Application to the controllability of the heat equation

Our previous Theorem gives immediatly the following corollary for solutions of the heat equation at low frequency.

Corollary 2.5.4. *Under the previous assumptions on P and the related ψ_i, λ_i. Let ω be an open subset of Ω. There exist C and $c>0$ so that we have the estimate uniform in $\lambda\geq 0$ and $T>0$*

$$\|u(T)\|^2_{L^2(\Omega)}\leq Ce^{c\sqrt{\lambda}}\frac{1}{T}\int_0^T\|u(t)\|^2_{L^2(\omega)}\,dt,$$

for any $f=\sum_{\lambda_j\leq\lambda} f_j\psi_j$ with u solution of the heat equation

$$\left\{\begin{array}{rcl}
\partial_t u-\Delta u &=& 0 \text{ on } [0,T]\times\Omega\\
u &=& 0 \text{ on } [0,T]\times\partial\Omega\\
u(0,x) &=& f(x) \text{ on } \Omega
\end{array}\right.$$

Proof. The decay of the energy gives $\|u(T)\|^2_{L^2(\Omega)} \leq \|u(t)\|^2_{L^2(\Omega)}$ for any $0 \leq t \leq T$. Moreover, for any $t \in [0,T]$, the spectral estimates can be written $\|u(t)\|^2_{L^2(\Omega)} \leq Ce^{c\sqrt{\lambda}} \|u\|^2_{L^2(\omega)}$. Integrating in time gives the previous estimates and using

$$T\|u(T)\|^2_{L^2(\Omega)} \leq \int_0^T \|u(t)\|^2_{L^2(\Omega)} \leq Ce^{c\sqrt{\lambda}} \int_0^T \|u\|^2_{L^2(\omega)}.$$

□

We will prove some observability estimate for the heat equation on a bounded domain. It is known that some observability estimates are equivalent to some result of control.

We have chosen to prove the observability and then to deduce the related result of control.

In the original paper, the idea was the following:

- Use the "observability inequality" of Theorem 2.5.3 to get some result about control of the low frequency up to λ in time $T/4$ with a cost $\approx \frac{4}{T}e^{\sqrt{\lambda}}$. This allows to control the λ first frequency to zero.
- Use the decay of the heat equation on the remaining eigenvalues to get decay of $e^{-\lambda T/4}$ in time $T/4$.
- Iterate the result on dyadic times and tending to T.

The important fact is that the power $1/2$ in λ in $e^{c\sqrt{\lambda}}$ is strictly smaller than the exponential decay $e^{-\lambda T/4}$. We will work directly on the observability estimate, but still using the decay provided by the heat equation.

Theorem 2.5.5 (Observability for the heat equation)**.** *Let $\omega \subset \Omega$ a non empty open set and $T > 0$. Then, there exists $C > 0$ so that we have the estimate*

$$\|u(T)\|^2_{L^2} \leq C\int_0^T \|u\|^2_{L^2(\omega)}$$

for any u solution of

$$\begin{cases} \partial_t u - \Delta u &= 0 \text{ on } [0,T]\times\Omega \\ u &= 0 \text{ on } [0,T]\times\partial\Omega \\ u(0,x) &= u_0(x) \text{ on } \Omega \end{cases}$$

with $u_0 \in L^2(\Omega)$.

Proof. The idea is that our spectral estimate gives good estimates only when there are few high frequencies, that is after the decay of the heat operator have operated, that is close to times T.

We will divide the interval $[0,T]$ as the union of the intervals $[T_{k+1}, T_k]$ with $T_0 = T$, $T_{k+1} = T_k - T2^{-k}$. We check that T_k converges to $T - \sum_{k\in\mathbb{N}^*} T2^{-k} = 0$. To simplify the notations, we denote $L_k = T2^{-k}$ the length of the interval.

For each interval $[T_{k+1}, T_k]$, we will select a frequency cutoff μ_k and decompose

$$u = u_{k,L} + u_{k,H} = \sum_{\lambda_j \leq \mu_k} + \sum_{\lambda_j > \mu_k}$$

We will cut $[T_{k+1}, T_k]$ in two pieces, $[T_{k+1}, T_{k+1} + L_k/2]$ where we only use the damping and $[T_{k+1} + L_k/2, T_k]$ where we observe (using that the high frequency have been damped). We apply Corollary 2.5.4 on $[T_{k+1} + L_k/2, T_k]$

$$\|u_{k,L}(T_k)\|^2_{L^2(\Omega)} \leq Ce^{c\sqrt{\mu_k}} \frac{2}{L_k} \int_{T_{k+1}+L_k/2}^{T_k} \|u_{k,L}(t)\|^2_{L^2(\omega)}\ dt. \tag{2.54}$$

So, by triangular inequality, noticing that the error we do from the cut off in frequency is small. For low frequencies, we simply write $\|u_{k,L}(t)\|_{L^2(\omega)} \leq \|u(t)\|_{L^2(\omega)} + \|u_{k,H}(t)\|_{L^2(\omega)}$ where, for high frequencies,

$$\|u_{k,H}(t)\|_{L^2(\omega)} \leq \|u_{k,H}(t)\|_{L^2(\Omega)} \leq \|u_{k,H}(T_{k+1} + L_k/2)\|_{L^2(\Omega)} \leq e^{-\mu_k L_k/2} \|u(T_{k+1})\|_{L^2(\Omega)} \tag{2.55}$$

where we have used the damping of high frequency.

Moreover, we have similarly

$$\|u_{k,H}(T_k)\|_{L^2(\omega)} \leq e^{-\mu_k L_k/2} \|u(T_{k+1})\|_{L^2(\Omega)}. \tag{2.56}$$

So, putting together (2.54), (2.55) and (2.56), we get

$$\begin{aligned}
\|u(T_k)\|^2_{L^2(\Omega)} &\leq Ce^{c\sqrt{\mu_k}} \frac{2}{L_k} \int_{T_{k+1}+L_k/2}^{T_k} \|u(t)\|^2_{L^2(\omega)} \ dt + Ce^{c\sqrt{\mu_k}} e^{-\mu_k L_k/2} \|u(T_{k+1})\|_{L^2(\Omega)} \\
&\leq Ce^{c\sqrt{\mu_k}} \frac{2}{L_k} \int_0^T \|u(t)\|^2_{L^2(\omega)} \ dt + Ce^{c\sqrt{\mu_k}} e^{-\mu_k L_k} \|u(T_{k+1})\|^2_{L^2(\Omega)}.
\end{aligned}$$

Now, we can choose μ_k. Recall that $L_k = T2^{-k}$ converges to zero. Pick for instance, $\mu_k = \sqrt{C_1} L_k^{-2}$ with C_1 large. We have $\mu_k L_k = \sqrt{C_1}\sqrt{\mu_k}$ and $\mu_{k+1} = 2\mu_k$. If C_1 is large enough, we have

$$Ce^{c\sqrt{\mu_k}} e^{-\mu_k L_k} \leq e^{-3c\sqrt{\mu_{k+1}}}.$$

Indeed,

$$-c\sqrt{\mu_k} + \mu_k L_k - 3c\sqrt{\mu_{k+1}} \quad = \quad \sqrt{\mu_k}\left(\sqrt{C_1} - c(1+3\sqrt{2})\right) = C_1 2^k \left(C_1 - c(1+3\sqrt{2})\right).$$

This can be made arbitrary large uniformly for $k \in \mathbb{N}$. Once C_1 and μ_k are fixed, we have one constant C so that

$$e^{c\sqrt{\mu_k}} \frac{2}{L_k} \leq Ce^{2c\sqrt{\mu_k}}$$

So, we obtain

$$\|u(T_k)\|^2_{L^2(\Omega)} \quad \leq \quad Ce^{2c\sqrt{\mu_k}} \int_0^T \|u(t)\|^2_{L^2(\omega)} \ dt + e^{-3c\sqrt{\mu_{k+1}}} \|u(T_{k+1})\|^2_{L^2(\Omega)}.$$

$$e^{-3c\sqrt{\mu_k}} \|u(T_k)\|^2_{L^2(\Omega)} \quad \leq \quad Ce^{-c\sqrt{\mu_k}} \int_0^T \|u(t)\|^2_{L^2(\omega)} \ dt + e^{-3c\sqrt{\mu_{k+1}}} \|u(T_{k+1})\|^2_{L^2(\Omega)}.$$

Denoting $z_n = e^{-3c\sqrt{\mu_k}} \|u(T_k)\|^2_{L^2(\Omega)}$, we get

$$z_n - z_{n+1} \leq Ce^{-c\sqrt{\mu_k}} \int_0^T \|u(t)\|^2_{L^2(\omega)} \ dt.$$

We recognize a telescopic series and that $e^{-c\sqrt{\mu_k}}$ is summable. So, by summing up, we get with a new constant, uniform in $k \in \mathbb{N}$,

$$e^{-3c\sqrt{\mu_0}} \|u(T)\|^2_{L^2(\Omega)} - e^{-3c\sqrt{\mu_k}} \|u(T_k)\|^2_{L^2(\Omega)} \leq \widetilde{C} \int_0^T \|u(t)\|^2_{L^2(\omega)} \ dt$$

Since $\|u(T_k)\|_{L^2(\Omega)}$ is bounded, $e^{-3c\sqrt{\mu_k}} \|u(T_k)\|^2_{L^2(\Omega)}$ converges to zero, which gives the result. □

Theorem 2.5.6 (Control to zero of the heat equation)**.** *Let $\omega \subset \Omega$ a non empty open set and $T > 0$. Let $u_0 \in L^2(\Omega)$. Then, there exists $g \in L^2([0,T], L^2(\omega))$ so that the solution of*

$$\begin{cases} \partial_t u - \Delta u &= g \text{ on } [0,T] \times \Omega \\ u &= 0 \text{ on } [0,T] \times \partial\Omega \\ u(0,x) &= u_0(x) \text{ on } \Omega \end{cases} \tag{2.57}$$

satisfies $u(T) = 0$.

Proof. We consider the dual to the heat equation, v is a solution of

$$\begin{cases} -\partial_t v - \Delta v &= 0 \text{ on } [0,T]\times\Omega \\ v &= 0 \text{ on } [0,T]\times\partial\Omega \\ v(T,x) &= v_T(x) \text{ on } \Omega \end{cases} \tag{2.58}$$

This is actually exactly the backward heat equation. It is made interesting, because at least for smooth solutions u and v with Dirichlet boundary conditions, we have the formula that can be easily obtained by multiplying the equation (2.57) by v (consider v real-valued for simplicity), integrating over $[0,T]\times\Omega$ and integrating by parts

$$\int_\Omega u(T)v(T) - \int_\Omega u(0)v(0) = \int_0^T \int_\Omega gv.$$

The formula can also be extended to the case where $u_0 \in L^2(\Omega)$, $g \in L^2([0,T];L^2(\Omega))$, $v_T \in L^2(\Omega)$ by a density argument.

Our hope if $u(T) = 0$ would be to get $\int_0^T gv = \int_\Omega u_0 v(0)$. Reciprocally, we can check that if $\int_0^T gv = \int_\Omega u_0 v(0)$ for any solution of (2.58) with $v_T \in L^2(\Omega)$, then $u(T) = 0$.

Now, consider the quadratic form

$$a(v_T, \widetilde{v}_T) = \int_0^T \int_\omega v\widetilde{v}dx\ dt.$$

where v, $\widetilde{v}$ are the associated solutions to (2.58). a is well defined for v_T, $\widetilde{v}_T \in L^2(\Omega)$ and defines a positive quadratic form. Our observability estimates says that it is a scalar product. Yet, it is weaker than the $L^2(\Omega)$ norm. We define the completion $\overline{H}$ of $L^2(\Omega)$ with respect to this norm.

Define the linear form

$$l(v_T) = \int_\Omega u_0 v(0).$$

Our observability estimates can be written

$$\|v(0)\|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |v|^2 dx\ dt.$$

This says exactly that l is linear continuous in $\overline{H}$, since $u_0 \in L^2(\Omega)$. By the Riesz representation (or Lax-Milgram), there exists $v_T^{u_0} \in \overline{H}$ so that

$$a(v_T, v_T^{u_0}) = l(v_T)$$

for all $v_T \in \overline{H}$.

The application $\theta : L^2(\Omega) \mapsto L^2([0,T]\times\omega)$ defined by $\theta(v_T) = v_{|[0,T]\times\omega}$ where v is solution of (2.58) is well defined in $L^2(\Omega)$, but also bounded for the norm a on $\overline{H}$. Therefore, it can be extended to $\overline{H}$.

Take $g = \theta(v_T^{u_0}) \in L^2([0,T]\times\omega)$. By choice, we have

$$\int_0^T \int_\Omega g\theta(v_T) = l(v_T)$$

for any $v_T \in \overline{H}$. If we take in particular $v_T \in L^2(\Omega)$, this gives

$$\int_0^T \int_\Omega g\theta(v_T) = \int_\Omega u_0 v(0)$$

for v solution of (2.58). This gives the expected result. □

2.6 Further results and problems

2.6.1 The general Theorem of Hörmander

Remark 2.6.1 (Regarding the Carleman estimate)**.** In the elliptic case, the "trick" of factorisation by τ of the imaginary part of Q_I can be avoided. Indeed, close to $\{\tau = 0\}$ the symbol p_Φ is actually close to p and is therefore non zero.

Note that a more general assumption for treating the behavior of $\{\overline{p_\Phi}, p_\Phi\}$ close to $\{\tau = 0\}$ is to use the principal normality assumption

$$|\{\overline{p}, p\}| \leq C|p||\xi|^{m-1}.$$

The inequality is obviously fulfilled in the following two situations:

- elliptic operators (not necessarily with real-valued coefficients) in which case the ellipticity implies $|p||\xi|^{m-1} \geq C|\xi|^{2m-1}$
- operators with real-valued principal symbol in which case $\{\overline{p}, p\} = 0$.

In the more general case of principal normality, for the behavior of $\frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\}$, close to $\tau = 0$, the inequality allows to take advantage of the term $|p_\phi|^2$ for proving some inequality related to (2.3). In that case, a variant of Lemma 2.1.6 remains true, but only close to the set $\{p_\Phi = 0\}$.

We refer to Section 2.6.1 for the statemenent of the Theorem and to Hörmander [Hör94, Sections 28.3-28.4] for the proof.

Theorem 2.6.2 (Hörmander's theorem)**.** *Let Ω an open set of $\mathbb{R}^n$ and $x_0 \in \Omega$. Let P be a differential operator of order m, possibly having complex-valued coefficients, having $C^\infty(\Omega)$ principal symbol and all coefficients in $L^\infty_{loc}(\Omega)$. Assume that P is principally normal, that is the principal symbol p of P satisfies: for any compact K of Ω, there is $C > 0$ such that*

$$|\{\overline{p}, p\}| \leq C|p||\xi|^{m-1}.$$

for all $(x, \xi) \in K \times \mathbb{R}^n$.

Let $\Phi \in C^2(\Omega)$ real-valued so that $\nabla\Phi(x_0) \neq 0$. Assume that it satisfies

$$\begin{aligned} &\operatorname{Re}\{\overline{p}, \{p, \Phi\}\}(x_0, \xi) > 0, && \text{if } p(x_0, \xi) = \{p, \Phi\}(x_0, \xi) = 0 \text{ and } \xi \neq 0; \\ &\frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\}(x_0, \xi, \tau) > 0, && \text{if } p_\Phi(x_0, \xi, \tau) = \{p_\Phi, \Phi\}(x_0, \xi, \tau) = 0 \text{ and } \tau > 0, \end{aligned}$$

where $p_\Phi(x, \xi) = p(x, \xi + i\tau d\Phi)$ and p is the principal symbol of P.

Then, there exists V one neighborhood of x_0 in Ω so that for any $u \in H^{m-1}_{loc}(V)$,

$$\begin{cases} Pu = 0 \text{ in } V, \\ u = 0 \text{ in } V \cap \{\Phi > \Phi(x_0)\} \end{cases} \Longrightarrow u = 0 \text{ on } V.$$

This is Theorem 28.3.4 of [Hör94].

2.6.2 A short bibliography

Unique continuation results have a long history going back to Carleman [Car39] who first had the idea to conjugate the operator with an exponential weight to get unique continuation. He proved the result in the case of elliptic operators of order 2 in dimension 2. Calderón [Cal58] extended the result to some operators with simple characteristics. Namely, that was in situations where $p_\phi = \{p_\phi, \phi\} = 0$ never happens. The general version was given by Hörmander [Hör63] for real operators and [Hör94]. Note that other works consider the limit case where there is some higher order of cancelation. We refer to Zuily [Zui83] for more details.

Theorem 2.4.10 was first proved by Donnelly and Fefferman [DF88] (under a stronger form). The proof presented here, as well as the proof and use of interpolation inequalities is from Lebeau-Robbiano [LR95].

The boundary Carleman estimates were proved by Lebeau-Robbiano [LR95] in order to give the same application that we give in Section 2.5.3, that is the controllability of the heat equation. They also proved the spectral estimates of Section 2.4.4.

Note that there is also another proof (independently) by Fursikov-Immanuvilov [FI96] of the controllability of the heat equation using directly some Carleman estimates for the heat equation. Details about this and some link with the elliptic Carleman estimate are given in [LRL12].

2.6.3 Further questions

Many things have not been written in an optimal way in the previous theorems and can be improved:

- the fact that the a^{ij} are real-valued is not necessary♣ .
- the regularity of u can be much lowered. Note also, that if the coefficients are regular enough, the regularity of u can often be recovered using classical elliptic regularity results, see Brézis [Bre83] for instance.
- the regularity of the coefficients is not optimal. The main coefficients should actually be C^1 while the lower order terms can be in some L^p spaces.
- the fact to be an exact solution of $Pu = 0$ can be replaced by some weaker assumption like $|Pu|(x) \leq C(|u(x)| + |\nabla u(x)|)$ for almost every $x \in \Omega$.

Counterexamples of Alinhac
Rough coefficients, nonlinear problems
boundary conditions, interfaces
Global result

2.7 Exercises on Chapter 2

Exercise 5 (warm up of the Exam of May, 2018)**.** Given smooth functions $\Psi : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$, compute $\operatorname{Hess}(G \circ \Psi)$ and $\Delta(G \circ \Psi)$ (in terms of derivatives of G and Ψ).

Correction 5. We have $\partial_i(G \circ \Psi) = (\partial_i \Psi) G' \circ \Psi$ and $\partial_j \partial_i (G \circ \Psi) = (\partial_j \partial_i \Psi) G' \circ \Psi + (\partial_i \Psi)(\partial_j \Psi) G'' \circ \Psi$, whence $\operatorname{Hess}(G \circ \Psi) = G' \circ \Psi \operatorname{Hess}(\Psi) + G'' \circ \Psi d\Psi \otimes d\Psi$, that is to say

$$\operatorname{Hess}(G \circ \Psi)(x)(\eta, \xi) = G' \circ \Psi(x) \operatorname{Hess}(\Psi)(x)(\eta, \xi) + G'' \circ \Psi(x)(d\Psi(x) \cdot \eta)(d\Psi(x) \cdot \xi).$$

Also, this implies $\Delta(G \circ \Psi)(x) = G' \circ \Psi(x)\ \Delta\Psi(x) + G'' \circ \Psi(x)|d\Psi(x)|^2$.

Exercise 6 (Perturbations of the $\bar{\partial}$ operator, part of the Exam of May, 2018)**.** We consider on $\mathbb{R}^2$ the operator $P = D_{x_1} + iD_{x_2} + V(x_1, x_2)$, where $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{C})$, and U a bounded open set of $\mathbb{R}^2$.

1. Given a function $\Phi \in C^\infty(\overline{U}; \mathbb{R})$, compute $P_\Phi = e^{\tau\Phi} P e^{-\tau\Phi}$. Decompose it as an operator in $\operatorname{Diff}^1_\tau(U)$ plus a lower order term. Give the principal symbol of the operator in $\operatorname{Diff}^1_\tau(U)$, which we call p_Φ.
2. Using a decomposition of the principal part of P_Φ as a selfadjoint part and a skewadjoint part, prove that for all functions $v \in C^\infty_c(U)$ and all $\tau \geq 0$, we have
$$\|P_\Phi v\|^2_{L^2} \geq \frac{\tau}{2}\big((\Delta\Phi)v, v\big)_{L^2} - \|Vv\|^2_{L^2}.$$
3. For $r > 0$ fixed, we consider the set $U = B(0, 3r) \setminus \overline{B}(0, r/2)$. Construct a *radial* (i.e. depending on $|x|$ only) function Φ being decreasing in the radial variable, and such that there exists a constant $c_0 > 0$ such that $\Delta\Phi \geq c_0$ uniformly on U. *Hint: one may choose Φ under the form $G \circ \Psi$ with G and Ψ to be determined.*
4. Deduce that, for such a function Φ, there exist two constants $C, \tau_0 > 0$ such that for all $\tau \geq \tau_0$ and $w \in C^\infty_c(U)$, we have
$$\left\|e^{\tau\Phi} P w\right\|^2_{L^2} \geq C \left\|e^{\tau\Phi} w\right\|^2_{L^2}.$$
5. Prove that for all $r > 0$, there exist $C > 0$ and $\delta \in (0, 1)$ such that for all $u \in C^\infty(\overline{B}(0, 3r))$, we have
$$\|u\|_{L^2(B(0,2r))} \leq C \left(\|u\|_{L^2(B(0,r))} + \|Pu\|_{L^2(B(0,3r))}\right)^\delta \|u\|^{1-\delta}_{L^2(B(0,3r))}. \tag{2.59}$$
6. Let $\omega \subset \mathbb{R}^2$ a bounded nonempty open set. Prove that every C^∞ solution u to $Pu = 0$ in $\mathbb{R}^2$ such that $u = 0$ in ω vanishes identically on $\mathbb{R}^2$.
7. In the case $V = 0$, give another proof of this result.

Correction 6.

1. We have $D_1(e^{-\tau\Phi} u) = e^{-\tau\Phi}(D_1 u + i\tau\partial_1\Phi u)$ so that $e^{\tau\Phi} D_1 e^{-\tau\Phi} = D_1 + i\tau\partial_1\Phi$ and
$$P_\Phi = D_1 + i\tau\partial_1\Phi + i\left(D_2 + i\tau\partial_2\Phi\right) + V.$$
We have $D_1 + i\tau\partial_1\Phi + i\left(D_2 + i\tau\partial_2\Phi\right) \in \operatorname{Diff}^1_\tau(U)$, with principal symbol $p_\Phi(x, \xi) = \xi_1 - \tau\partial_2\Phi(x) + i(\xi_2 + \tau\partial_1\Phi(x))$. The term $V(x)$ will be considered as a lower order term.
2. We write $P_\Phi = P_R + iP_I + V$ where
$$P_R = \frac{1}{2}(P_\Phi + P_\Phi^*) = D_1 - \tau\partial_2\Phi(x), \quad P_I = \frac{1}{2i}(P_\Phi - P_\Phi^*) = D_2 + \tau\partial_1\Phi(x),$$

are both formally selfadjoint. Then, we have

$$\|(P_\Phi - V)v\|_{L^2}^2 = ((P_R + iP_I)v, (P_R + iP_I)v)_{L^2} = \|P_R v\|_{L^2}^2 + \|P_I v\|_{L^2}^2 + (i[P_R, P_I]v, v)_{L^2}. \quad (2.60)$$

We now compute $[P_R, P_I] = [D_1 - \tau\partial_2\Phi, D_2 + \tau\partial_1\Phi] = [D_1, \tau\partial_1\Phi] - [\tau\partial_2\Phi, D_2]$, since $[D_1, D_2] = 0 = [\partial_2\Phi, \partial_1\Phi]$. That is to say, $[P_R, P_I]v = \tau(D_1\partial_1\Phi)v + \tau(D_2\partial_2\Phi)v = \frac{\tau}{i}\Delta\Phi v$.

Together with (2.60), we have now obtained

$$2\|P_\Phi v\|_{L^2}^2 + 2\|Vv\|_{L^2}^2 \geq \|(P_\Phi - V)v\|_{L^2}^2 = \|P_R v\|_{L^2}^2 + \|P_I v\|_{L^2}^2 + \tau(\Delta\Phi v, v)_{L^2} \geq \tau(\Delta\Phi v, v)_{L^2},$$

and hence, for all $v \in C_c^\infty(U)$ and all $\tau \geq 0$,

$$\|P_\Phi v\|_{L^2}^2 \geq \frac{\tau}{2}\big((\Delta\Phi)v, v\big)_{L^2} - \|Vv\|_{L^2}^2.$$

3. We set $\Psi(x) := -|x|$ (Euclidean distance to zero, the sign does not matter in this question and in the next question, but turns out to be a key point in Question 5), $G(s) := e^{\lambda s}$ and $\Phi(x) := G \circ \Psi = e^{-\lambda|x|}$. We have $\nabla\Psi(x) = -\frac{x}{|x|}$, so that $|\nabla\Psi(x)| = 1$ does not vanish, and $\Delta\Psi$ is a smooth function on $\mathbb{R}^2 \setminus \{0\}$. According to the computation in Exercice 5, we have

$$\Delta\Phi(x) = \lambda e^{-\lambda|x|}\ \Delta\Psi(x) + \lambda^2 e^{-\lambda|x|} = e^{-\lambda|x|}(\lambda^2 + \lambda\ \Delta\Psi(x)) \geq e^{-\lambda|x|}(\lambda^2 - \lambda\|\Delta\Psi\|_{L^\infty(U)}),$$

where $U = B(0, 3r) \setminus \overline{B}(0, r/2)$. Choosing e.g. $\lambda_r := 2\|\Delta\Psi\|_{L^\infty(U)} + 1 > 0$, we have $\lambda_r^2 - \lambda_r\|\Delta\Psi\|_{L^\infty(U)} \geq 1$ and hence $\Delta\Phi(x) \geq e^{-\lambda_r 3r} =: c_0 > 0$ for all $x \in \overline{U}$.

4. From the previous two questions, we have obtained for all $v \in C_c^\infty(U)$ that

$$\begin{aligned}\|P_\Phi v\|_{L^2(U)}^2 &\geq \frac{\tau}{2}\big((\Delta\Phi)v, v\big)_{L^2(U)} - \|Vv\|_{L^2(U)}^2 \geq \frac{\tau}{2}c_0\|v\|_{L^2(U)}^2 - \|V\|_{L^\infty(U)}^2\|v\|_{L^2}^2 \\ &\geq \frac{\tau}{4}c_0\|v\|_{L^2(U)}^2,\end{aligned}$$

for all $\tau \geq \tau_0 = \frac{4}{c_0}\|V\|_{L^\infty(U)}^2$. Recalling that $P_\Phi = e^{\tau\Phi}Pe^{-\tau\Phi}$ and applying this inequality to $v = e^{\tau\Phi}w$ (which belongs to $C_c^\infty(U)$ for $w \in C_c^\infty(U)$), we have obtained

$$\left\|e^{\tau\Phi}Pw\right\|_{L^2}^2 \geq C\tau\left\|e^{\tau\Phi}w\right\|_{L^2}^2, \qquad \text{for all } w \in C_c^\infty(U), \tau \geq \tau_0.$$

5. The proof of this local interpolation inequality proceeds exactly as that of Theorem 2.4.3. The only difference is that here P is of order one. Hence, commutators $[P, \chi]$ are of order zero and this results in the fact that only L^2 norms appear in (2.59) (as opposed to the statement of Theorem 2.4.3 in which the operator is of order two, commutators are of order one, and the interpolation inequality formulates with H^1 norms).

6. Choose a point $x_0 \in \omega$. In Question 5 (and after a translation; the assumptions on V are translation-invariant), we proved that for all $r > 0$, there exist $C > 0$ and $\delta \in (0, 1)$ such that for all $u \in C^\infty(\overline{B}(0, 3r))$, we have

$$\|u\|_{L^2(B(x_0,2r))} \leq C\|u\|_{L^2(B(x,r))}^\delta\|u\|_{L^2(B(x_0,3r))}^{1-\delta}, \quad (2.61)$$

where we have used that $Pu = 0$ on $\mathbb{R}^2$.

Now assume that u does not vanish identically, that is $\mathrm{supp}(u) \neq \emptyset$. We set $r_0 := \sup\{r \geq 0, B(0, r) \cap \mathrm{supp}(u) = \emptyset\}$. We have $r_0 > 0$ since $x_0 \in \omega$ (which is open) and $r_0 < +\infty$ since $\mathrm{supp}(u) \neq \emptyset$. Moreover, (2.61) with $r = r_0$ yields $\|u\|_{L^2(B(x_0,2r_0))} \leq 0$, so that $B(0, r_0) \cap \mathrm{supp}(u) = \emptyset$. This implies $2r_0 = r_0$ which contradicts $0 < r_0 < +\infty$.

7. In case $V = 0$, we have $P = \overline{\partial}$ (the Cauchy-Riemann operator) and solutions to $Pu = 0$ are real-analytic on $\mathbb{R}^2$. Therefore, they satisfy the analytic continuation principle: a solution to $Pu = 0$ vanishing on an open set (even any set containing an accumulation point) vanishes on all $\mathbb{R}^2$.

Exercise 7 (Carleman estimates for the Laplace operator, part of the Exam of May, 2018)**.** We consider in $\mathbb{R}^n$ the operator $P = -\Delta + V$, where $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R})$, and U a bounded open set of $\mathbb{R}^n$.

1. Given a function $\Phi \in C^\infty(\overline{U};\mathbb{R})$, compute $P_\Phi = e^{\tau\Phi} P e^{-\tau\Phi}$. Decompose it as an operator in $\mathrm{Diff}^2_\tau(U)$ plus a lower order term. Give the principal symbol of the operator in $\mathrm{Diff}^2_\tau(U)$, which we call p_Φ.

2. Compute $\frac{1}{4\tau}\{\mathrm{Re}(p_\Phi), \mathrm{Im}(p_\Phi)\}$.

3. We now set $\Phi = G \circ \Psi$, with $\Psi : \overline{U} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$. Express $\frac{1}{4\tau}\{\mathrm{Re}(p_\Phi), \mathrm{Im}(p_\Phi)\}$ in terms of G, Ψ.

4. Show that if $d\Psi \neq 0$ on $\overline{U}$, if $G' > 0$, $G'' > 0$ on $\Psi(\overline{U})$, and if $\frac{G''}{G'}$ is sufficiently large on $\Psi(\overline{U})$, then we have
$$\Big((x,\xi,\tau) \in \overline{U} \times \mathbb{R}^n \times \mathbb{R}^+, (\xi,\tau) \neq (0,0), \text{ and } \mathrm{Re}(p_\Phi)(x,\xi,\tau) = 0\Big)$$
$$\implies \frac{1}{4\tau}\{\mathrm{Re}(p_\Phi), \mathrm{Im}(p_\Phi)\}(x,\xi,\tau) > 0.$$
Hint: one may prove that the term containing $|d\Psi|^4$ is large, whereas the other terms either have the right sign, or are sufficiently small when $\frac{G''}{G'}$ is large.

5. Deduce a Carleman estimate for P with such a weight function Φ. Only explain the main steps of the proofs, and omit the details.

We now want to give another proof of the same Carleman estimate.

6. Let $f : \overline{U} \to \mathbb{R}$ be a smooth function. Check that P_Φ decomposes under the form
$$P_\Phi = Q_2 + iQ_1, \qquad Q_2 = (P_R - \tau f), \quad Q_1 = (P_I - i\tau f),$$
with $P_R = \frac{P_\Phi + P_\Phi^*}{2}$, $P_I = \frac{P_\Phi - P_\Phi^*}{2i}$. Give the principal symbol of P_R, P_I, Q_2, Q_1.

7. Prove that we have
$$\|P_\Phi v\|^2_{L^2} = \|Q_2 v\|^2_{L^2} + \|Q_1 v\|^2_{L^2} + \tau(Lv, v)_{L^2},$$
for all $v \in C^\infty_c(U)$, where L belongs to $\mathrm{Diff}^m_\tau(U)$ is to be determined, as well as its order m and its principal symbol (in terms of p_Φ and f).

8. Assume in this question the existence of a constant $C > 0$ such that for all $(x,\xi,\tau) \in \overline{U} \times \mathbb{R}^n \times \mathbb{R}^+$, we have
$$\frac{1}{\tau}\{\mathrm{Re}(p_\Phi), \mathrm{Im}(p_\Phi)\}(x,\xi,\tau) + 2f(x)\,\mathrm{Re}(p_\Phi)(x,\xi,\tau) \geq C(|\xi|^2 + \tau^2). \tag{2.62}$$
Deduce a Carleman estimate for P. Only explain the main steps of the proofs, and omit the details.

9. We again consider $\Phi = G \circ \Psi$ with $d\Psi \neq 0$ on $\overline{U}$, and $G' > 0$, $G'' > 0$ on $\Psi(\overline{U})$. Prove that for all $\mu \in (0,1)$, there exists $\lambda_0 > 0$ such that if $\frac{G''}{G'} \geq \lambda_0$ on $\Psi(\overline{U})$, there is $C > 0$ such that
$$\frac{1}{4\tau}\{\mathrm{Re}(p_\Phi), \mathrm{Im}(p_\Phi)\} + \mu\, G'' \circ \Psi |d\Psi|^2\, \mathrm{Re}(p_\Phi) \geq C(|\xi|^2 + \tau^2),$$
on $\overline{U} \times \mathbb{R}^n \times \mathbb{R}^+$. *Hint: One may re-use the computations and the strategy of Question 4.*

10. Conclude.

Correction 7. 1. We have already computed in Example 1.3.11 that

$$e^{\tau\Phi}(-\Delta)e^{-\tau\Phi}u = -\Delta u - \tau^2|\partial\Phi|^2 u + 2\tau\partial\Phi\cdot\partial u + \tau(\Delta\Phi)u,$$

and hence

$$P_\Phi = -\Delta - \tau^2|\partial\Phi|^2 + 2\tau\partial\Phi\cdot\partial + \tau(\Delta\Phi) + V,$$

where only $V \notin \mathrm{Diff}^2_\tau(U)$ (since we do not assume smoothness). We also have $\tau(\Delta\Phi) \in \mathrm{Diff}^1_\tau(U)$ and thus

$$p_\Phi(x,\xi,\tau) = |\xi|^2 - \tau^2|\partial\Phi|^2 + 2i\tau\partial\Phi\cdot\xi$$

2. We have $\mathrm{Re}(p_\Phi)(x,\xi,\tau) = |\xi|^2 - \tau^2|\partial\Phi|^2$ and $\mathrm{Im}(P_\Phi)(x,\xi,\tau) = 2\tau\partial\Phi\cdot\xi$ so that

$$\{\mathrm{Re}(p_\Phi),\mathrm{Im}(p_\Phi)\} = \sum_j\sum_k \left(4\tau\xi_j\partial_j\partial_k\Phi\xi_k + 4\tau^3\partial_j\Phi\partial_j\partial_k\Phi\partial_k\Phi\right),$$

so that

$$\frac{1}{4\tau}\{\mathrm{Re}(p_\Phi),\mathrm{Im}(p_\Phi)\} = \mathrm{Hess}(\Phi)(\xi,\xi) + \tau^2\,\mathrm{Hess}(\Phi)(d\Phi,d\Phi).$$

3. According to Exercise 5 we have

$$\mathrm{Hess}(\Phi)(\eta,\xi) = G'\circ\Psi\,\mathrm{Hess}(\Psi)(\eta,\xi) + G''\circ\Psi(d\Psi\cdot\eta)(d\Psi\cdot\xi).$$

and hence

$$\begin{aligned}\frac{1}{4\tau}\{\mathrm{Re}(p_\Phi),\mathrm{Im}(p_\Phi)\} = {}& G'\circ\Psi\,\mathrm{Hess}(\Psi)(\xi,\xi) + G''\circ\Psi(d\Psi\cdot\xi)^2\\ &+\tau^2(G'\circ\Psi)^2\left(G'\circ\Psi\,\mathrm{Hess}(\Psi)(d\Psi,d\Psi) + G''\circ\Psi|d\Psi|^4\right)\end{aligned}$$

4. Remark first that $\mathrm{Re}(p_\Phi)(x,\xi,\tau) = |\xi|^2 - \tau^2|d\Phi|^2$ so that the condition

$$(\xi,\tau)\neq(0,0), \text{ and } \mathrm{Re}(p_\Phi)(x,\xi,\tau) = 0$$

implies actually that $\tau > 0$, which we assume from now on.
Since $G'' > 0$ on $\Psi(\overline{U})$, we first have

$$\begin{aligned}\frac{1}{4\tau}\{\mathrm{Re}(p_\Phi),\mathrm{Im}(p_\Phi)\} \geq {}& G'\circ\Psi\,\mathrm{Hess}(\Psi)(\xi,\xi)\\ &+\tau^2(G'\circ\Psi)^2\left(G'\circ\Psi\,\mathrm{Hess}(\Psi)(d\Psi,d\Psi) + G''\circ\Psi|d\Psi|^4\right)\end{aligned}$$

Using the assumption that

$$0 = \mathrm{Re}(p_\Phi)(x,\xi,\tau) = |\xi|^2 - \tau^2|d\Phi|^2 = |\xi|^2 - \tau^2(G'\circ\Psi)^2|d\Psi|^2,$$

and denoting $|A| = \sup_{|\xi|=1} A(\xi,\xi)$, we deduce

$$\begin{aligned}\frac{1}{4\tau}\{\mathrm{Re}(p_\Phi),\mathrm{Im}(p_\Phi)\} \geq {}& -G'\circ\Psi|\,\mathrm{Hess}(\Psi)||\xi|^2\\ &+\tau^2(G'\circ\Psi)^2\left(-G'\circ\Psi|\,\mathrm{Hess}(\Psi)||d\Psi|^2 + G''\circ\Psi|d\Psi|^4\right)\\ \geq {}& -(G'\circ\Psi)^3|\,\mathrm{Hess}(\Psi)|\tau^2|d\Psi|^2\\ &+\tau^2(G'\circ\Psi)^2\left(-G'\circ\Psi|\,\mathrm{Hess}(\Psi)||d\Psi|^2 + G''\circ\Psi|d\Psi|^4\right)\\ \geq {}& \tau^2(G'\circ\Psi)^3\Big(\frac{G''\circ\Psi}{G'\circ\Psi}|d\Psi|^4 - 2|\,\mathrm{Hess}(\Psi)||d\Psi|^2\Big).\end{aligned}$$

Assuming $d\Psi \neq 0$ on $\overline{U}$, $G' > 0$, $G'' > 0$ on $\Psi(\overline{U})$, if we choose G such that $\frac{G''}{G'} \geq \lambda_0 := 3\max_{\overline{U}}\frac{|\,\mathrm{Hess}(\Psi)|}{|d\Psi|^2}$ on $\Psi(\overline{U})$, we have obtained that $\frac{1}{4\tau}\{\mathrm{Re}(p_\Phi),\mathrm{Im}(p_\Phi)\} > 0$ on the set $\{(x,\xi,\tau)\in\overline{U}\times\mathbb{R}^n\times\mathbb{R}^+, (\xi,\tau)\neq(0,0), \mathrm{Re}(p_\Phi)(x,\xi,\tau) = 0\}$.

5. We first notice that the homogeneity of $(\operatorname{Re} p_\Phi)^2$ of degree four, of $\left\{\operatorname{Re} p_\Phi, \frac{\operatorname{Im} p_\Phi}{\tau}\right\}$ of degree two, together with the use of Lemma 2.1.8 on the compact set $\overline{U} \times \{(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}^+, |\xi|^2 + \tau^2 = 1\}$ yields the existence of $C_1, C_2 > 0$ such that for all $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}_+^*$,

$$\frac{C_1}{|\xi|^2 + \tau^2}(\operatorname{Re} p_\Phi)^2 + \frac{1}{\tau}\{\operatorname{Re} p_\Phi, \operatorname{Im} p_\Phi\} \geq C_2\left(|\xi|^2 + \tau^2\right).$$

(Note that this inequality is actually stronger than (2.3), for we here do not need the term $\frac{C_1}{|\xi|^2+\tau^2}(\operatorname{Im} p_\Phi)^2$ on the left hand-side. Note also that the explicit computation of the previous question could actually directly yield the sought result, without making use of the compactness argument of Lemma 2.1.8). The proof then follows that of Theorem 2.1.1, except that we have to use the semiglobal version of the Gårding inequality, namely Proposition 1.3.18. We finally obtain the existence of $C, \tau_0 > 0$ so that

$$\tau \|v\|_{H^1_\tau}^2 \leq C \|P_\Phi v\|_{L^2}^2, \quad \text{for all } v \in C_c^\infty(U), \tau \geq \tau_0.$$

In turn, after "unconjugating" (i.e. writing $v = e^{\tau\Phi}u$), this translates into

$$\tau^3 \left\|e^{\tau\Phi}u\right\|_{L^2}^2 + \tau \left\|e^{\tau\Phi}\nabla u\right\|_{L^2}^2 \leq C \left\|e^{\tau\Phi}Pu\right\|_{L^2}^2, \quad \text{for all } u \in C_c^\infty(U), \tau \geq \tau_0. \tag{2.63}$$

6. We have as usual $P_\Phi = P_R + iP_I$ with $P_R = \frac{P_\Phi + P_\Phi^*}{2}, P_I = \frac{P_\Phi - P_\Phi^*}{2i}$. Now with $Q_2 = (P_R - \tau f)$ and $Q_1 = (P_I - i\tau f)$ we have

$$Q_2 + iQ_1 = (P_R - \tau f) + i(P_I - i\tau f) = P_R + iP_I - \tau f + \tau f = P_R + iP_I = P_\Phi.$$

We have $P_R, P_I \in \operatorname{Diff}_\tau^2(U)$ whereas $\tau f \in \operatorname{Diff}_\tau^1(U)$. Therefore, the principal symbol of both P_R and Q_2 is $\operatorname{Re}(p_\Phi)$ and the principal symbol of both P_I and Q_1 is $\operatorname{Im}(p_\Phi)$.

7. Beware that Q_2 is selfadjoint but Q_1 is not. We have

$$\begin{aligned}
\|P_\Phi v\|_{L^2}^2 &= ((Q_2 + iQ_1)v, (Q_2 + iQ_1)v)_{L^2} \\
&= \|Q_2 v\|_{L^2}^2 + \|Q_1 v\|_{L^2}^2 + (iQ_1 v, Q_2 v)_{L^2} + (Q_2 v, iQ_1 v)_{L^2} \\
&= \|Q_2 v\|_{L^2}^2 + \|Q_1 v\|_{L^2}^2 + (i(Q_2 Q_1 - Q_1^* Q_2)v, v)_{L^2}.
\end{aligned}$$

We then remark that $Q_1^* = P_I + i\tau f = Q_1 + 2i\tau f$ and hence

$$i(Q_2 Q_1 - Q_1^* Q_2) = i(Q_2 Q_1 - (Q_1 + 2i\tau f)Q_2) = i[Q_2, Q_1] + 2\tau f Q_2.$$

Remarking that τ factorizes in the operator Q_1 (it does for P_I and for τf), we may write this as

$$\|P_\Phi v\|_{L^2}^2 = \|Q_2 v\|_{L^2}^2 + \|Q_1 v\|_{L^2}^2 + (\tau L v, v)_{L^2},$$

where $L = i\left[Q_2, \frac{Q_1}{\tau}\right] + 2fQ_2$ is in $\operatorname{Diff}_\tau^2(U)$, with principal symbol $\frac{1}{\tau}\{\operatorname{Re}(p_\Phi), \operatorname{Im}(p_\Phi)\} + 2f\operatorname{Re}(p_\Phi)$ (according to the previous question together with the symbolic calculus).

8. If we now assume the symbolic estimate (2.62), we simply deduce from the previous question that

$$\|P_\Phi v\|_{L^2}^2 \geq \tau (Lv, v)_{L^2},$$

where $L \in \operatorname{Diff}_\tau^2(U)$ has principal symbol $\frac{1}{\tau}\{\operatorname{Re}(p_\Phi), \operatorname{Im}(p_\Phi)\} + 2f\operatorname{Re}(p_\Phi) \geq C(|\xi|^2 + \tau^2)$. The semiglobal version of the Gårding inequality of Proposition 1.3.18 directly yields the existence of $C, \tau_0 > 0$ so that

$$\tau \|v\|_{H^1_\tau}^2 \leq C \|P_\Phi v\|_{L^2}^2, \quad \text{for all } v \in C_c^\infty(U), \tau \geq \tau_0,$$

which yields the usual Carleman estimate (2.63).

9. In Question 4, we have obtained that $\mathrm{Re}(p_\Phi)(x,\xi,\tau) = |\xi|^2 - \tau^2 (G'\circ\Psi)^2 |d\Psi|^2$ and

$$\frac{1}{4\tau}\{\mathrm{Re}(p_\Phi), \mathrm{Im}(p_\Phi)\} \geq \tau^2 (G'\circ\Psi)^3 \Big(\frac{G''\circ\Psi}{G'\circ\Psi}|d\Psi|^4 - 2|\operatorname{Hess}(\Psi)||d\Psi|^2\Big).$$

As a consequence,

$$\begin{aligned}
&\frac{1}{4\tau}\{\mathrm{Re}(p_\Phi), \mathrm{Im}(p_\Phi)\} + \mu\, G''\circ\Psi |d\Psi|^2 \mathrm{Re}(p_\Phi) \\
&\geq \mu\, G''\circ\Psi |d\Psi|^2 \Big(|\xi|^2 - \tau^2 (G'\circ\Psi)^2 |d\Psi|^2\Big) + \tau^2 (G'\circ\Psi)^3 \Big(\frac{G''\circ\Psi}{G'\circ\Psi}|d\Psi|^4 - 2|\operatorname{Hess}(\Psi)||d\Psi|^2\Big) \\
&= \mu\, G''\circ\Psi |d\Psi|^2 |\xi|^2 + \tau^2 (G'\circ\Psi)^3 \Big((1-\mu)\frac{G''\circ\Psi}{G'\circ\Psi}|d\Psi|^4 - 2|\operatorname{Hess}(\Psi)||d\Psi|^2\Big)
\end{aligned}$$

Henceforth, if $\mu \in (0,1)$, both coefficients in front of $|\xi|^2$ and of $\frac{G''\circ\Psi}{G'\circ\Psi}$ are positive (recall that we still assume $d\Psi \neq 0$ on $\overline{U}$, $G' > 0$, $G'' > 0$ on $\Psi(\overline{U})$). We assume $\mu \in (0,1)$ from now on. If we then choose G such that $\frac{G''}{G'} \geq \lambda_0 := 3 \max_{\overline{U}} \frac{|\operatorname{Hess}(\Psi)|}{(1-\mu)|d\Psi|^2}$ on $\Psi(\overline{U})$, both coefficients in front of $|\xi|^2$ and τ^2 are positive (and independent of (ξ,τ)) and therefore there is a constant $C>0$ such that

$$\frac{1}{4\tau}\{\mathrm{Re}(p_\Phi), \mathrm{Im}(p_\Phi)\} + \mu\, G''\circ\Psi |d\Psi|^2 \mathrm{Re}(p_\Phi) \geq C(|\xi|^2 + \tau^2),$$

on $\overline{U}\times\mathbb{R}^n\times\mathbb{R}^+$.

10. In the end, we have obtained another proof of the Carleman estimate (2.63). Starting from any function Ψ with $d\Psi \neq 0$ on $\overline{U}$, it may be "convexified" so that to yield an appropriate Carleman weight $\Phi = G\circ\Psi$. Note that this can always be achieved by starting from G_0 a convex increasing function, then the function $G(s) = G_0(\lambda s)$ will do the job for λ sufficiently large. We usually take $G_0(s) = e^s$ for convenience. The functions Ψ and Φ have the same levelsets.

    Note that this strategy has the advantage of only using a Gårding inequality for a genuine *differential* operator L, whereas the usual one makes use the Fourier multiplier $(-\Delta+\tau^2)^{-1}$ (in order to downgrade the order of the operator P_R^2). This strategy however only works with elliptic differential operators.

Exercise 8 (?)**.** Let Ω be and open subset of $\mathbb{R}^n$ and $x_0 \in \Omega$. Let $P \in \mathrm{Diff}^2(\Omega)$ be a (classical) differential operator with *real-valued principal symbol* p_2 and $\Phi \in C^\infty(\overline{\Omega};\mathbb{R})$. Assume that Assumption (2.3) is satisfied.

1. Prove that

$$\tau \|v\|^2_{H^1_\tau} + \tau^{-1}\|Pv\|^2_{L^2} \leq C\|P_\Phi v\|^2_{L^2}, \quad \text{for all } v \in C^\infty_c(B(x_0,r)), \tau \geq \tau_0;$$

2. Prove that the same estimate is true for P replaced by $P+V$ for $V \in L^{\frac{n}{2}}$.

3. In case P is elliptic, prove that

$$\tau^{-1}\|v\|^2_{H^2_\tau} \leq C\|P_\Phi v\|^2_{L^2}, \quad \text{for all } v \in C^\infty_c(B(x_0,r)), \tau \geq \tau_0;$$

4. Still assuming P is elliptic, prove that the same estimate is true for P replaced by $P + W\cdot\nabla + V$ for $V \in L^{\frac{n}{2}}$ and $W \in L^{???}$ ♣ `avec des inj de Sobolev`

Correction 8. 1. Recall that from (2.2), we have

$$\tau \|v\|^2_{H^1_\tau} \leq C\|P_\Phi v\|^2_{L^2}, \quad \text{for all } v \in C^\infty_c(B(x_0,r)), \tau \geq \tau_0.$$

$\tilde{P}_\Phi = e^{\tau\Phi}\tilde{P}e^{-\tau\Phi}$ and prove (2.4). To this aim, we notice that it is again equivalent to prove the result with $\tilde{P} = \sum_{i,j=1}^n D_i a^{ij}(x) D_j$ instead of P. We decompose the operator $\tilde{P}_\Phi$ again as in (2.5) as $\tilde{P}_\Phi = Q_R + iQ_I$ with $Q_R^* = Q_R$, $Q_I^* = Q_I$, $Q_R, Q_I \in \mathrm{Diff}^2_\tau$ given by

$$Q_R = \sum_{i,j=1}^n D_i a^{ij} D_j - \tau^2 a^{ij}(\partial_i\Phi)(\partial_j\Phi) = \tilde{P} - \tau^2 \sum_{i,j=1}^n a^{ij}(\partial_i\Phi)(\partial_j\Phi) = P + R - \tau^2 \sum_{i,j=1}^n a^{ij}(\partial_i\Phi)(\partial_j\Phi)$$

$$Q_I = i\tau \sum_{i,j=1}^n D_i a^{ij}(x)(\partial_j\Phi) + (\partial_i\Phi)a^{ij}(x)D_j$$

with $R \in \mathrm{Diff}^1_\tau$. In particular, we have, using $Q_R = \tilde{P}_\Phi - iQ_I$ in the second equality

$$\begin{aligned}\|Pv\|_{L^2} &\le \|Q_R v\|_{L^2} + \|Rv\|_{L^2} + C\tau^2\|v\|_{L^2} \\ &\le \|\tilde{P}_\Phi v\|_{L^2} + \|Q_I v\|_{L^2} + C\|v\|_{H^1_\tau} + C\tau^2\|v\|_{L^2} \le \|\tilde{P}_\Phi v\|_{L^2} + C\tau\|v\|_{H^1_\tau}.\end{aligned}$$

Hence

$$\tau^{-1}\|Pv\|^2_{L^2} \le \|\tilde{P}_\Phi v\|^2_{L^2} + C\tau\|v\|^2_{H^1_\tau} \le C\|\tilde{P}_\Phi v\|^2_{L^2},$$

where we have used the Carleman inequality in the last place. This implies the sought estimate.

2. We write the Hölder inequality $\|Vv\|^2_{L^2} \le \||V|^2\|_{L^{p'}}\||v|^2\|_{L^p}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Next, if $n \ge 3$ the Sobolev embedding is $H^1 \hookrightarrow L^{\frac{2n}{n-2}}$, and thus $\||v|^2\|_{L^{\frac{d}{d-2}}} = \|v\|^2_{L^{\frac{2d}{d-2}}} \le C\|v\|^2_{H^1}$. Choosing $p = \frac{d}{d-2}$, we obtain $p' = \frac{d}{2}$ and thus

$$\|Vv\|_{L^2} \le \|V\|_{L^{\frac{d}{2}}}\|v\|_{H^1}.$$

Therefore...

3. In this case, the result follows from local elliptic regularity (ref?): there is $C > 0$ depending on the coefficients of P and $r > 0$ such that

$$\|v\|_{H^2} \le C\,\|Pv\|_{L^2} + C\|v\|_{H^1} \quad \text{for all } v \in C^\infty_c(B(x_0, r)).$$

Combined with the first question, this yields the sought result.

Exercise 9 (Agmon estimates)**.** Let $V \in C^\infty_b(\mathbb{R}^n)$, bounded as well as all of its derivatives and real-valued. We consider the operator P_τ depending a large parameter τ, defined by

$$P_\tau u = -\Delta u + \tau^2 V u$$

Let U an open bounded subset of $\mathbb{R}^n$ so that

$$V(x) \ge \varepsilon > 0 \quad \text{for all} \quad x \in U,$$

and let W be an open set such that $U \Subset W$.

Let $\psi, \phi \in C^\infty_c(W)$ so that $0 \le \psi, \phi \le 1$ and $\psi = 1$ on U, $\phi = 1$ on $\mathrm{supp}(\psi)$.

Assume also

$$\mathrm{supp}(\phi) \Subset W \Subset \{V(x) > \varepsilon\}.$$

1. Compute $e^{\delta\tau\psi}P_\tau e^{-\delta\tau\psi}$ for one $\delta > 0$ to be chosen later on. Compute its principal symbol $p_{\tau,\psi}$.
2. Prove that for δ small enough and one $c_0 > 0$, $|p_{\tau,\psi}|^2 \ge c_0$ for all $x \in W$, $|\xi|^2 + \tau^2 = 1$.
3. We admit the following elliptic type estimate.

   Let $Q_\tau \in \mathrm{Diff}^m_\tau$ with principal symbol q so that $q \ne 0$ for all $x \in K$, $(\xi, \tau) \ne 0$, then, for τ large enough

$$\|v\|_{H^m_\tau} \le C\,\|Q_\tau v\|_{L^2}.$$

for all $v \in C^\infty(\mathbb{R}^n)$ supported in K.

Prove the estimate

$$\left\| e^{\delta\tau\psi}\phi u \right\|_{H^2_\tau} \leq C \left\| e^{\delta\tau\psi} P_\tau \phi u \right\|_{L^2}$$

for all $u \in C^\infty_c(\mathbb{R}^n)$, $\tau \geq \tau_0$.

4. Using the properties of the support of $\nabla\phi$, prove the estimate $\left\| e^{\delta\tau\psi}[P_\tau, \phi]u \right\|_{L^2} \leq C \left\| u \right\|_{H^1_\tau(\widetilde{W})}$ with $\widetilde{W}$ so that $supp(\phi) \Subset \widetilde{W} \Subset W$.

   Here, we use the notation $\|u\|^2_{H^1_\tau(\widetilde{W})} = \|\nabla u\|^2_{L^2(\widetilde{W})} + \tau^2 \|u\|^2_{L^2(\widetilde{W})}$.

5. Let $U_1 \Subset U_2$ open subset. Prove that

$$\|u\|_{H^2_\tau(U_1)} \leq C \|P_\tau u\|_{L^2(U_2)} + C\tau^2 \|u\|_{L^2(U_2)}.$$

   for $u \in C^\infty_c(U_2)$.

6. Conclude that

$$\|u\|_{H^2(U)} \leq Ce^{-\delta\tau/2} \|u\|_{L^2(W)} + C \|P_\tau u\|_{L^2(W)}.$$

7. Give an interpretation of the previous estimate.

Correction 9. ♣ `To be written one day.`

For the last question :

We write $h = 1/\tau$ the (small) semiclassical parameter, and consider the semiclassical stationary Schrödinger operator $\mathcal{P}_h := h^2 P_{1/h} = -h^2\Delta + V(x)$. We have proven the following statement so far: for all $\varepsilon > 0$, all bounded open set $U \subset \{V \geq \varepsilon\}$ and all W such that $U \Subset W$, there is $\delta > 0$ such that we have

$$\|u\|_{H^2(U)} \leq Ce^{-\delta/h} \|u\|_{L^2(W)} + Ch^{-2} \|\mathcal{P}_h u\|_{L^2(W)}.$$

One can also check that this statement remains valid if V is replaced by $V - E_h$ with $E_h \in \mathbb{R}$ and $E_h \to E \in \mathbb{R}$. This implies in particular that for any bounded open set U such that $\overline{U} \subset \{V > E\}$ and all W such that $U \Subset W$, there is $\delta > 0$ such that we have

$$\|u_h\|_{L^2(U)} \leq Ce^{-\delta/h} \|u_h\|_{L^2(W)}, \quad \text{for } u_h \text{ such that } (-h^2\Delta + V)u_h = E_h u_h, \quad E_h \to E, h \leq h_0. \tag{2.64}$$

This can be interpreted as follows: A classical particle in the potential well $V(x)$ has classical Hamiltonian $H(x,\xi) = |\xi|^2 + V(x)$ (x is the position of the particle and ξ its momentum). If the particle has energy E, it lives in the energy layer $\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, H(x,\xi) = E\}$ (this follows from the fact that H is preserved along the Hamiltonian flow it generates). In particular, the position of the particle is necessarily in the so-called *classically allowed region* at energy E defined by $K_E = \{x \in \mathbb{R}^n, V(x) \leq E\}$ (projection of the energy layer on the x-variable).

If we now consider a *quantum* particle at energy E, its wave-function $u_h(x)$ solves the semiclassical stationary Schrödinger equation $(-h^2\Delta + V)u_h = E_h u_h$ with $E_h \to E$. If the eigenfunction is normalized $\|u_h\|_{L^2(\mathbb{R}^n)} = 1$, the square of its modulus $|u_h(x)|^2$ is a probability density, which models the likelihood of the particle to be at position $x \in \mathbb{R}^n$. For an open set $U \subset \mathbb{R}^n$, the quantity $\|u_h\|^2_{L^2(U)}$ models the probability of finding the quantum particle/state u_h in U. What we have proved in (2.64) may be reformulated as

$$\overline{U} \subset \{V > E\} \implies \|u_h\|^2_{L^2(U)} \leq Ce^{-2\delta/h}.$$

That is to say, if U is in the *classically forbidden region* at energy E, namely $\mathbb{R}^n \setminus K_E = \{V > E\}$, the probability of finding a particle at energy E in U is exponentially small in the semiclassical limit $h \to 0^+$ (i.e. in the limit from quantum mechanics to classical mechanics).

Exercise 10 (Warm up, part of the Exam of May, 2019)**.** Given a bounded open set $\Omega \subset \mathbb{R}^n$, we consider, for $m \in \mathbb{N}$, the class $\mathrm{Diff}^m_{\tau,\gamma}(\Omega)$ consisting of operators (depending on two large parameters $\tau, \gamma > 0$) of the form

$$P = \sum_{|\alpha|+\beta+\delta\le m} p_{\alpha,\beta,\delta}(x)\tau^\beta\gamma^\delta D^\alpha, \quad (\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}, \delta \in \mathbb{N}),$$

with coefficients $p_{\alpha,\beta,\delta} \in C^\infty(\overline{\Omega})$ bounded as well as all their derivatives. We define by $p_m(x,\xi,\tau,\gamma) = \sum_{|\alpha|+\beta+\delta=m} p_{\alpha,\beta,\delta}(x)\tau^\beta\gamma^\delta\xi^\alpha$ its principal symbol in this class.

Prove that:

- if $P_1 \in \mathrm{Diff}^{m_1}_{\tau,\gamma}(\Omega), P_2 \in \mathrm{Diff}^{m_2}_{\tau,\gamma}(\Omega)$, then $P_1P_2 \in \mathrm{Diff}^{m_1+m_2}_{\tau,\gamma}(\Omega)$ with principal symbol $p_{m_1}p_{m_2}$;
- if $P_1 \in \mathrm{Diff}^{m_1}_{\tau,\gamma}(\Omega), P_2 \in \mathrm{Diff}^{m_2}_{\tau,\gamma}(\Omega)$, then $[P_1,P_2] \in \mathrm{Diff}^{m_1+m_2-1}_{\tau,\gamma}(\Omega)$ with principal symbol $\frac{1}{i}\{p_{m_1},p_{m_2}\}$;
- if $P \in \mathrm{Diff}^m_{\tau,\gamma}(\Omega)$, then $P^* \in \mathrm{Diff}^m_{\tau,\gamma}(\Omega)$ with principal symbol $\overline{p_m}$.

Hint: one might consider first the case of operators of the form $P = a(x)\tau^\beta\gamma^\delta D^\alpha$ and use known results.

Correction 10. ♣ `to be written one day`

Exercise 11 (Limiting weights and limiting Carleman estimates, part of the Exam of May, 2019)**.** This Exercise is *not independent* from Exercises 3 and 10. The inequality $2ab \le a^2 + b^2$ might be useful in this exercise. The goal of this exercise is to prove the following Carleman estimate.

Theorem 2.7.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume $P \in \mathrm{Diff}^2(\Omega)$ is* elliptic *with real principal symbol p_2. Assume that $\psi \in C^\infty(\overline{\Omega};\mathbb{R})$ satisfies $d\psi \ne 0$ on $\overline{\Omega}$ together with*

$$\{\overline{p_\psi}, p_\psi\}(x,\xi,\tau) = 0 \quad \text{for all } (x,\xi,\tau) \in \Omega\times\mathbb{R}^n\times\mathbb{R}^+, \tag{2.65}$$

where $p_\psi(x,\xi,\tau) = p_2(x,\xi + i\tau d\psi(x))$. Then, there exist $C, \tau_0 > 0$ such that

$$\tau^2 \left\|e^{\tau\psi}u\right\|^2_{L^2} + \left\|e^{\tau\psi}\nabla u\right\|^2_{L^2} \le C\left\|e^{\tau\psi}Pu\right\|^2_{L^2}, \quad \text{for all } u \in C^\infty_c(\Omega), \tau \ge \tau_0. \tag{2.66}$$

1. Compare this Carleman estimate with the usual one.
2. Is this Carleman estimate insensitive to the change of P into $P+V(x)$? respectively into $P+W(x)\cdot\nabla$?
3. Prove the following lemma.

   Lemma 2.7.2. *Let $G \in C^\infty(\mathbb{R})$ and $\psi \in C^\infty(\overline{\Omega};\mathbb{R})$, and set $\phi = G\circ\psi$. Assume $p_2(x,\xi)$ is a homogeneous symbol of order two with real-valued coefficients, and set $p_\phi(x,\xi,\tau) = p_2(x,\xi+i\tau d\phi(x))$. Then we have with $\xi = (G'\circ\psi)(x)\eta$*

   $$\begin{aligned}\frac{1}{2i}\{\overline{p_\phi},p_\phi\}(x,\xi,\tau) =& \tau(G''\circ\psi)(x)(G'\circ\psi)^2(x)\Big(|\{p_2,\psi\}(x,\eta)|^2 + 4\tau^2 p_2(x,d\psi(x))^2\Big)\\ &+ (G'\circ\psi)^3(x)\frac{1}{2i}\{\overline{p_\psi},p_\psi\}(x,\eta,\tau).\end{aligned}$$

4. In all the exercise, we only consider the function $G(s) = G_{\gamma,\tau}(s) = s + \frac{\gamma}{2\tau}s^2$, and hence $\phi = \phi_{\gamma,\tau} = G\circ\psi = \psi + \frac{\gamma}{2\tau}\psi^2$.
   (a) Prove the existence of $C_0 > 0$ such that for $\tau \ge C_0\gamma$, we have $1/2 \le G'\circ\psi \le 3/2$ on Ω.
   (b) Compute $d\phi$ in terms of $d\psi$. Prove that there is $C > 0$ such that $|\partial_j\phi| \le C$ holds for $j \in \{1,\cdots,n\}$, uniformly for γ,τ in the range $\tau \ge C_0\gamma$.

5. Prove that if ψ satisfies (2.65), then

$$\frac{1}{2i}\{\overline{p_\phi}, p_\phi\}(x,\xi,\tau,\gamma) = \gamma|\{p_2,\psi\}(x,\xi)|^2 + 4\gamma(\tau+\gamma\psi(x))^2 p_2(x, d\psi(x))^2,$$

for all $(x,\xi) \in \Omega\times\mathbb{R}^n, \tau,\gamma > 0$. What is the homogeneity of $\frac{1}{2i}\{\overline{p_\phi}, p_\phi\}$ in the variables (ξ,τ,γ)?

From now on, we assume that $P = P^*$, that is, $P = \sum_{i,j=1}^n D_i a^{ij}(x) D_j$ with $a^{ij}(x) = a^{ji}(x)$ real-valued and uniformly elliptic. We denote $P_\phi = e^{\tau\phi}Pe^{-\tau\phi}$, $P_R = \frac{P_\phi + P_\phi^*}{2}$ and $P_I = \frac{P_\phi - P_\phi^*}{2i}$.

6. Give the explicit expression of P_ϕ, P_R and P_I in terms of ϕ and its derivatives.

7. Give the explicit expression of P_ϕ, P_R and P_I in terms of ψ and its derivatives. Deduce that $P_\phi, P_R, P_I \in \mathrm{Diff}^2_{\tau,\gamma}(\Omega)$ and compute their principal symbols p_ϕ, p_R, p_I in this class.

8. Prove that $(P_R v, v)_{L^2} = \sum_{i,j=1}^n (a^{ij}D_j v, D_i v)_{L^2} - (p_2(\cdot,\tau d\phi)v, v)_{L^2}$ for all $v \in C^\infty_c(\Omega)$. Deduce the existence of a constant $C>0$ such that

$$\|\nabla v\|^2_{L^2} \le C\tau^2 \|v\|^2_{L^2} + C\tau^{-2}\|P_R v\|^2_{L^2}, \quad \text{for all } \tau>0, \quad \gamma\in(0,\tau/C_0], \quad v\in C^\infty_c(\Omega).$$

9. Prove that we have $i[P_R, P_I] = \frac{1}{2i}\{\overline{p_\phi}, p_\phi\}(x,D,\tau,\gamma) + R$, with $R \in \mathrm{Diff}^2_{\tau,\gamma}$. Here, $\{\overline{p_\phi}, p_\phi\}(x,D,\tau,\gamma)$ denotes the differential operator having $\{\overline{p_\phi}, p_\phi\}$ as full symbol.

10. Prove that for all $R\in \mathrm{Diff}^2_{\tau,\gamma}$, there is $C>0$ such that

$$|(Rv,v)_{L^2}| \le C\left(\tau^2\|v\|^2_{L^2} + \|\nabla v\|^2_{L^2}\right) \quad \text{for all } \tau\ge 1, \quad \gamma\in(0,\tau/C_0], \quad v\in C^\infty_c(\Omega).$$

11. Prove that $(i[P_R,P_I]v,v)_{L^2(\Omega)} = 4\gamma\|(\tau+\gamma\psi(x))p_2(x,d\psi(x))v\|^2_{L^2} + \gamma(R_2 v, v)_{L^2} + (Rv,v)_{L^2}$ for all $v\in C^\infty_c(\Omega)$, where $R_2 \in \mathrm{Diff}^2_{\tau,\gamma}$ is to be determined as well as its principal symbol.

12. Deduce that there is $C>0$ such that

$$\gamma(R_2 v,v)_{L^2} + (Rv,v)_{L^2} \ge -C\left(\tau^2\|v\|^2_{L^2} + \|\nabla v\|^2_{L^2}\right) \quad \text{for all } \tau\ge 1, \quad \gamma\in(0,\tau/C_0], \quad v\in C^\infty_c(\Omega).$$

13. Deduce that there are $\gamma_0, C>0$ such that

$$\gamma\tau^2\|v\|^2_{L^2} \le C\left(i[P_R,P_I]v,v\right)_{L^2(\Omega)} + C\|\nabla v\|^2_{L^2} \quad \text{for all } \gamma\ge\gamma_0, \quad \tau\ge C_0\gamma, \quad v\in C^\infty_c(\Omega).$$

14. Prove that there are $\gamma_0, C>0$ such that

$$\gamma\tau^2\|v\|^2_{L^2} + \gamma\|\nabla v\|^2_{L^2} \le C\left(i[P_R,P_I]v,v\right)_{L^2(\Omega)} + C\gamma\tau^{-2}\|P_R v\|^2_{L^2} \quad \text{for all } \gamma\ge\gamma_0, \tau\ge C_0\gamma, \quad v\in C^\infty_c(\Omega).$$

15. Deduce that there are $\gamma_0, C>0$ such that

$$\gamma\tau^2\|v\|^2_{L^2} + \gamma\|\nabla v\|^2_{L^2} \le C\|P_\phi v\|^2_{L^2} \quad \text{for all } \gamma\ge\gamma_0, \quad \tau\ge C_0\gamma, \quad v\in C^\infty_c(\Omega).$$

Prove that this estimate remains true if P is replaced by $P + W(x)\cdot\nabla + V(x)$ (up to changing the constants involved).

16. Prove that there are $\tau_0, C>0$ such that

$$\tau^2\left\|e^{\tau\phi}u\right\|^2_{L^2} + \left\|e^{\tau\phi}\nabla u\right\|^2_{L^2} \le C\left\|e^{\tau\phi}Pu\right\|^2_{L^2}, \quad \text{for all } u\in C^\infty_c(\Omega), \tau\ge\tau_0,$$

and conclude the proof of (2.66).

Correction 11. 1. ♣ `to be written one day`

2.

3. We start from Lemma 2.2.3, implying

$$\frac{1}{2i\tau}\{\overline{p_\phi}, p_\phi\}(x,\xi,\tau) = \frac{1}{\tau}\operatorname{Im}\left[\partial_\xi p_2(x,\xi - i\tau d\phi(x))\cdot \partial_x p_2(x,\xi+i\tau d\phi(x))\right] \\ + \operatorname{Hess}(\phi)(x)\left[\partial_\xi p_2(x,\xi - i\tau d\phi(x)); \partial_\xi p_2(x,\xi+i\tau d\phi(x))\right].$$

Recalling that $d\phi = G'\circ\psi d\psi$, taking $\xi = (G'\circ\psi)\eta$ and recalling that p_2 (resp. $\partial_\xi p_2$) is homogeneous of degree 2 (resp. degree 1) w.r.t the second variable, we obtain

$$\frac{1}{2i\tau}\{\overline{p_\phi}, p_\phi\}(x,\xi,\tau) = \frac{1}{\tau}(G'\circ\psi)^3\operatorname{Im}\left[\partial_\xi p_2(x,\eta - i\tau d\psi(x))\cdot \partial_x p_2(x,\eta+i\tau d\psi(x))\right] \\ + (G'\circ\psi)^2\operatorname{Hess}(\phi)(x)\left[\partial_\xi p_2(x,\eta - i\tau d\psi(x)); \partial_\xi p_2(x,\eta+i\tau d\psi(x))\right].$$

We have $\operatorname{Hess}(\phi) = (G'\circ\psi)\operatorname{Hess}(\psi) + (G''\circ\psi)d\psi\otimes d\psi$ so that

$$\begin{aligned}\frac{1}{2i\tau}\{\overline{p_\phi}, p_\phi\}(x,\xi,\tau) &= \frac{1}{\tau}(G'\circ\psi)^3\operatorname{Im}\left[\partial_\xi p_2(x,\eta - i\tau d\psi(x))\cdot \partial_x p_2(x,\eta+i\tau d\psi(x))\right] \\ &\quad + (G'\circ\psi)^3\operatorname{Hess}(\psi)(x)\left[\partial_\xi p_2(x,\eta - i\tau d\psi(x)); \partial_\xi p_2(x,\eta+i\tau d\psi(x))\right] \\ &\quad + (G'\circ\psi)^2(G''\circ\psi)\big(\partial\psi(x)\cdot\partial_\xi p_2(x,\eta - i\tau d\psi(x))\big)\big(\partial\psi(x)\cdot\partial_\xi p_2(x,\eta + i\tau d\psi(x))\big) \\ &= (G'\circ\psi)^3\frac{1}{2i\tau}\{\overline{p_\psi}, p_\psi\}(x,\eta,\tau) + (G'\circ\psi)^2(G''\circ\psi) \\ &\quad \times\big(\partial\psi(x)\cdot\partial_\xi p_2(x,\eta - i\tau d\psi(x))\big)\big(\partial\psi(x)\cdot\partial_\xi p_2(x,\eta + i\tau d\psi(x))\big)\end{aligned}$$

Using that $\partial_\xi p_2$ is homogeneous of degree 1, we may expand

$$\big(\partial\psi(x)\cdot\partial_\xi p_2(x,\eta - i\tau d\psi(x))\big)\big(\partial\psi(x)\cdot\partial_\xi p_2(x,\eta + i\tau d\psi(x))\big) \\ = \big(\partial\psi(x)\cdot\partial_\xi p_2(x,\eta)\big)^2 + \tau^2\big(\partial\psi(x)\cdot\partial_\xi p_2(x,d\psi(x))\big)^2.$$

Then, we notice that $\{p_2,\psi\}(x,\eta) = \partial_\xi p_2(x,\eta)\cdot\partial_x\psi(x)$. Moreover, since p_2 is a real-valued quadratic form, we can write $p_2(x,\xi) = \sum a^{ij}\xi_i\xi_j$ with $a^{ji} = a^{ij}$ real, and thus $\xi\cdot\partial_\xi p_2(x,\xi) = \sum 2a^{ij}\xi_i\xi_j = 2p_2(x,\xi)$, whence $\partial\psi(x)\cdot\partial_\xi p_2(x,d\psi(x)) = 2p_2(x,d\psi(x))$. Combining all these computations, we have obtained

$$\begin{aligned}\frac{1}{2i\tau}\{\overline{p_\phi}, p_\phi\}(x,\xi,\tau) &= (G'\circ\psi)^3\frac{1}{2i\tau}\{\overline{p_\psi}, p_\psi\}(x,\eta,\tau) \\ &\quad + (G'\circ\psi)^2(G''\circ\psi)\left(\big(\partial\psi(x)\cdot\partial_\xi p_2(x,\eta)\big)^2 + \tau^2\big(\partial\psi(x)\cdot\partial_\xi p_2(x,d\psi(x))\big)^2\right) \\ &= (G'\circ\psi)^3\frac{1}{2i\tau}\{\overline{p_\psi}, p_\psi\}(x,\eta,\tau) \\ &\quad + (G'\circ\psi)^2(G''\circ\psi)\left(\big(\{p_2,\psi\}(x,\eta)\big)^2 + \tau^2\big(2p_2(x,d\psi(x))\big)^2\right),\end{aligned}$$

which proves Lemma 2.7.2 after multiplication by τ.

4.

♣ `to be written one day`

Exercise 12 (Carleman estimates with linear weight, warm-up of the exam of May, 2020)**.** In this exercise, we consider the flat Laplace operator $P = -\Delta$ on a connected bounded open set $\Omega \subset \mathbb{R}^n$. We take $\alpha \in \mathbb{R}^n \setminus \{0\}$ a fixed vector and consider the weight function $\Phi(x) = \alpha\cdot x$.

1. Compute $P_\Phi = e^{\tau\Phi}Pe^{-\tau\Phi}$, its full symbol, and its principal symbol p_Φ.

2. Write $P_\Phi = P_R + iP_I$ where P_R and P_I are both selfadjoint. Compute the principal symbols p_R, p_I of P_R, P_I respectively, as well as their Poisson bracket $\{p_R, p_I\}$. What can we deduce, as far as classical Carleman estimates are concerned?

3. Let $L > 0$. Prove that for all $v \in C_c^\infty((0,L)\times\mathbb{R}^{n-1})$ and all $x_1 \in (0,L)$, we have
$$\int_{\mathbb{R}^{n-1}} |v(x_1,x')|^2 dx' \leq 2\left\|v\right\|_{L^2([0,L]\times\mathbb{R}^{n-1})}\left\|\partial_{x_1}v\right\|_{L^2([0,L]\times\mathbb{R}^{n-1})}$$

4. Deduce that there is C depending only on Ω, α such that $\left\|P_I v\right\|_{L^2(\Omega)} \geq C\tau\left\|v\right\|_{L^2(\Omega)}$ for all $\tau \in \mathbb{R}$ and all $v \in C_c^\infty(\Omega)$. *Hint: one may reduce to the case α is proportional to $e_1 = (1,0,\cdots,0)$, and consider $\Omega \subset [0,L]\times\mathbb{R}^{n-1}$.*

5. Prove that there is $C > 0$ such that
$$\left\|e^{\tau\alpha\cdot x}\Delta u\right\|_{L^2(\Omega)}^2 \geq C\tau^2\left\|e^{\tau\alpha\cdot x}u\right\|_{L^2(\Omega)}^2, \quad \text{for all } u \in C_c^\infty(\Omega), \tau \in \mathbb{R}.$$

6. Conclude that there is $C > 0$ such that
$$\tau^2\left\|e^{\tau\alpha\cdot x}u\right\|_{L^2(\Omega)}^2 + \left\|e^{\tau\alpha\cdot x}\nabla u\right\|_{L^2(\Omega)}^2 \leq C\left\|e^{\tau\alpha\cdot x}\Delta u\right\|_{L^2(\Omega)}^2, \quad \text{for all } u \in C_c^\infty(\Omega), |\tau| \geq 1.$$

7. Compare with usual Carleman estimates. Is this Carleman estimate sufficient for proving the following result: for a nonempty open set $\omega \subset \Omega$,
$$\Big(w \in C^\infty(\Omega),\ \Delta w = 0 \text{ in } \Omega,\ w = 0 \text{ in } \omega\Big) \implies w = 0 \text{ on } \Omega?$$

Correction 12. 1. We have $P_\Phi = -\Delta - \tau^2|\alpha|^2 + 2\tau\alpha\cdot\nabla$, its principal symbol and its full symbol are equal and given by $p_\phi(x,\xi,\tau) = |\xi|^2 - \tau^2|\alpha|^2 + 2i\tau\alpha\cdot\xi$.

2. We have $P_R = -\Delta - \tau^2|\alpha|^2$ and $P_I = 2\tau\alpha\cdot D$, with respective principal symbols $p_R(x,\xi,\tau) = |\xi|^2 - \tau^2|\alpha|^2$ and $p_I(x,\xi,\tau) = 2\tau\alpha\cdot\xi$. Finally, we have $\{p_R, p_I\} = 0$ identically (neither depends on x). The classical Hörmander subellipticity condition (2.3) is not satisfied: on the characteristic set $\{p_\phi = 0\} = \{(x,\xi,\tau) \in \Omega\times\mathbb{R}^n\times\mathbb{R}_+, \xi \perp \alpha, |\xi| = \tau|\alpha|\}$, the Poisson bracket $\{p_R, p_I\}$ vanishes. Hence the usual Carleman estimate cannot be true.

3. We have $|v(0,x')|^2 = 0$ and thus
$$|v(x_1,x')|^2 = \int_0^{x_1}\partial_s|v(s,x')|^2 ds = \int_0^{x_1} 2\,\mathrm{Re}\big(v(s,x')\partial_s\bar{v}(s,x')\big)ds,$$
and hence
$$\int_{\mathbb{R}^{n-1}}|v(x_1,x')|^2dx' = \int_0^{x_1}\int_{\mathbb{R}^{n-1}} 2\,\mathrm{Re}\big(v(s,x')\partial_s\bar{v}(s,x')\big)dsdx' \leq \int_0^L\int_{\mathbb{R}^{n-1}} 2|v(s,x')||\partial_s\bar{v}(s,x')|dsdx'$$
$$\leq 2\left\|v\right\|_{L^2([0,L]\times\mathbb{R}^{n-1})}\left\|\partial_{x_1}v\right\|_{L^2([0,L]\times\mathbb{R}^{n-1})}.$$

4. Then integrating in x_1, and dividing by $\left\|v\right\|_{L^2([0,L]\times\mathbb{R}^{n-1})}$ (if nonzero) implies for all $v \in C_c^\infty((0,L)\times\mathbb{R}^{n-1})$,
$$\left\|v\right\|_{L^2([0,L]\times\mathbb{R}^{n-1})} \leq 2L\left\|\partial_{x_1}v\right\|_{L^2([0,L]\times\mathbb{R}^{n-1})}.$$
Next, the operator Δ is rotationally/translationally invariant, so we may assume that $\alpha = |\alpha|e_1$ (that is to say $\alpha\cdot D = |\alpha|D_1$) and $\Omega \subset [0,L]\times\mathbb{R}^{n-1}$. We thus obtain, for all $v \in C_c^\infty(\Omega)$,
$$\left\|v\right\|_{L^2(\Omega)} \leq 2L\left\|D_1 v\right\|_{L^2(\Omega)} = \frac{2L}{|\alpha|}\left\|\alpha\cdot Dv\right\|_{L^2(\Omega)}.$$
Recalling that $P_I = 2\tau\alpha\cdot D$, we have obtained
$$\left\|P_I v\right\|_{L^2(\Omega)} = 2\tau\left\|\alpha\cdot Dv\right\|_{L^2(\Omega)} \geq \tau\frac{L}{|\alpha|}\left\|v\right\|_{L^2(\Omega)}.$$

5. We remark that $[P_R, P_I] = 0$, so that, for all $v \in C_c^\infty(\Omega)$ and $\tau \in \mathbb{R}$,

$$\begin{aligned}\|P_\Phi v\|^2_{L^2(\Omega)} &= \|P_I v\|^2_{L^2(\Omega)} + \|P_R v\|^2_{L^2(\Omega)} + (i[P_R, P_I]v, v)_{L^2(\Omega)} \\ &= \|P_I v\|^2_{L^2(\Omega)} + \|P_R v\|^2_{L^2(\Omega)} \geq \|P_I v\|^2_{L^2(\Omega)} \geq \tau^2 \frac{L^2}{|\alpha|^2} \|v\|^2_{L^2(\Omega)} .\end{aligned}$$

taken for $v = e^{\tau\alpha\cdot x}u$ with $u \in C_c^\infty(\Omega)$, this yields the sought result.

6. Then, we remark that

$$(P_R v, v)_{L^2(\Omega)} = (-\Delta v, v)_{L^2(\Omega)} - \tau^2|\alpha|^2 \|v\|^2_{L^2(\Omega)} = \|\nabla v\|^2_{L^2(\Omega)} - \tau^2|\alpha|^2 \|v\|^2_{L^2(\Omega)} .$$

As a consequence,

$$\|\nabla v\|^2_{L^2(\Omega)} = (P_R v, v)_{L^2(\Omega)} + \tau^2|\alpha|^2 \|v\|^2_{L^2(\Omega)} \leq \frac{1}{4|\alpha|^2} \|P_R v\|^2_{L^2(\Omega)} + |\alpha|^2 \|v\|^2_{L^2(\Omega)} + \tau^2|\alpha|^2 \|v\|^2_{L^2(\Omega)} .$$

Recalling that $\|P_\Phi v\|^2_{L^2(\Omega)} = \|P_I v\|^2_{L^2(\Omega)} + \|P_R v\|^2_{L^2(\Omega)}$, and using the previous question, we have thus obtained, for all $|\tau| \geq 1$,

$$\|\nabla v\|^2_{L^2(\Omega)} \leq \frac{1}{4|\alpha|^2} \|P_\Phi v\|^2_{L^2(\Omega)} + 2\frac{|\alpha|^4}{L^2} \|P_\Phi v\|^2_{L^2(\Omega)} .$$

Undoing the conjugation, we have $v = e^{\tau\alpha\cdot x}u$ hence $\nabla v = e^{\tau\alpha\cdot x}\nabla u + \tau\alpha v$ and

$$\|e^{\tau\alpha\cdot x}\nabla u\|^2_{L^2(\Omega)} \leq 2\|\nabla v\|^2_{L^2(\Omega)} + 2\tau^2|\alpha|^2 \|v\|^2_{L^2(\Omega)} \leq \left(\frac{1}{2|\alpha|^2} + 4\frac{|\alpha|^4}{L^2} + 2\frac{|\alpha|^4}{L^2}\right) \|e^{\tau\alpha\cdot x}Pu\|^2_{L^2(\Omega)} .$$

Together with the previous question, we obtain, for all $|\tau| \geq 1$,

$$\tau^2|\alpha|^2 \|e^{\tau\alpha\cdot x}u\|^2_{L^2(\Omega)} + \|e^{\tau\alpha\cdot x}\nabla u\|^2_{L^2(\Omega)} \leq \left(\frac{1}{2|\alpha|^2} + 7\frac{|\alpha|^4}{L^2}\right) \|e^{\tau\alpha\cdot x}Pu\|^2_{L^2(\Omega)} .$$

7. We have a loss of a power of τ in the left handside. This is a consequence of the fact that the subellipticity condition (2.3) is not satisfied. This Carleman estimate allows to propagate uniqueness in the direction α, but not in all other directions. Hence it does not imply straightforwardly unique continuation from any nonempty open set. For example, to fix ideas, if $\Omega = (-1,1)^2 \subset \mathbb{R}^2$ and $\alpha = e_1$, the Carleman estimate from which we want to extract information essentially reads $\|e^{\tau x_1}\chi w\|_{L^2(\Omega)} \leq C\|e^{\tau x_1}[\Delta, \chi]w\|_{L^2(\Omega)}$. But if $\omega = (-1,1) \times (-\delta, \delta)$, we cannot choose $\chi \in C_c^\infty((-1,1)^2)$ such that $\operatorname{supp}(\nabla\chi)$ has two connected components, one included in ω, and both contained between two levelsets of $\Phi = x_1$.

   Note that this Carleman estimate is actually a particular case of Theorem 2.7.1 in Exercice 11. However, the two proofs are very different. Here, we have an exact cancellation of the commutator $[P_R, P_I]$ and the positivity comes from coercivity of P_I (a Poincaré inequality). In Exercice 11, only $\{p_R, p_I\} = 0$ (hence $[P_R, P_I] \in \mathrm{Diff}^2_\tau$ instead of Diff^3_τ) and we convexify (in a subtle way) the weight function to have positivity (in class Diff^2_τ only, whence the loss of a power of τ) from the commutator $[P_R, P_I]$.

Exercise 13 (Carleman estimate for the Laplace operator with singular weight, part of the exam of May, 2020)**.** In this exercise, we consider the flat Laplace operator $P = -\Delta$ on $\mathbb{R}^n \setminus \{0\}$, together with the weight function $\Phi(x) = -\log|x|$. We also denote (with a slight abuse of notation) by $F(x)$ the operator of multiplication by $F(x)$, e.g. the operator $D_j|x|^\gamma$ is defined by $\big(D_j|x|^\gamma u\big)(x) = D_j(|x|^\gamma u(x))$. The derivative of the function will be denoted with parentheses, e.g. $D_j x_j^2$ is an operator ($= x_j^2 D_j + \frac{2}{i}x_j$) whereas $D_j(x_j^2) = \frac{2}{i}x_j$ is a function.

1. Compute $D_{j,\tau} = |x|^{-\tau} D_j |x|^\tau$.

2. Compute $|x|^{1-\tau} D_j^2 |x|^{\tau+1} = P_{R,j} + iP_{I,j}$ where both $P_{R,j}$ and $P_{I,j}$ are formally selfadjoint. Write $P_{I,j}$ in terms of the operator $x_j D_j + D_j x_j$.

3. Write $-|x|^{1-\tau}\Delta|x|^{\tau+1} = P_R + iP_I$ where both P_R and P_I are formally selfadjoint. Write P_I in terms of the operator
$$A := \frac{1}{2}\left(x \cdot D + D \cdot x\right) = x \cdot D - \frac{in}{2},$$
where we have taken the gradient/divergence notation $(x \cdot D)u(x) = \sum_{j=1}^n x_j D_j u(x)$ and $(D \cdot x)u(x) = \sum_{j=1}^n D_j(x_j u(x))$. Compare with the usual computation of $P_\Phi := e^{\tau\Phi} P e^{-\tau\Phi}$.

4. (About the operator A). Compute principal symbol $a(x,\xi)$ of A and the Hamiltonian flow of the a. For a smooth function $f : (0,+\infty) \to \mathbb{R}$, compute $\{a, f(|x|)\}$.

5. Compute the principal symbols p_R, p_I of P_R, P_I respectively, as well as their Poisson bracket $\{p_R, p_I\}$. What can we deduce, as far as classical Carleman estimates are concerned?

6. Compute all following commutators: $[iA, D_j], [iA, x_j], [iA, D_j^2], [iA, \Delta], [iA, |x|]$.

7. Compute $[P_R, P_I]$.

8. (a) In this question, we consider the operator $P_\lambda := -\Delta + \lambda$ for $\lambda \in \mathbb{R}$. Prove that for all $u \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, all $\tau \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, we have
$$\left\| |x|^{1-\tau} P_\lambda u \right\|_{L^2(\mathbb{R}^n)}^2 \geq 4\tau\lambda \left\| |x|^{-\tau} u \right\|_{L^2(\mathbb{R}^n)}^2. \tag{2.67}$$
   (b) In this question, we consider the operator $P_V := -\Delta + V(x)$ for $V \in L^\infty_{\text{loc}}(\mathbb{R}^n)$. Prove that for all $R > 0$, there exists $\tau_0, C > 0$ such that
$$\left\| |x|^{1-\tau} P_V u \right\|_{L^2(\mathbb{R}^n)}^2 \geq \tau C \left\| |x|^{-\tau} u \right\|_{L^2(\mathbb{R}^n)}^2, \quad \text{for all } u \in C_c^\infty(B(0,R) \setminus \{0\}), \tau \geq \tau_0. \tag{2.68}$$

9. (Positivity estimates for the operator A) Using the positivity of $\|(A - if(|x|))\, v\|_{L^2(\mathbb{R}^n)}^2$ for real-valued functions f, prove that for all $R > 0$, there is $C > 0$ such that
$$\|Av\|_{L^2(\mathbb{R}^n)}^2 \geq C \left\| \sqrt{|x|} v \right\|_{L^2(\mathbb{R}^n)}^2, \quad \text{for all } v \in C_c^\infty(B(0,R) \setminus \{0\}),$$
$$\|Av\|_{L^2(\mathbb{R}^n)}^2 \geq C \left\| \frac{1}{\log(R) - \log(|x|)} v \right\|_{L^2(\mathbb{R}^n)}^2, \quad \text{for all } v \in C_c^\infty(B(0,R) \setminus \{0\}).$$
Compare these two inequalities with each other.

10. Deduce that for all $R > 0$, there is $C > 0$ such that
$$\left\| |x|^{1-\tau} \Delta u \right\|_{L^2(\mathbb{R}^n)}^2 \geq C\tau^2 \left\| |x|^{-\tau-\frac{1}{2}} u \right\|_{L^2(\mathbb{R}^n)}^2, \quad \text{for all } u \in C_c^\infty(B(0,R) \setminus \{0\}), \tau \in \mathbb{R}$$
$$\left\| |x|^{1-\tau} \Delta u \right\|_{L^2(\mathbb{R}^n)}^2 \geq C\tau^2 \left\| \frac{|x|^{-\tau-1}}{\log(R) - \log(|x|)} u \right\|_{L^2(\mathbb{R}^n)}^2, \quad \text{for all } u \in C_c^\infty(B(0,R) \setminus \{0\}), \tau \in \mathbb{R}.$$
Compare these two inequalities with each other, and compare with (2.67)-(2.68).

11. We now want to obtain an estimate on the gradient term.
   (a) Compute $[-\Delta, f(|x|)]$ on $\mathbb{R}^n \setminus \{0\}$ for a smooth function $f : (0,+\infty) \to \mathbb{R}$.
   (b) Prove that for all $R > 0$, there is a constant $C > 0$ such that
$$\left\| D|x|^{3/2} v \right\|_{L^2}^2 \leq C \left\| |x|^{1-\tau} \Delta |x|^{\tau+1} v \right\|_{L^2}^2, \text{ for all } v \in C_c^\infty(B(0,R) \setminus \{0\}), |\tau| \geq 1,$$
$$\left\| D \left(\log\left(\frac{R}{|x|}\right) \right)^{-1} |x| v \right\|_{L^2}^2 \leq C \left\| |x|^{1-\tau} \Delta |x|^{\tau+1} v \right\|_{L^2}^2, \text{ for all } v \in C_c^\infty(B(0,R/2) \setminus \{0\}), |\tau| \geq 1.$$
   NB: as above, $Df(|x|)|x|$ stands for the composition of the (vector-valued) operator $D = \frac{1}{i}\nabla$ and the multiplication by $f(|x|)|x|$.

12. Prove the following Carleman estimates: for all $R > 0$, there is $C > 0$ such that

$$\tau^2 \left\| |x|^{-\tau-\frac{1}{2}} u \right\|^2_{L^2(\mathbb{R}^n)} + \left\| |x|^{-\tau+\frac{1}{2}} Du \right\|^2_{L^2(\mathbb{R}^n)} \leq C \left\| |x|^{1-\tau} \Delta u \right\|^2_{L^2(\mathbb{R}^n)},$$
$$\text{for all } u \in C^\infty_c(B(0,R) \setminus \{0\}), |\tau| \geq 1,$$
$$\tau^2 \left\| |x|^{-\tau-1} \left(\log\left(\frac{R}{|x|}\right)\right)^{-1} u \right\|^2_{L^2(\mathbb{R}^n)} + \left\| |x|^{-\tau} \left(\log\left(\frac{R}{|x|}\right)\right)^{-1} Du \right\|^2_{L^2(\mathbb{R}^n)} \leq C \left\| |x|^{1-\tau} \Delta u \right\|^2_{L^2(\mathbb{R}^n)},$$
$$\text{for all } u \in C^\infty_c(B(0,R/2) \setminus \{0\}), |\tau| \geq 1.$$

13. (Bonus: only treat this question if some time is left) Reprove these Carleman estimates in polar coodinates $(r, \omega) \in (0, +\infty) \times \mathbb{S}^{n-1}$, where $r = |x|$. We recall that the Laplace operator is given by

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}} = \frac{1}{r}\partial_r r \partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}},$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the Laplace operator on the sphere in the variable ω.

14. (Bonus: only treat this question if some time is left) What can we say if we replace $-\Delta$ by $P = \sum_{j,k} a^{jk}(x) D_j D_k \in \mathrm{Diff}^2(\mathbb{R}^n)$ such that $C|\xi|^2 \geq \sum_{j,k} a^{jk}(x)\xi_j\xi_k \geq C|\xi|^2$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$?

Correction 13. 1. We compute $D_{j,\tau} := |x|^{-\tau} D_j |x|^\tau$ by

$$\partial_j(|x|^\tau u) = |x|^\tau \partial_j u + u \partial_j(|x|^\tau) = |x|^\tau \partial_j u + u\tau x_j |x|^{\tau-2},$$

and hence

$$D_{j,\tau} = |x|^{-\tau} D_j |x|^\tau = D_j - i\tau \frac{x_j}{|x|^2}.$$

♣ `parler des facteurs de` $|x|$ `ajoutes pour avoir de l'invariance par scaling.`

2. Hence we can compute

$$|x|^{1-\tau} D_j^2 |x|^{\tau+1} = |x| \Big(|x|^{-\tau} D_j |x|^\tau \Big) \Big(|x|^{-\tau} D_j |x|^\tau \Big) |x| = |x| D_{j,\tau} D_{j,\tau} |x|$$
$$= |x| \left(D_j - i\tau \frac{x_j}{|x|^2} \right) \left(D_j - i\tau \frac{x_j}{|x|^2} \right) |x|$$

Expanding this expression, we obtain

$$|x|^{1-\tau} D_j^2 |x|^{\tau+1} = |x| D_j^2 |x| - \tau^2 \frac{x_j^2}{|x|^2} - i\tau \left(\frac{x_j}{|x|} D_j |x| + |x| D_j \frac{x_j}{|x|} \right)$$

Now, remarking that $\frac{1}{|x|}\partial_j(|x|) + |x|\partial_j(\frac{1}{|x|}) = \frac{1}{|x|}\frac{x_j}{|x|} + |x|\left(-\frac{x_j}{|x|^3}\right) = 0$, we deduce that

$$\frac{x_j}{|x|} D_j |x| + |x| D_j \frac{x_j}{|x|} = x_j D_j + \frac{x_j}{|x|} D_j(|x|) + D_j x_j + |x| D_j(\frac{x_j}{|x|}) = x_j D_j + D_j x_j$$

is selfadjoint, so that

$$|x|^{1-\tau} D_j^2 |x|^{\tau+1} = P_{j,R} + iP_{j,I}, \quad P_{j,R} = |x| D_j^2 |x| - \tau^2 \frac{x_j^2}{|x|^2}, \quad P_{j,I} = -\tau \left(x_j D_j + D_j x_j \right).$$

3. We deduce that

$$-|x|^{1-\tau} \Delta |x|^{\tau+1} = P_R + iP_I, \quad \text{with}$$
$$P_R = -|x| \Delta |x| - \tau^2 = |x| |D|^2 |x| - \tau^2,$$
$$P_I = -\tau \sum_{j=1}^n (x_j D_j + D_j x_j) = -\tau (x \cdot D + D \cdot x) = -2\tau A,$$

Note that the choice $\Phi(x) = -\log|x|$ leads to $e^{-\tau\Phi(x)} = |x|^\tau$. Hence, the usual computation is $P_\Phi := e^{\tau\Phi}Pe^{-\tau\Phi} = -|x|^\tau\Delta|x|^\tau$, so that

Advantage : scale invariant.

4. We have $a(x,\xi) = x\cdot\xi$. Its Hamiltonian flow is defined by $(x_t,\xi_t)(x_0,\xi_0)$ where

$$\dot{x}_t = \partial_\xi a(x_t,\xi_t) = x_t, \quad \dot{\xi}_t = -\partial_x a(x_t,\xi_t) = -\xi_t,$$

hence we have $(x_t,\xi_t)(x_0,\xi_0) = (e^t x_0, e^{-t}\xi_0)$. This linear flow is (for positive t) a dilation in the space variable x and a contraction in the frequency variable ξ. Finally, we have $\{a, f(|x|)\} = \partial_\xi a\cdot\partial_x(f(|x|)) = x\cdot\frac{x}{|x|}f'(|x|) = |x|f'(|x|)$. In particular, $\{a,\Phi\} = \{a, -\log|x|\} = -1$.

5. We have $p_R(x,\xi) = |x|^2|\xi|^2 - \tau^2$ and $p_I(x,\xi) = -2\tau x\cdot\xi$ and thus (on the whole $\mathbb{R}^n\times\mathbb{R}^n$)

$$\begin{aligned}\{p_R,p_I\}(x,\xi) &= \partial_\xi p_R(x,\xi)\cdot\partial_x p_I(x,\xi) - \partial_x p_R(x,\xi)\cdot\partial_\xi p_I(x,\xi)\\ &= 2|x|^2\xi\cdot(-2\tau\xi) - 2|\xi|^2 x\cdot(-2\tau x) = -4\tau^2|x|^2|\xi|^2 + 4\tau^2|\xi|^2|x|^2 = 0.\end{aligned}$$

6. We have

$$\begin{aligned}[iA, D_j] &= [x_j\partial_j, D_j] = -D_j(x_j)\partial_j = -D_j,\\ [iA, x_j] &= [x_j\partial_j, x_j] = x_j\partial_j(x_j) = x_j,\\ [iA, D_j^2] &= D_j[iA,D_j] + [iA,D_j]D_j = -2D_j^2,\\ [iA,\Delta] &= -\sum_{j=1}^n [iA, D_j^2] = 2\sum_{j=1}^n D_j^2 = -2\Delta,\\ [iA,|x|] &= \sum_{k=1}^n [x_k\partial_k, |x|] = \sum_{k=1}^n x_k\partial_k(|x|) = \sum_{k=1}^n \frac{x_k^2}{|x|} = |x|.\end{aligned}$$

7. Recalling that $P_I = -2\tau A$, we instead compute using the last question

$$\begin{aligned}[iA, -P_R] &= [iA, |x|\Delta|x|] = |x|\,[iA,\Delta|x|] + [iA,|x|]\,\Delta|x|\\ &= |x|\,[iA,\Delta]\,|x| + |x|\Delta\,[iA,|x|] + [iA,|x|]\,\Delta|x|\\ &= |x|(-2\Delta)|x| + |x|\Delta|x| + |x|\Delta|x| = 0,\end{aligned}$$

that is to say, $[P_R,P_I] = 0$. Note that this is consistent with the fact that $\{p_R,p_I\} = 0$ (but this is even stronger!).

8. (a) Now we consider $P_\lambda = -\Delta + \lambda$, and we perform the same decomposition:

$$|x|^{1-\tau}P_\lambda|x|^{\tau+1} = -|x|^{1-\tau}\Delta|x|^{\tau+1} + \lambda|x|^2 = P_{R,\lambda} + iP_I,$$

where $P_{R,\lambda} := P_R + \lambda|x|^2$ is formally selfadjoint. As usual, we have for all $v\in C_c^\infty(\mathbb{R}^n\setminus\{0\})$,

$$\begin{aligned}\left\||x|^{1-\tau}P_\lambda|x|^{\tau+1}v\right\|_{L^2(\mathbb{R}^n)}^2 &= \left((P_{R,\lambda}+iP_I)\,v, (P_{R,\lambda}+iP_I)\,v\right)_{L^2(\mathbb{R}^n)}\\ &= \|P_{R,\lambda}v\|_{L^2(\mathbb{R}^n)}^2 + \|P_I v\|_{L^2(\mathbb{R}^n)}^2 + (i[P_{R,\lambda},P_I]v,v)_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Here,

$$i[P_{R,\lambda},P_I] = i\,[P_R,P_I] + i\left[\lambda|x|^2, P_I\right] = 0 + i\left[\lambda|x|^2, -2\tau A\right] = 2\tau\lambda[iA,|x|^2],$$

where $[iA,|x|^2] = |x|[iA,|x|] + [iA,|x|]|x| = 2|x|^2$. Hence, $i[P_{R,\lambda},P_I] = 4\tau\lambda|x|^2$ and we have the inequality:

$$\left\||x|^{1-\tau}P_\lambda|x|^{\tau+1}v\right\|_{L^2(\mathbb{R}^n)}^2 \geq (i[P_{R,\lambda},P_I]v,v)_{L^2(\mathbb{R}^n)} = \left(4\tau\lambda|x|^2 v, v\right)_{L^2(\mathbb{R}^n)} = 4\tau\lambda\,\||x|v\|_{L^2(\mathbb{R}^n)}^2.$$

As a consequence, writing $u = |x|^{\tau+1}v$, that is $v = |x|^{-\tau-1}u$ (for $u\in C_c^\infty(\mathbb{R}^n\setminus\{0\})$, that is, supported away from zero) we obtain the inequality (2.67).

Note that this simple inequality has applications to the absence of embedded eigenvalues. We refer to Proposition 14.7.1 in [Hör83] and Theorem 14.7.2 (which is a unique continuation statement from infinity) stating that no eigenvalues are embedded in the continuous spectrum of $-\Delta + V$ if V is a suitable short range perturbation of the flat Laplace operator Δ.

(b) We first fix $\lambda = 1$. If $P_V = -\Delta + V(x) = P_1 + (V(x) - 1)$, application of (2.67) with $\lambda = 1$ yields for all $\tau \in \mathbb{R}$ and $u \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$,

$$4\tau \left\| |x|^{-\tau} u \right\|_{L^2(\mathbb{R}^n)}^2 \leq \left\| |x|^{1-\tau} P_1 u \right\|_{L^2(\mathbb{R}^n)}^2 \leq 2 \left\| |x|^{1-\tau} P_V u \right\|_{L^2(\mathbb{R}^n)}^2 + 2 \left\| |x|^{1-\tau} (V(x) - 1) u \right\|_{L^2(\mathbb{R}^n)}^2$$

Using that $\mathrm{supp}(u) \subset B(0, R)$, we further have

$$\begin{aligned} \left\| |x|^{1-\tau} (V(x) - 1) u \right\|_{L^2(\mathbb{R}^n)} &\leq \| |x| (V(x) - 1) \|_{L^\infty(B(0,R))} \left\| |x|^{-\tau} u \right\|_{L^2(\mathbb{R}^n)} \\ &\leq \left\| |x|^{-\tau} u \right\|_{L^2(\mathbb{R}^n)} \end{aligned}$$

These two inequalities imply

$$\left(4\tau - 2R^2 (\|V\|_{L^\infty(B(0,R))} + 1)^2\right) \left\| |x|^{-\tau} u \right\|_{L^2(\mathbb{R}^n)}^2 \leq 2 \left\| |x|^{1-\tau} P_V u \right\|_{L^2(\mathbb{R}^n)}^2,$$

whence the sought result when taking $\tau \geq \tau_0$ with $\tau_0 = 2R^2 (\|V\|_{L^\infty(B(0,R))} + 1)^2$.

9. As above, we have

$$[iA, F(x)] = x \cdot \nabla F(x), \quad [iA, f(|x|)] = x \cdot \left(f'(|x|) \frac{x}{|x|} \right) = f'(|x|)|x|. \tag{2.69}$$

We see that $[iA, F(x)]$ is maximal when F is radial and increasing. The second commutator is then more precise.

Then we compute, for a real-valued function f,

$$\begin{aligned} \| (A - i f(|x|)) v \|_{L^2(\mathbb{R}^n)}^2 &= \|Av\|_{L^2(\mathbb{R}^n)}^2 + \|f(|x|) v\|_{L^2(\mathbb{R}^n)}^2 - (i[A, f(|x|)] v, v)_{L^2(\mathbb{R}^n)} \\ &= \|Av\|_{L^2(\mathbb{R}^n)}^2 + \left(f(|x|)^2 v, v\right)_{L^2(\mathbb{R}^n)} - (f'(|x|)|x| v, v)_{L^2(\mathbb{R}^n)} \end{aligned}$$

Hence, we may write

$$\|Av\|_{L^2(\mathbb{R}^n)}^2 \geq \left(\left(f'(|x|)|x| - f(|x|)^2\right) v, v \right)_{L^2(\mathbb{R}^n)} \tag{2.70}$$

We now choose the function f. Taking $f(s) = \frac{s}{2R}$ implies

$$f'(s)s - f(s)^2 = \frac{s}{2R} - \frac{s^2}{4R^2} \geq \frac{s}{2R} - \frac{sR}{4R^2} = \frac{s}{4R} \quad \text{on the set } \{0 < s < R\}.$$

Together with (2.70), this yields

$$\|Av\|_{L^2(\mathbb{R}^n)}^2 \geq \left(\frac{|x|}{4R} v, v \right)_{L^2(\mathbb{R}^n)} = \frac{1}{4R} \left\| \sqrt{|x|} v \right\|_{L^2(\mathbb{R}^n)}^2, \quad \text{for all } v \in C_c^\infty(B(0, R) \setminus \{0\}).$$

If we try to optimize and instead choose f so that $f'(|x|)|x| = 2f(|x|)^2$, this will imply

$$\|Av\|_{L^2(\mathbb{R}^n)}^2 \geq \|f(|x|) v\|_{L^2(\mathbb{R}^n)}^2.$$

This means $f'(s)s = 2f(s)^2$, that is to say, for $R > 0$ and $s \in (0, R]$, $f(s) = \frac{1}{2(\log(R) - \log(s))} = \left(2 \log\left(\frac{R}{s}\right)\right)^{-1}$. Coming back to the last inequality, this yields

$$\|Av\|_{L^2(\mathbb{R}^n)}^2 \geq \frac{1}{4} \left\| \frac{1}{\log(R) - \log(|x|)} v \right\|_{L^2(\mathbb{R}^n)}^2, \quad \text{for all } v \in C_c^\infty(B(0, R) \setminus \{0\}).$$

These two inequalities have exactly the same form, except for the weight in the right handside. Notice that the function $s \mapsto \frac{1}{\log(R) - \log(s)} = \left(\log\left(\frac{R}{s}\right)\right)^{-1}$ may be extended by continuity by 0 at $s = 0$, tends to $+\infty$ as $s \to R^-$ and is strictly increasing on $[0, R)$. Near 0 it is bigger than every power of s^γ, $\gamma > 0$ (the smaller γ, the better the estimate is). Hence, the second estimate is better than the first one!

10. The usual computation together with the nullity of the commutator in Question 7 implies that for all $v \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$,

$$\left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|_{L^2(\mathbb{R}^n)}^2 = \|P_R v\|_{L^2(\mathbb{R}^n)}^2 + \|P_I v\|_{L^2(\mathbb{R}^n)}^2 + (i[P_R, P_I]v, v)_{L^2(\mathbb{R}^n)}$$
$$= \|P_R v\|_{L^2(\mathbb{R}^n)}^2 + \|P_I v\|_{L^2(\mathbb{R}^n)}^2 .$$

We have moreover proved that $P_I = -2\tau A$, so that

$$\left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|_{L^2(\mathbb{R}^n)}^2 \geq \|P_I v\|_{L^2(\mathbb{R}^n)}^2 = 4\tau^2 \|Av\|_{L^2(\mathbb{R}^n)}^2 .$$

We deduce from the previous question that

$$\left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|_{L^2(\mathbb{R}^n)}^2 \geq R^{-1}\tau^2 \left\|\sqrt{|x|}v\right\|_{L^2(\mathbb{R}^n)}^2, \quad \text{for all } v \in C_c^\infty(B(0,R)\setminus\{0\}), \tau \in \mathbb{R} \tag{2.71}$$

$$\left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|_{L^2(\mathbb{R}^n)}^2 \geq \tau^2 \left\|\frac{1}{\log(R)-\log(|x|)}v\right\|_{L^2(\mathbb{R}^n)}^2, \quad \text{for all } v \in C_c^\infty(B(0,R)\setminus\{0\}), \tau \in \mathbb{R}. \tag{2.72}$$

As a consequence, writing $u = |x|^{\tau+1}v$, that is $v = |x|^{-\tau-1}u$ (for $u \in C_c^\infty(B(0,R)\setminus\{0\})$) we obtain

$$\left\||x|^{1-\tau}\Delta u\right\|_{L^2(\mathbb{R}^n)}^2 \geq R^{-1}\tau^2 \left\||x|^{-\tau-\frac{1}{2}}u\right\|_{L^2(\mathbb{R}^n)}^2, \quad \text{for all } u \in C_c^\infty(B(0,R)\setminus\{0\}), \tau \in \mathbb{R}$$

$$\left\||x|^{1-\tau}\Delta u\right\|_{L^2(\mathbb{R}^n)}^2 \geq \tau^2 \left\|\frac{|x|^{-\tau-1}}{\log(R)-\log(|x|)}u\right\|_{L^2(\mathbb{R}^n)}^2, \quad \text{for all } u \in C_c^\infty(B(0,R)\setminus\{0\}), \tau \in \mathbb{R}.$$

As above, the second inequality is better than the first one. When comparing with (2.68), both of these inequalities have a better power of τ (τ^2 instead of τ), and moreover a better power of $|x|$ near zero ($|x|^{-\tau-\frac{1}{2}}$ or even $\frac{|x|^{-\tau-1}}{\log(R)-\log(|x|)}$ instead of $|x|^{-\tau}$). The case of a power $|x|^{-\tau-1}$ is critical with respect to scaling and requires a finer analysis.

Note however that (2.67) does not require functions to be supported in a compact set, and can also be useful near infinity as already mentioned.

11. (a) We have $[D_j, f(|x|)] = D_j(f(|x|)) = \frac{1}{i}\frac{x_j}{|x|}f'(|x|)$ and hence

$$\begin{aligned}
[D_j^2, f(|x|)] &= D_j[D_j, f(|x|)] + [D_j, f(|x|)]D_j = D_j\frac{1}{i}\frac{x_j}{|x|}f'(|x|) + \frac{1}{i}\frac{x_j}{|x|}f'(|x|)D_j \\
&= D_j[D_j, f(|x|)] + [D_j, f(|x|)]D_j \\
&= \frac{1}{i}\frac{1}{|x|}f'(|x|)D_j x_j + \frac{1}{i}f'(|x|)x_j D_j\frac{1}{|x|} + \frac{1}{i}\frac{x_j}{|x|}D_j(f'(|x|)) + \frac{1}{i}\frac{x_j}{|x|}f'(|x|)D_j \\
&= \frac{1}{i}\frac{1}{|x|}f'(|x|)D_j x_j - f'(|x|)\frac{x_j^2}{|x|^3} - \frac{x_j^2}{|x|^2}f''(|x|) + \frac{1}{i}\frac{1}{|x|}f'(|x|)x_j D_j.
\end{aligned}$$

Summing in j and recognizing the definition of A (in the first + last terms), we have obtained

$$[|D|^2, f(|x|)] = \frac{1}{i}\frac{1}{|x|}f'(|x|)2A + \frac{f'(|x|)}{|x|} - f''(|x|). \tag{2.73}$$

(b) We want to use that $|x||D|^2|x| = P_R + \tau^2$ and that we have control on $P_R v$ and on v (in appropriate weighted norms). To this aim, we compute for real-valued f

$$\begin{aligned}
\|Df(|x|)|x|v\|_{L^2(\mathbb{R}^n)}^2 &= (Df(|x|)|x|v, Df(|x|)|x|v)_{L^2(\mathbb{R}^n)} = \left(|x||D|^2 f(|x|)|x|v, f(|x|)v\right)_{L^2(\mathbb{R}^n)} \\
&= \left(|x||D|^2|x|v, f(|x|)^2 v\right)_{L^2(\mathbb{R}^n)} + \left(|x|[|D|^2, f(|x|)]|x|v, f(|x|)v\right)_{L^2(\mathbb{R}^n)} \\
&= \left(P_R v, f(|x|)^2 v\right)_{L^2(\mathbb{R}^n)} + \tau^2 \|f(|x|)v\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + \left(|x|[|D|^2, f(|x|)]|x|v, f(|x|)v\right)_{L^2(\mathbb{R}^n)} .
\end{aligned}$$

Note that the quantity on the left handside is real, so that we may take real parts:

$$\|Df(|x|)|x|v\|^2_{L^2(\mathbb{R}^n)} = \mathrm{Re}\left(P_R v, f(|x|)^2 v\right)_{L^2(\mathbb{R}^n)} + \tau^2 \|f(|x|)v\|^2_{L^2(\mathbb{R}^n)} + \mathrm{Re}\left([|D|^2, f(|x|)]|x|v, f(|x|)|x|v\right)_{L^2(\mathbb{R}^n)}. \tag{2.74}$$

Now, if A, B, C are selfadjoint, $[A,B]^* = -[A,B]$ and $\mathrm{Re}(C[A,B]) = \frac{1}{2}\big((C[A,B]) + (C[A,B])^*\big) = \frac{1}{2}\big(C[A,B] - [A,B]C\big) = \frac{1}{2}[C,[A,B]]$. This, together with (2.73) implies

$$\begin{aligned}\mathrm{Re}\left(f(|x|)[|D|^2, f(|x|)]\right) &= \frac{1}{2}\left[f(|x|), [|D|^2, f(|x|)]\right] = \frac{1}{2}\left[f(|x|), \frac{1}{i}\frac{1}{|x|}f'(|x|)2A\right] \\ &= \frac{1}{i}\frac{1}{|x|}f'(|x|)\left[f(|x|), A\right] = \frac{1}{|x|}f'(|x|)f'(|x|)|x| = f'(|x|)^2,\end{aligned}$$

after having used (2.69). Coming back to (2.74), we deduce

$$\begin{aligned}\|Df(|x|)|x|v\|^2_{L^2(\mathbb{R}^n)} &= \mathrm{Re}\left(P_R v, f(|x|)^2 v\right)_{L^2(\mathbb{R}^n)} + \tau^2 \|f(|x|)v\|^2_{L^2(\mathbb{R}^n)} \\ &\quad + \left(f'(|x|)^2|x|v, |x|v\right)_{L^2(\mathbb{R}^n)} \\ &= \mathrm{Re}\left(P_R v, f(|x|)^2 v\right)_{L^2(\mathbb{R}^n)} + \tau^2 \|f(|x|)v\|^2_{L^2(\mathbb{R}^n)} + \|f'(|x|)|x|v\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

We have seen that $\left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|^2_{L^2(\mathbb{R}^n)} = \|P_R v\|^2_{L^2(\mathbb{R}^n)} + \|P_I v\|^2_{L^2(\mathbb{R}^n)}$, so that

$$\mathrm{Re}\left(P_R v, f(|x|)^2 v\right)_{L^2} \le \left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|_{L^2}\left\|f(|x|)^2 v\right\|_{L^2},$$

and hence

$$\|Df(|x|)|x|v\|^2_{L^2} \le \left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|_{L^2}\left\|f(|x|)^2 v\right\|_{L^2} + \tau^2 \|f(|x|)v\|^2_{L^2(\mathbb{R}^n)} + \|f'(|x|)|x|v\|^2_{L^2(\mathbb{R}^n)}. \tag{2.75}$$

We may now choose the function f. Taking for instance $f(s) = (s/R)^{1/2}$ and recalling that functions are supported in $B(0,R)\setminus\{0\}$, we obtain

$$\begin{aligned}R^{-1}\left\|D|x|^{3/2}v\right\|^2_{L^2} &\le \left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|_{L^2} R^{-1}\||x|v\|_{L^2} + \tau^2 R^{-1}\left\|\sqrt{|x|}v\right\|^2_{L^2} + (4R)^{-1}\left\|\sqrt{|x|}v\right\|^2_{L^2} \\ &\le \frac{1}{2}\left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|^2_{L^2} + \frac{1}{2}R^{-2}\||x|v\|_{L^2} + \left(\tau^2 + \frac{1}{4}\right)R^{-1}\left\|\sqrt{|x|}v\right\|^2_{L^2} \\ &\le \frac{1}{2}\left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|^2_{L^2} + \left(\tau^2 + \frac{3}{4}\right)R^{-1}\left\|\sqrt{|x|}v\right\|^2_{L^2}\end{aligned}$$

Using our estimate (2.71), we have now obtained, for $v \in C_c^\infty(B(0,R)\setminus\{0\})$ and $|\tau| \ge 1$

$$R^{-1}\left\|D|x|^{3/2}v\right\|^2_{L^2} \le \frac{1}{2}\left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|^2_{L^2} + 2\left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|^2_{L^2} = \frac{5}{2}\left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|^2_{L^2} \tag{2.76}$$

(c) We also remark that $D|x|^{3/2}v = \frac{3}{2i}\frac{x}{|x|^{1/2}}v + |x|^{3/2}Dv$ so that using again (2.71), we have, for $|\tau| \ge 1$

$$R^{-1}\left\||x|^{3/2}Dv\right\|^2_{L^2} \le R^{-1}\frac{3}{2}\left\|\sqrt{|x|}v\right\|^2_{L^2(\mathbb{R}^n)} + \frac{5}{2}\left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|^2_{L^2} \le 4\left\||x|^{1-\tau}\Delta|x|^{\tau+1}v\right\|^2_{L^2}$$

Coming back to $u = |x|^{\tau+1}v$, we have $Du = |x|^{\tau+1}Dv + (\tau+1)|x|^{\tau-1}xv$ and thus

$$\left\||x|^{-\tau+1/2}Du\right\|^2_{L^2} \le 2\left\||x|^{3/2}Dv\right\|^2_{L^2} + 2(\tau+1)^2\left\||x|^{1/2}v\right\|^2_{L^2} \le 2\left\||x|^{3/2}Dv\right\|^2_{L^2} + 8\tau^2\left\||x|^{1/2}v\right\|^2_{L^2}$$

for $|\tau| \geq 1$. Together with (2.76) and (2.71), this is

$$R^{-1} \left\| |x|^{-\tau+1/2} Du \right\|_{L^2}^2 \leq 5 \left\| |x|^{1-\tau} \Delta u \right\|_{L^2}^2 + 8 \left\| |x|^{1-\tau} \Delta u \right\|_{L^2}^2 = 13 \left\| |x|^{1-\tau} \Delta u \right\|_{L^2}^2 .$$

Adding this with (2.71) finally yields

$$R^{-1} \left\| |x|^{-\tau+1/2} Du \right\|_{L^2}^2 + R^{-1}\tau^2 \left\| |x|^{-\tau-1/2} u \right\|_{L^2}^2 \leq 14 \left\| |x|^{1-\tau} \Delta u \right\|_{L^2}^2 .$$

(d) If we now choose for $s \in (0, R]$, $f(s) = \frac{1}{2(\log(R)-\log(s))} = \left(2\log\left(\frac{R}{s}\right)\right)^{-1}$ and recall that $f'(s)s = 2f(s)^2$. For functions supported in $B(0,R) \setminus \{0\}$, we obtain from (2.75) that

$$\begin{aligned} \|Df(|x|)|x|v\|_{L^2}^2 &\leq \left\| |x|^{1-\tau} \Delta |x|^{\tau+1} v \right\|_{L^2} \left\| f(|x|)^2 v \right\|_{L^2} + \tau^2 \|f(|x|)v\|_{L^2}^2 + \left\| f(|x|)^2 v \right\|_{L^2}^2 \\ &\leq \frac{1}{2} \left\| |x|^{1-\tau} \Delta |x|^{\tau+1} v \right\|_{L^2}^2 + \frac{3}{2} \left\| f(|x|)^2 v \right\|_{L^2}^2 + \tau^2 \|f(|x|)v\|_{L^2}^2 . \end{aligned}$$

Using that f is increasing, we have $f(|x|) \leq f(R/2) = (2\log 2)^{-1} < +\infty$ for $x \in B(0,R)$. Hence, for $v \in C_c^\infty(B(0,R/2) \setminus \{0\})$, we have

$$\begin{aligned} \|Df(|x|)|x|v\|_{L^2}^2 &\leq \frac{1}{2} \left\| |x|^{1-\tau} \Delta |x|^{\tau+1} v \right\|_{L^2}^2 + \left(\frac{3}{2}(2\log 2)^{-2} + \tau^2 \right) \|f(|x|)v\|_{L^2}^2 \\ &\leq \frac{1}{2} \left\| |x|^{1-\tau} \Delta |x|^{\tau+1} v \right\|_{L^2}^2 + 2\tau^2 \|f(|x|)v\|_{L^2}^2 , \end{aligned}$$

for $|\tau| \geq 1$ after having used $\frac{3}{2}(2\log 2)^{-2} \leq 1$.
Using our estimate (2.72), we have now obtained, for $v \in C_c^\infty(B(0,R/2) \setminus \{0\})$ and $|\tau| \geq 1$

$$\|Df(|x|)|x|v\|_{L^2}^2 \leq \frac{1}{2} \left\| |x|^{1-\tau} \Delta |x|^{\tau+1} v \right\|_{L^2}^2 + 2\frac{1}{4} \left\| |x|^{1-\tau} \Delta |x|^{\tau+1} v \right\|_{L^2}^2 = \left\| |x|^{1-\tau} \Delta |x|^{\tau+1} v \right\|_{L^2}^2 . \tag{2.77}$$

We then remark that $Df(|x|)|x|v = f(|x|)|x|Dv + \frac{1}{i}\left(xf'(|x|) + \frac{x}{|x|}f(|x|)\right)v$ so that using again (2.72), we have for $v \in C_c^\infty(B(0,R/2) \setminus \{0\})$ and $|\tau| \geq 1$

$$\|f(|x|)|x|Dv\|_{L^2}^2 \leq 3 \|Df(|x|)|x|v\|_{L^2}^2 + 3 \|f(|x|)v\|_{L^2}^2 + 3 \left\| |x|f'(|x|)v \right\|_{L^2}^2$$

Using $f'(s)s = 2f(s)^2 \leq 2f(R/2)f(s) = 2(2\log 2)^{-1} f(s)$ in the last term, we deduce

$$\|f(|x|)|x|Dv\|_{L^2}^2 \leq 3 \|Df(|x|)|x|v\|_{L^2}^2 + (3 + \frac{3}{4}(\log 2)^{-2}) \|f(|x|)v\|_{L^2}^2$$

Combining with (2.77) and (2.72) (and using $\frac{3}{4}(\log 2)^{-2} \leq 2$) finally yields

$$\begin{aligned} \|f(|x|)|x|Dv\|_{L^2}^2 + \|f(|x|)v\|_{L^2}^2 &\leq 3 \|Df(|x|)|x|v\|_{L^2}^2 + 6 \|f(|x|)v\|_{L^2}^2 \\ &\leq 3 \left\| |x|^{1-\tau} \Delta |x|^{\tau+1} v \right\|_{L^2}^2 + 6\frac{1}{4} \left\| |x|^{1-\tau} \Delta |x|^{\tau+1} v \right\|_{L^2}^2 , \end{aligned}$$

that is to say

$$\left\| \left(\log\left(\frac{R}{|x|}\right)\right)^{-1} |x|Dv \right\|_{L^2}^2 + \left\| \left(\log\left(\frac{R}{|x|}\right)\right)^{-1} v \right\|_{L^2}^2 \leq 18 \left\| |x|^{1-\tau} \Delta |x|^{\tau+1} v \right\|_{L^2}^2 .$$

Coming back to $u = |x|^{\tau+1} v$ as above yields the sought result.

Exercise 14 (Strong unique continuation, part of the exam of May, 2020)**.** Let Ω be a connected open set containing 0. Let $V \in C^\infty(\Omega)$ and let $u \in C^\infty(\Omega)$ be a solution to $(-\Delta + V)u = 0$ in Ω. Assume that $\partial^\alpha u(0) = 0$ for all $\alpha \in \mathbb{N}^n$. Prove that $u = 0$ in Ω. *Hint: use one of the Carleman estimates of the previous exercise.*

Correction 14. We write $P_V = -\Delta + V$. Fix $R > 0$ such that $B(0, \pi R) \subset \Omega$. If $u \in C^\infty(\Omega)$ satisfies $\partial^\alpha u(0) = 0$ for all $\alpha \in \mathbb{N}^n$, then $u = \mathcal{O}_N(|x|^N)$ for all $N \in \mathbb{N}$. Hence, for all $\tau > 0$ $|x|^{-\tau} u \in L^2(B(0,R))$. Moreover, the same holds for $P_V u$ so that $|x|^{-\tau+1} P_V u \in L^2(B(0,R))$ as well. We now take $\chi_R \in C_c^\infty(\Omega)$ such that $\text{supp}(\chi_R) \subset B(0,R)$ and $\chi_R = 1$ in $B(0, R/2)$.

Next, we need to remark that the Carleman estimate (2.68) generalizes to all functions $v \in C_c^\infty(B(0,R))$ such that $\partial^\alpha v(0) = 0$ for all $\alpha \in \mathbb{N}^n$ (here, any of the Carleman estimates proved in the previous exercise works). Indeed, for such a function v, set $v_n = (1 - \chi(x/n))v \in C_c^\infty(B(0,R) \setminus \{0\})$ for $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\chi = 1$ in a neighborhood of zero. For all $\tau \in \mathbb{R}$, we have $|x|^{-\tau} v_n \to |x|^{-\tau} v$ in $C^\infty(\overline{B}(0,R))$ and since $v = \mathcal{O}_N(|x|^N)$ for all $N \in \mathbb{N}$. Similarly, $|x|^{1-\tau}\Delta v_n \to |x|^{1-\tau}\Delta v$ in $C^\infty(\overline{B}(0,R))$ (this uses that $\||x|^{-\tau}[\Delta, \chi(x/n)]v\|_{L^2(B(0,R))} \to 0$). As a consequence, the Carleman estimate (2.68) for $\tau \geq \tau_0$ applies to v_n:

$$\left\||x|^{1-\tau} P_V v\right\|^2_{L^2(B(0,R))} \leftarrow \left\||x|^{1-\tau} P_V v_n\right\|^2_{L^2(B(0,R))} \geq \tau C \left\||x|^{-\tau} v_n\right\|^2_{L^2(B(0,R))} \to \tau C \left\||x|^{-\tau} v\right\|^2_{L^2(B(0,R))}$$

We may then apply this inequality with $v = \chi_R u$, where u is the function satisfying $(-\Delta + V)u = 0$ in Ω vanishing at infinite order at 0. This yields

$$\left\||x|^{1-\tau}[P_V, \chi_R]u\right\|^2_{L^2(B(0,R))} \geq \tau C \left\||x|^{-\tau}\chi_R u\right\|^2_{L^2(B(0,R))}, \quad \text{for all } \tau \geq \tau_0.$$

But $[P_V, \chi_R]$ is a differential operator of order one with coefficients supported in $B(0,R) \setminus B(0,R/2)$. On that set, we have $|x|^{-\tau+1} \leq \left(\frac{R}{2}\right)^{-\tau+1}$. Therefore, we have

$$\left\||x|^{1-\tau}[P_V, \chi_R]u\right\|_{L^2(B(0,R))} \leq \left(\frac{R}{2}\right)^{-\tau+1} \|u\|_{H^1(B(0,R)\setminus B(0,R/2))} = \left(\frac{R}{2}\right)^{-\tau} C_{u,R},$$

where $C_{u,R}$ does not depend on τ. Coming back to the above Carleman inequality, we have obtained, for some $\tau_0 \geq 1$, and for another constant $\tilde{C}_{u,R} > 0$ independent of τ,

$$\left\||x|^{-\tau}\chi_R u\right\|^2_{L^2(B(0,R))} \leq \left(\frac{R}{2}\right)^{-\tau} \tilde{C}_{u,R}, \quad \text{for all } \tau \geq \tau_0.$$

This rewrites as

$$\left\|e^{\tau \log\left(\frac{R}{2|x|}\right)}\chi_R u\right\|^2_{L^2(B(0,R))} \leq \tilde{C}_{u,R}, \quad \text{for all } \tau \geq \tau_0.$$

The same argument as in the course shows that this implies $\chi_R u = 0$ on the set $\{\log\left(\frac{R}{2|x|}\right) > 0\} = \{|x| < \frac{R}{2}\}$. On this set, $\chi_R = 1$, so that $u = 0$ on $B(0, R/2)$. This is an open subset of Ω, which is connected; hence, from a result of the course, this implies $u = 0$ on the whole Ω (recall that it is connected).

Exercise 15 (Uniqueness under conditional pseudoconvexity, part of the exam of May, 2020)**.** In this exercise, we aim at proving the following result.

Theorem 2.7.3. *Let Ω be a bounded open set of $\mathbb{R}^n$ and $x_0 \in \Omega$. Let $P \in \text{Diff}^2$ and $Q \in \text{Diff}^1$ with respective principal symbols p_2 and q_1, with p_2 real-valued. Assume that the oriented hypersurface $S = \{\Psi = \Psi(x_0)\}$ satisfies ($d\Psi(x_0) \neq 0$ and) $p_2(x_0, d\Psi(x_0)) \neq 0$ and*

$$\{p_2, \{p_2, \Psi\}\}(x_0, \xi) > 0 \quad \textit{for all } \xi \in \mathbb{R}^n \setminus \{0\} \textit{ such that } p_2(x_0,\xi) = \{p_2, \Psi\}(x_0, \xi) = q_1(x_0, \xi) = 0. \quad (2.78)$$

Then, there exists a neighborhood V of x_0 so that for all $u \in C^\infty(\Omega)$, if we have

$$\begin{cases} |Pu(x)| \leq C\left(|\nabla u(x)| + |u(x)|\right) \text{ for all } x \in \Omega, \\ |Qu(x)| \leq C|u(x)| \text{ for all } x \in \Omega, \\ u = 0 \text{ in } \Omega \cap \{\Psi > \Psi(x_0)\}, \end{cases} \quad (2.79)$$

then we have $u = 0$ in V.

We write and $p_\Psi(x, \xi, \tau) = p_2(x, \xi + i\tau d\Psi(x))$ and assume (2.78) all along the exercise.

1. Given a function Φ, compute the operator $Q_\Phi := e^{\tau\Phi} Q e^{-\tau\Phi}$, give its order and its principal symbol q_Φ.
2. Prove that $\{p_\Psi, \Psi\}(x_0, \xi, \tau) \neq 0$ for all $\xi \in \mathbb{R}^n$ and $\tau > 0$ such that $p_\Psi(x_0, \xi, \tau) = 0$.
3. We define

$$c_\Psi(\xi, \tau) = \frac{1}{i\tau}\{\overline{p_\Psi}, p_\Psi\}(x_0, \xi, \tau), \text{ for } \tau > 0 \quad \text{and} \quad c_\Psi(\xi, 0) = 2\{p_2, \{p_2, \Psi\}\}(x_0, \xi).$$

Prove the existence of $C_1, C_2 > 0$ so that

$$c_\Psi(\xi, \tau) + C_1 \left(|\{p_\Psi, \Psi\}(x_0, \xi, \tau)|^2 + |p_\Psi(x_0, \xi, \tau)|^2 + |q_\Psi(x_0, \xi, \tau)|^2 \right) \geq C_2,$$

for all $(\xi, \tau) \in \mathbb{R}^{n+1}$ such that $\tau \geq 0$ and $|\xi|^2 + \tau^2 = 1$.

4. We now set $\Phi = e^{\lambda\Psi}$. Deduce that there exist constants $\lambda_0, C_1, C_2 > 0$ such that for all $\lambda \geq \lambda_0$ we have

$$c_\Phi(\xi, \tau) + C_1 \left(\frac{|p_\Phi(x_0, \xi, \tau)|^2}{|\xi|^2 + \tau^2} + |q_\Phi(x_0, \xi, \tau)|^2 \right) \geq C_2(|\xi|^2 + \tau^2), \tag{2.80}$$

for all $(\xi, \tau) \in \mathbb{R}^{n+1}, \tau \geq 0$.

5. We set $\tilde{P} = \frac{P+P^*}{2}$ and $\tilde{P}_\Phi = e^{\tau\Phi}\tilde{P}e^{-\tau\Phi}$. Prove that if Φ satisfies (2.80), then there exist $C, r, \tau_0 > 0$ so that

$$\tau \|v\|^2_{H^1_\tau} \leq C \left\|\tilde{P}_\Phi v\right\|^2_{L^2} + C\tau \|Q_\Phi v\|^2_{L^2}, \quad \text{for all } v \in C^\infty_c(B(x_0, r)), \tau \geq \tau_0.$$

Hint: one may write $\tilde{P}_\Phi = P_R + i\tau\tilde{P}_I$ *where* P_R *and* $\tilde{P}_I$ *two formally selfadjoint operators to be determined, compute* $\frac{1}{\tau}\left\|\tilde{P}_\Phi v\right\|^2_{L^2}$*, and link the principal symbols of the operators involved with those appearing in* (2.80).

6. Deduce that there exist $C, \tau_0 > 0$ so that

$$\tau^3 \left\|e^{\tau\Phi} w\right\|^2_{L^2} + \tau \left\|e^{\tau\Phi}\nabla w\right\|^2_{L^2} \leq C \left\|e^{\tau\Phi} P w\right\|^2_{L^2} + C\tau \left\|e^{\tau\Phi} Q w\right\|^2_{L^2},$$

for all $w \in C^\infty_c(B(x_0, r)), \tau \geq \tau_0$.

7. Assume $u \in C^\infty(\Omega)$ satisfies $Pu = 0$ and $Qu = 0$, and let $\chi \in C^\infty_c(B(0, r))$. Then prove that there exist $C, \tau_0 > 0$ so that for all $\tau \geq \tau_0$

$$\tau^3 \left\|e^{\tau\Phi}\chi u\right\|^2_{L^2} + \tau \left\|e^{\tau\Phi}\nabla(\chi u)\right\|^2_{L^2} \leq C \left\|e^{\tau\Phi}[P, \chi]u\right\|^2_{L^2}. \tag{2.81}$$

8. Prove Theorem 2.7.3 in this case. Only explain the main steps of the proofs, and omit the details.
9. Assume now that u only satisfies (2.79) and prove Theorem 2.7.3. Only explain the difference with respect to the preceding question.

Correction 15. 1. We have $Q_\Phi := e^{\tau\Phi} Q e^{-\tau\Phi} \in \text{Diff}^1_\tau$, with principal symbol $q_\Phi(x, \xi, \tau) = q_1(x, \xi + i\tau d\Phi(x))$.

2. This is a reformulation of Lemma 2.2.8.
3. We first recall that c_Ψ is continuous up to $\tau = 0$ according to Lemma 2.1.6. We thus consider on $\mathbb{S}^n_+ = \{(\xi, \tau), |\xi|^2 + \tau^2 = 1, \tau \geq 0\}$ the two continuous functions $c_\Psi(\xi, \tau)$ and

$$f(\xi, \tau) = |\{p_\Psi, \Psi\}(x_0, \xi, \tau)|^2 + |p_\Psi(x_0, \xi, \tau)|^2 + |q_\Psi(x_0, \xi, \tau)|^2.$$

We have $f \geq 0$ and $\mathbb{S}^n_+$ compact: if we prove that $f(\xi,\tau) = 0 \implies c_\Psi(\xi,\tau) > 0$, the result follows from Lemma 2.1.8. Question 2 gives that $f(\xi,\tau) = 0 \implies \tau = 0$. But if $\tau = 0$, we have $p_\Psi(x_0,\xi,0) = p_2(x_0,\xi)$, $\{p_\Psi,\Psi\}(x_0,\xi,0) = \{p_2,\Psi\}(x_0,\xi)$ and $q_\Psi(x_0,\xi,0) = q_1(x_0,\xi)$ so that

$$f(\xi,0) = |\{p_2,\Psi\}(x_0,\xi)|^2 + |p_2(x_0,\xi)|^2 + |q_1(x_0,\xi)|^2.$$

Condition (2.78), then precisely implies $c_\Psi(\xi,0) = 2\{p_2,\{p_2,\Psi\}\}(x_0,\xi) > 0$. Lemma 2.1.8 thus yields (2.80).

4. Similar to the proof of Proposition 2.2.5 from Lemma 2.2.6.
5. idem cours
6. ♣ . . .

Chapter 3

The wave equation with coefficients constant in time

In this chapter, we focus our attention to very specific operators of wave type $\partial_t^2 + Q$ where $Q = q(x, D_x)$ is a positive elliptic operator. In Remark 2.3.6, we have seen that, in the framework of regular coefficient, the unique continuation theorem of the previous chapter, starting from sets of the form $\{(t,x), \Psi(x) \geq 0\}$ $\{\varphi(x) \leq 0\}$ requires some (strong) convexity assumptions of the surface. Yet, in the Holmgren Theorem 1.2.4, the condition is less restrictive (in particular, unique continuation always holds across any surface of the type $\{(t,x), \Psi(x) = 0\}$). It only requires the surface to be *non-characteristic*. In the case of the flat wave operator $\partial_t^2 - \Delta$ for instance, the latter reads $|\partial_t \Psi|^2 \neq |\partial_x \Psi|^2$. It means that the surface should not be tangent to the light cone. More or less, this is the weakest condition that one could expect but would not contradict the finite speed of propagation. But we would like to relax the analyticity assumption. The counterexamples of Alinhac and Baouendi [AB79, Ali83, AB95] actually prevent from relaxing this assumption completely.

It turns out that some analyticity with respect to part of the variables can be sufficient, for instance the time in our example of the wave equation. The results presented in this chapter have been proved in [Tat95], and revisited/generalized in [Hör97, RZ98, Tat99]. Our presentation is inspired by [Hör97].

3.1 Setting and statement of the unique continuation result

In the following, the variable will be $z = (t, x) \in \mathbb{R}^{1+n}$ with dual variable $\xi = (\xi_t, \xi_x) \in \mathbb{R}^{1+n}$. To keep the notation coherent with the elliptic case, we will denote $\xi_t = \xi_0$ and ξ_x will be written $\xi_x = (\xi_1, \cdots, \xi_n)$.

The main theorem of this chapter will be the following.

Theorem 3.1.1 (Wave type operator with coefficients constant in time)**.** *Let $T > 0$ and Ω_x an open set of $\mathbb{R}^n$. Denote $\Omega =]-T, T[\times\Omega_x$.*

Let

$$Q = \sum_{i,j=1}^{n} a^{ij}(x) D_i D_j + \sum_k b_k(x) D_k + c(x)$$

be a differential operator of order 2 with $a^{ij} \in C^\infty(\Omega_x)$ real-valued, b_k, $c \in L^\infty(\Omega_x)$. Assume also that Q is positive elliptic, that is there exists $C > 0$ so that

$$q(x, \xi_x) := \sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j \geq C|\xi_x|^2, \text{ for all } (x, \xi_x) \in \Omega_x \times \mathbb{R}^n.$$

Define $P = \partial_t^2 + Q$ on Ω, having principal symbol $p(t, x, \xi_t, \xi_x) = -\xi_t^2 + q(x, \xi_x)$. Let $z_0 = (t_0, x_0) \in \Omega$ and $\Psi \in C^2(\Omega)$ with $d\Psi(z_0) \neq 0$ so that $p(z_0, d_z\Psi(z_0)) \neq 0$, i.e.

$$(\partial_t \Psi(z_0))^2 \neq \sum_{i,j} a^{ij}(x_0)(\partial_i \Psi(z_0))(\partial_j \Psi(z_0)).$$

Then, there exists a neighborhood V of z_0 so that for any $u \in C^\infty(\Omega)$,

$$\left\{ \begin{array}{rcl} Pu & = & 0 \text{ in } \Omega, \\ u & = & 0 \text{ in } \Omega \cap \{\Psi > \Psi(z_0)\} \end{array} \right. \Longrightarrow u = 0 \text{ on } V. \tag{3.1}$$

The main tool will be an inequality of Carleman type, but with an additional weight in the Fourier variable. Namely, we let $e^{-\varepsilon \frac{|D_t|^2}{2\tau}}$ be the Fourier multiplier defined naturally by

$$\mathcal{F}\left(e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u\right)(\xi) = e^{-\varepsilon \frac{|\xi_t|^2}{2\tau}} \widehat{u}(\xi), \quad u \in \mathcal{S}(\mathbb{R}^n),$$

where ξ_t is the Fourier variable corresponding to the variable t and $\xi = (\xi_t, \xi_x)$. Note that this amounts to solving the heat equation with t as a "spatial" variable, during a "time" $\frac{\varepsilon}{2\tau}$. Using the explicit expression of the Fourier transform of the Gaussian $e^{-\varepsilon \frac{|\xi_t|^2}{2\tau}}$, this may be rewritten as a convolution with a heat kernel:

$$\left(e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u\right)(t,x) = \left(\frac{\tau}{2\pi\varepsilon}\right)^{1/2} \int_{\mathbb{R}} e^{-\frac{\tau}{2\varepsilon}(t-s)^2} u(s,x) ds.$$

This operator has several interesting features: it localizes close to $D_t = 0$ (i.e. in low frequencies w.r.t. the time variable t), in an analytic way (the function $e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u$ produced is an entire function in the t-variable). However (and consequently), note that $e^{-\frac{\varepsilon}{2\tau}|D_t|^2}$ is not local; in particular, $e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u$ is not compactly supported, even if u is.

For a smooth real-valued weight function Φ (later on, we will assume that it is polynomial of order 2), the Carleman estimate below will make use of the operator

$$Q^\Phi_{\varepsilon,\tau} u = e^{-\varepsilon \frac{|D_t|^2}{2\tau}} e^{\tau\Phi} u.$$

The following is an analogue of the Definition 2.1.5, under which the Carleman estimate of Theorem 2.1.1 holds. Here, the condition is weaker for it is only restricted to $\xi_t = 0$.

Definition 3.1.2 (Pseudoconvex function in $\xi_t = 0$)**.** With the above assumptions for P, let Φ be smooth and real-valued. We say that Φ is a pseudoconvex function with respect to P in $\xi_t = 0$ at z_0 if

$$\{p, \{p, \Phi\}\}(z_0, \xi) > 0, \qquad \text{if } p(z_0,\xi) = 0, \quad \xi_t = 0, \quad \xi \neq 0; \tag{3.2}$$

$$\frac{1}{i\tau}\{\overline{p}_\Phi, p_\Phi\}(z_0,\xi,\tau) > 0, \quad \text{if } p_\Phi(z_0,\xi,\tau) = 0, \quad \xi_t = 0, \quad \tau > 0, \tag{3.3}$$

where $p_\Phi(z,\xi,\tau) = p(z, \xi + i\tau d\Phi(z))$.

Theorem 3.1.3 (Carleman estimate for wave type operators with coefficients constant in time)**.** *With the above assumptions for P, let Φ be a* quadratic real-valued polynomial *such that Φ is a pseudoconvex function with respect to P in $\xi_t = 0$ at z_0, in the sense of Definition 3.1.2.*

Then, there exist $r, \varepsilon, \mathsf{d}, C, \tau_0 > 0$ such that for all $\tau \geq \tau_0$ and $u \in C^\infty_c(B(z_0, r))$, we have

$$\tau \|Q^\Phi_{\varepsilon,\tau} u\|^2_{H^1_\tau} \leq C \left\|Q^\Phi_{\varepsilon,\tau} Pu\right\|^2_{L^2} + Ce^{-\mathsf{d}\tau} \left\|e^{\tau\Phi} u\right\|^2_{H^1_\tau}. \tag{3.4}$$

Note that if we set $\varepsilon = 0$, this would be a classical Carleman estimate. Yet, the role of the Fourier multiplier will be to kill the high frequency in the variable t. So, we will just need to look at the very small frequency in ξ_t. That is why the pseudoconvexity assumption is only made in $\xi_t = 0$.

3.2 Proving unique continuation using the Carleman estimate

In this section, we assume that Theorem 3.1.3 is proved and we will prove Theorem 3.1.1. Some part will be similar to the classical case, that means constructing an approriate function Φ, pseudoconvex for functions in $\xi_t = 0$ from the function Ψ defining the surface $S = \{\Psi = \Psi(z_0)\}$.

The main differences are the following:

- the pseudoconvexity is only on $\xi_t = 0$, so it requires a small adaptation of the convexification procedure. Moreover, we want Φ to be quadratic.

- the Carleman estimate implies an additional Fourier multiplier ($e^{-\varepsilon \frac{|D_t|^2}{2\tau}}$) that changes the proof of unique continuation. The additional difficulty comes from the fact that the Carleman estimate (3.4) only dominates the low frequencies in ξ_t of the function u.

3.2.1 Convexification

Quite similarly to the classical case, the natural assumption for the unique continuation Theorem 3.1.1 is a strong pseudoconvexity condition similar to that of Definition 2.2.1, but restricted to the set $\{\xi_t = 0\}$. We define this notion, and then check that any noncharacteristic surface is strongly pseudoconvex in $\xi_t = 0$.

Definition 3.2.1 (Pseudoconvex surface in $\xi_t = 0$)**.** Let $\Omega \ni z_0$ be an open set, $P \in \mathrm{Diff}^2(\Omega)$ with real-valued principal symbol p_2 and $\Psi \in C^\infty(\Omega)$ real-valued. We say that the *oriented hypersurface* $S = \{\Psi = \Psi(z_0)\} \ni z_0$ is strongly pseudoconvex with respect to P at z_0 in $\xi_t = 0$ if

$$\{p_2, \{p_2, \Psi\}\}(z_0, \xi) > 0, \qquad \text{if } p_2(z_0,\xi) = \{p_2, \Psi\}(z_0,\xi) = \xi_t = 0 \text{ and } \xi \neq 0; \tag{3.5}$$

$$\frac{1}{i\tau}\{\overline{p_\Psi}, p_\Psi\}(z_0,\xi,\tau) > 0, \quad \text{if } p_\Psi(z_0,\xi,\tau) = \{p_\Psi, \Psi\}(z_0,\xi,\tau) = \xi_t = 0 \text{ and } \tau > 0, \tag{3.6}$$

where $p_\Psi(z,\xi,\tau) = p_2(z, \xi + i\tau d\Psi(z))$.

The next lemma explains that the noncharacteristicity condition assumed in Theorem 3.1.1 is a particular case of Definition 3.2.1.

Lemma 3.2.2 (noncharacteristicity implies strong pseudoconvexity in $\xi_t = 0$)**.** *Let Ω, P as in Theorem 3.1.1. If the surface $S = \{\Psi = \Psi(z_0)\} \ni z_0$ is noncharacteristic for P at z_0 ($p(z_0, d_z\Psi(z_0)) \neq 0$), then it is strongly pseudoconvex with respect to P at z_0 in $\xi_t = 0$.*

Proof. The principal symbol of P is $p(t,x,\xi_t,\xi_x) = -\xi_t^2 + q(x,\xi_x)$ where $q(x,\xi_x) = \sum_{i,j} a^{ij}(x)\xi_i\xi_j$.

So, we notice that for $\xi_t = 0$, we have $p(t,x,0,\xi_x) = q(x,\xi_x)$. Since q is assumed to be elliptic, the assumption $p(z_0,\xi) = \xi_t = 0$ implies $\xi = 0$ and therefore Condition (3.5) is empty.

We now check that (3.6) is also empty (note that we have actually already proved that (3.6) is empty if $p(x, d\Psi) \neq 0$ in Proposition 2.2.7). Denoting $\tilde{p}(z_0,\cdot,\cdot)$ the symmetric polar bilinear form of $p(z_0,\cdot)$, the computations of Lemma 2.2.11 give

$$\{p_\Psi, \Psi\}(z_0,\xi,\tau) = 2\tilde{p}(z_0,\xi,d\Psi(z_0)) + 2i\tau p(z_0, d\Psi(z_0)).$$

The assumtion is $p(z_0, d\Psi(z_0)) \neq 0$, hence $\mathrm{Im}\{p_\Psi, \Psi\}$ never cancels for $\tau > 0$, and (3.6) is also empty. □

Next, we will follow the same previous steps of convexification as Section 2.2.1.

Proposition 3.2.3 (Analytic convexification)**.** *Let Ω, P satisfy the assumptions of Theorem 3.1.1. Assume that the surface $S = \{\Psi = \Psi(z_0)\}$ is strongly pseudoconvex with respect to P at z_0 in $\xi_t = 0$, in the sense of Definition 3.2.1. Then there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, the function $\Phi = e^{\lambda\Psi}$ is a pseudoconvex function with respect to P at z_0 in $\xi_t = 0$, in the sense of Definition 3.1.2.*

Note however that, as opposed to the classical case, the Carleman estimate of Theorem 3.1.3 does not apply to the weight function Φ since it is not quadratic.

Proof. The proof is very similar to Proposition 2.2.5. Again, we assume that $\Psi(z_0) = 0$ for simplicity, and denote

$$c_\Psi(\xi,\tau) = \frac{1}{i\tau}\{\overline{p_\Psi}, p_\Psi\}(z_0,\xi,\tau), \text{ for } \tau > 0 \quad \text{ and } \quad c_\Psi(\xi,0) = 2\{p_2,\{p_2,\Psi\}\}(z_0,\xi),$$

with a similar definition for $c_\Phi(\xi,\tau)$. Lemma 2.1.6 still applies and $c_\Psi(\xi,\tau)$ and $c_\Phi(\xi,\tau)$ are both continuous on the whole $\mathbb{R}^n \times \mathbb{R}^+$. Then, using Lemma 2.1.8, Definition 3.2.1 may be equivalently reformulated as the existence of some constants C_1, $C_2 > 0$ so that

$$c_\Psi(\xi,\tau) + C_1\left[|\{p_\Psi,\Psi\}(z_0,\xi,\tau)|^2 + \frac{|p_\Psi(z_0,\xi,\tau)|^2}{|\xi|^2+\tau^2} + |\xi_t|^2\right] \geq C_2(|\xi|^2+\tau^2).$$

Lemma 2.2.6 still applies, giving

$$c_\Phi(\xi,\tau) = \lambda c_\Psi(\xi,\lambda\tau) + 2\lambda^2 \left|\{p_\Psi,\Psi\}(z_0,\xi,\lambda\tau)\right|^2 .$$

The same arguments then lead to

$$c_\Phi(\xi,\tau) + C_1 \left[\frac{\left|p_\Phi(z_0,\xi,\tau)\right|^2}{|\xi|^2+\tau^2} + |\xi_t|^2\right] \geq C_2(|\xi|^2+\tau^2),$$

for λ large enough. This implies the result. □

It remains to perform the Geometric convexification and to ensure that we can take the weight function Φ quadratic.

Proposition 3.2.4 (Geometric convexification)**.** *Let Φ be a pseudoconvex function for P at z_0 in $\xi_t = 0$, in the sense of Definition 3.1.2. Assume further that $\Phi(z_0) = 0$.*

Then there exists a function φ such that

1. *φ pseudoconvex function for P at z_0 in $\xi_t = 0$,*
2. *φ is a quadratic polynomial,*
3. *$\varphi(z_0) = 0$ and there exists $R_0 > 0$ such that for any $0 < R < R_0$, there exists $\eta > 0$ so that $\varphi(z) \leq -\eta$ for $z \in \{\Phi \leq 0\} \cap \{R/2 \leq |z - z_0| \leq R\}$.*

Proof. For $\delta > 0$, we take

$$\varphi(z) = \Phi_T(z) - \delta|z - z_0|^2.$$

where

$$\Phi_T(z) = \sum_{|\alpha|\leq 2} \frac{1}{\alpha!}(\partial^\alpha \Phi)(z_0)(z - z_0)^\alpha$$

that is Φ_T is the Taylor expansion of Φ at order 2. Indeed, this is almost the same construction as in the classical case, except that we have replaced Φ by its Taylor expansion at order 2.

First, we notice that the pseudoconvexity condition only involves derivative up to order 2 at z_0. Hence, Φ_T is also a strongly pseudoconvex function in $\xi_t = 0$ at z_0. Moreover, the same stability argument as in Proposition 2.1.9 applies. So, for δ small enough, φ is as well a strongly pseudoconvex function in $\xi_t = 0$ at z_0. We fix $\delta > 0$ sufficiently small. It remains to prove the geometric properties.

Since Φ_T is the Taylor expansion of Φ at order 2, there exists R_0 small enough so that $|\Phi_T - \Phi| \leq |z - z_0|^2\delta/2$ for $|z - z_0| \leq R_0$. Now, take $R \leq R_0$.

Let $z \in \{\Phi \leq 0\} \cap \{R/2 \leq |z - z_0| \leq R\}$. Since $\Phi(z) \leq 0$, we have $\Phi_T(z) \leq |z - z_0|^2\delta/2$. Therefore,

$$\varphi(z) \leq -\delta|z - z_0|^2/2.$$

So, in particular since $|z - z_0|^2 \geq R^2/4$, we get $\varphi(z) \leq -\delta R^2/8$ and we can take $\eta = \delta R^2/8$. □

3.2.2 Unique continuation

In this section, we conclude the proof of the unique continuation of Theorem 3.1.1 assuming the Carleman estimate of Theorem 3.1.3.

Proof of Theorem 3.1.1. Let u solution of $Pu = 0$ in Ω so that $u = 0$ on $\Omega \cap \{\Psi > 0\}$. The surface $S = \{\Psi = \Psi(z_0)\}$ is strongly pseudoconvex at z_0 in $\xi_t = 0$. Propositions 3.2.3 and 3.2.4 allow to produce some quadratic function Φ (it is the function called φ in Proposition 3.2.4, which we now rename Φ) that satisfies the pseudoconvexity for functions at z_0 in $\xi_t = 0$. In particular, Theorem 3.1.3 applies. We therefore obtain the following properties

1. there exists $R, C, \mathsf{d}, \varepsilon > 0$ and $\tau_0 > 0$ so that we have the following estimate
$$\tau\|Q^{\Phi}_{\varepsilon,\tau}w\|^2_{H^1_\tau} \leq C\left\|Q^{\Phi}_{\varepsilon,\tau}Pw\right\|^2_{L^2} + Ce^{-\mathsf{d}\tau}\left\|e^{\tau\Phi}w\right\|^2_{H^1_\tau} \tag{3.7}$$
for any $w \in C^\infty(B(z_0, R))$ and $\tau \geq \tau_0$.
2. $\Phi(z_0) = 0$ and there exists $\eta > 0$ so that $\Phi(z) \leq -\eta$ for $z \in \{\Psi \leq 0\} \cap \{|z - z_0| \geq R/2\}$,
3. $\Phi(z) \leq \mathsf{d}/4$ in $B(z_0, R)$.

All the properties were already obtained. We only added Item 3, which follows from a continuity statement (holding up to reducing R) using $\Phi(z_0) = 0$.

Pick $\chi \in C^\infty_c(B(z_0, R))$ so that $\chi = 1$ on $B(z_0, R/2)$. As before, we want to apply the Carleman estimate to $w = \chi u \in C^\infty_c(B(z_0, R))$, solution of $Pw = \chi Pu + [P, \chi]u = [P, \chi]u$. Again, $[P, \chi]$ is a classical differential operator of order 1 with coefficients supported in the set $\{\frac{R}{2} \leq |z - z_0| \leq R\}$. Moreover, we have $\text{supp}(u) \subset \{\Psi \leq 0\}$, and thus $[P, \chi]u$ is supported in $\{\Psi \leq 0\} \cap \{\frac{R}{2} \leq |z - z_0|\}$, where we have $\Phi \leq -\eta$. In particular, we have $\left\|Q^{\Phi}_{\varepsilon,\tau}Pw\right\|_{L^2} \leq \left\|e^{\tau\Phi}Pw\right\|_{L^2} \leq Ce^{-\tau\eta}\|u\|_{H^1}$.

For the second term in the right hand side, we use Property 3 to get
$$e^{-\mathsf{d}\tau}\left\|e^{\tau\Phi}w\right\|^2_{H^1_\tau} \leq e^{-\mathsf{d}\tau}e^{\mathsf{d}\tau/2}\|w\|^2_{H^1_\tau} \leq e^{-\mathsf{d}\tau/2}\tau^2\|w\|^2_{H^1} \leq e^{-\mathsf{d}\tau/4}\|u\|^2_{H^1},$$
for τ large enough.

So, we have obtained that there exist $C, \delta, \tau_0 > 0$ so that
$$\|Q^{\Phi}_{\varepsilon,\tau}w\|_{L^2} \leq Ce^{-\delta\tau}, \qquad \text{for all } \tau \geq \tau_0. \tag{3.8}$$
We will use the following lemma that we prove below. It contains the main novelty with respect to the proof of Theorem 2.3.2.

Lemma 3.2.5. *Let $\Phi \in C^\infty(\Omega)$ be a real-valued function such that $d\Phi \neq 0$ on Ω. Let $v \in C^\infty_c(\Omega)$ and assume there exists $C_0, \tau_0, \varepsilon > 0$ such that*
$$\|Q^{\Phi}_{\varepsilon,\tau}v\|_{L^2} \leq C_0 \quad \textit{for all } \tau \geq \tau_0. \tag{3.9}$$
Then, v is supported in $\{\Phi \leq 0\}$.

To apply the lemma, we rewrite (3.8) as $\|Q^{\Phi}_{\varepsilon,\tau}e^{\delta\tau}w\|_{L^2} \leq C$ i.e. $\|Q^{\Phi+\delta}_{\varepsilon,\tau}w\|_{L^2} \leq C$. Lemma 3.2.5 applied to the function $\Phi + \delta$ implies that w is supported in the set $\{\Phi + \delta \leq 0\} = \{\Phi \leq -\delta\}$. Since we have $\Phi(z_0) = 0$ and $\chi = 1$ on $B(z_0, R/2)$, the set $V = B(z_0, R/2) \cap \{\Phi > -\delta\}$ is a neighborhood of z_0 on which $u = w = 0$, concluding the proof of the theorem. $\square$

We need to prove Lemma 3.2.5. Note that if we had $\varepsilon = 0$, the proof would be (easy and) another formulation of the proof of Theorem 2.3.2. Before describing the details of the proof, we first give a sketch of it to present the main new ideas.

1. Proving that $\text{supp}(v) \subset \{\Phi \leq 0\} \iff$ Proving $v = 0$ on $\{\Phi \geq 0\} \iff$ Proving that $z \mapsto \chi \circ \Phi(z)v(z)$ vanishes identically on $\mathbb{R}^{1+n}$ for all test function $\chi \in C^\infty(\mathbb{R})$, such that $\text{supp}(\chi) \subset [0, +\infty)$. Again, this may be reformulated equivalently in a weak form (still for all $\chi \in C^\infty(\mathbb{R})$ such that $\text{supp}(\chi) \subset [0, +\infty)$) as
$$\int_{\mathbb{R}^{n+1}} f(z)v(z)\chi(\Phi(z))dz = 0, \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^{1+n}).$$
2. We change slightly the point of view and, considering f fixed, see this quantity as a distribution on $\mathbb{R}$, with χ as test function:
$$\langle h_f, \chi\rangle_{\mathcal{E}'(\mathbb{R}), C^\infty(\mathbb{R})} = \langle fv, \chi(\Phi)\rangle_{\mathcal{E}'(\mathbb{R}^{n+1}), C^\infty(\mathbb{R}^{n+1})} = \int_{\mathbb{R}^{n+1}} f(z)v(z)\chi(\Phi(z))dz \tag{3.10}$$
This corresponds in fact to make a kind of foliation along the level sets of Φ: if we want to measure v, we rather define the distribution $h_f = \Phi_*(fv)$. Heuristically, $h_f(s)$ is the integral of fv on the level set $\{\Phi(x) = s\}$. According to the first point, the sought result $\text{supp}(v) \subset \{\Phi \leq 0\}$ is now equivalent to proving that
$$\text{supp}(h_f) \subset (-\infty, 0].$$

3. We shall see that the Fourier transform of h_f is

$$\widehat{h_f}(\zeta) = \langle h_f, s \mapsto e^{-i\zeta s}\rangle_{\mathcal{E}'(\mathbb{R}),C^\infty(\mathbb{R})} = \int_{\mathbb{R}^{n+1}} f(z)v(z)e^{-i\zeta\Phi(z)}dz,$$

and can be extended to the complex domain if v is compactly supported (which is assumed here). In particular, for $\zeta \in i\mathbb{R}^+$, $\zeta = i\tau$, we have $\widehat{h_f}(i\tau) = \langle f, ve^{\tau\Phi}\rangle$. The assumption (3.9) gives an information on the norm of $e^{\tau\Phi}v$ for τ large which can be translated in a uniform bound on $|\widehat{h_f}|$ on the upper imaginary axis. A Phragmén-Lindelöf type argument allows to transfer this uniform bound on $|\widehat{h_f}|$ to the *whole upper half plan.*

4. From the bound $|\widehat{h_f}| \leq C$ on the whole upper half plan, a Paley-Wiener theorem (roughly saying $\text{supp}(g) \subset (-\infty, 0] \Longleftrightarrow |\hat{g}| \leq C$ uniformly on the upper half complex plane) allows to conclude that $\text{supp}(h_f) \subset (-\infty, 0]$ for all f, which is the sought result according to the first two points.

Let us now proceed to the details of the proof.

Proof of Lemma 3.2.5. We will work by duality. Let $f \in \mathcal{S}(\mathbb{R}^{n+1})$ with Fourier transform $\widehat{f}$ compactly supported in $B(0,R)$ for R large. We define the distribution $h_f \in \mathcal{E}'(\mathbb{R})$ by (3.10). Note that h_f is a distribution of order zero since

$$|\langle h_f, \chi\rangle_{\mathcal{E}'(\mathbb{R}),C^\infty(\mathbb{R})}| \leq \int_{\mathbb{R}^{n+1}} |f(z)v(z)||\chi(\Phi(z))|dz \leq \|f\|_{L^2} \|v\|_{L^2} \sup_{\Phi(\text{supp}(v))} |\chi|,$$

and is indeed compactly supported because $\text{supp}(h_f) \subset \Phi(\text{supp}(v)) = \{\Phi(z); z \in \text{supp}(v)\}$ which is compact. We shall prove below that h_f is in fact a smooth function (using that v is).

Since $h_f \in \mathcal{E}'(\mathbb{R})$, the Fourier transform of h_f can be computed for $\zeta \in \mathbb{R}$ by

$$\widehat{h_f}(\zeta) = \left\langle h_f, e^{-is\zeta}\right\rangle_{\mathcal{E}'(\mathbb{R}_s),C^\infty(\mathbb{R}_s)} = \langle fv, e^{-i\zeta\Phi}\rangle_{\mathcal{E}'(\mathbb{R}^{n+1}),C^\infty(\mathbb{R}^{n+1})} = \int_{\mathbb{R}^{n+1}} f(z)v(z)e^{-i\zeta\Phi(z)}dz.$$

We notice that this formula still defines a function for $\zeta \in \mathbb{C}$ satisfying the bound

$$|\widehat{h_f}(\zeta)| \leq \int_{\text{supp}(v)} |f(z)v(z)|e^{\text{Im}(\zeta)\Phi(z)}dz \leq e^{C_1|\text{Im}(\zeta)|} \|f\|_{L^2} \|v\|_{L^2}, \quad C_1 = \max_{\text{supp}(v)} |\Phi|. \tag{3.11}$$

Its derivatives satisfy similar bounds, and we may derivate under the integral. The holomorphicity of the integrand with respect to ζ implies that $\widehat{h_f}(\zeta)$ satisfies the Cauchy Riemann equations and is thus holomorphic on the whole $\mathbb{C}$.

For $\zeta \in \mathbb{R}$, the Cauchy-Schwarz inequality (3.11) yields the general bound

$$|\widehat{h_f}(\zeta)| \leq \|f\|_{L^2} \|v\|_{L^2} = C_{f,v}.$$

Now, we use the assumption of the lemma, namely (3.9), to obtain a bound on the upper imaginary axis. Indeed, for $\zeta = i\tau$, and $\tau \geq \tau_0$, (3.9) implies

$$\begin{aligned}
|\widehat{h_f}(i\tau)| &= \left|\langle fv, e^{\tau\Phi}\rangle\right|_{\mathcal{E}'(\mathbb{R}^{n+1}),C^\infty(\mathbb{R}^{n+1})} \\
&= \left|\langle f, ve^{\tau\Phi}\rangle_{\mathcal{S}'(\mathbb{R}^{n+1}),\mathcal{S}(\mathbb{R}^{n+1})}\right| \\
&= \left|\langle e^{\varepsilon\frac{|D_t|^2}{2\tau}} f, e^{-\varepsilon\frac{|D_t|^2}{2\tau}} ve^{\tau\Phi}\rangle_{\mathcal{S}'(\mathbb{R}^{n+1}),\mathcal{S}(\mathbb{R}^{n+1})}\right| \\
&\leq \left\|e^{\varepsilon\frac{|D_t|^2}{2\tau}} f\right\|_{L^2(\mathbb{R}^{n+1})} \left\|e^{-\varepsilon\frac{|D_t|^2}{2\tau}} ve^{\tau\Phi}\right\|_{L^2(\mathbb{R}^{n+1})} \\
&\leq \left\|e^{\varepsilon\frac{|\xi_t|^2}{2\tau}}\right\|_{L^\infty(\text{supp}(\widehat{f}))} \|f\|_{L^2(\mathbb{R}^{n+1})} \left\|Q^\Phi_{\varepsilon,\tau} v\right\|_{L^2(\mathbb{R}^{n+1})} \leq Ce^{\frac{\varepsilon R^2}{2\tau}} \|f\|_{L^2(\mathbb{R}^{n+1})} \\
&\leq C_{\varepsilon,f,\tau_0} C_0.
\end{aligned}$$

Note that at that point, the term $e^{\varepsilon\frac{|D_t|^2}{2\tau}}$ was harmless because the Fourier transform of f is compactly supported in $B(0,R)$. Otherwise $e^{\varepsilon\frac{|D_t|^2}{2\tau}}f$ does not have any meaning, even for $f\in\mathcal{S}(\mathbb{R}^{n+1})$. That is why we had to work by duality.

Moreover, for $\tau\in[0,\tau_0]$, the estimate

$$|\widehat{h_f}(i\tau)|\leq C$$

follows by compactness and continuity, with some appropriate constant C independent on τ.

Now, $|\widehat{h_f}|$ has a uniform bound on $\mathbb{R}\cup i\mathbb{R}_+$, as well as an a priori subexponential growth (3.11). We are thus in position to transfer the uniform bounds to the whole upper half plane by the Phragmén-Lindelöf Theorem.

Lemma 3.2.6 (Phragmén-Lindelöf Theorem)**.** *Let g be a holomorphic function in $Q_1=\{x+iy; x>0, y>0\}$, continuous in $\bar{Q}_1$. Assume that there exist $c>0$ and $C>0$ such that*

$$|g(z)|\leq Ce^{c|z|},\quad \textit{for all } z\in Q_1,$$
$$|g(z)|\leq 1,\quad \textit{for all } z\in\partial Q_1=\mathbb{R}_+\cup i\mathbb{R}_+.$$

Then, we have $|g(z)|\leq 1$ for all $z\in Q_1$.

Applying this result to the function $g=\widehat{h_f}$ on both Q_1 and the quarter plane $\{x+iy; x<0, y>0\}$, we obtain that

$$|\widehat{h_f}(\zeta)|\leq C\quad \text{for all } \zeta\in\mathbb{C}, \mathrm{Im}(\zeta)\geq 0.$$

We now want to apply the Paley-Wiener Theorem 3.2.8. We need first to prove that h_f is in fact a smooth function. To this aim, let us study the derivative of h_f,

$$\langle h_f',\chi\rangle_{\mathcal{E}'(\mathbb{R}),C^\infty(\mathbb{R})}=-\langle h_f,\chi'\rangle_{\mathcal{E}'(\mathbb{R}),C^\infty(\mathbb{R})}=-\int_{\mathbb{R}^{n+1}}f(z)v(z)\chi'(\Phi(z))dz.$$

Taking advantage of the assumption $d\Phi\neq 0$ on Ω, we may assume (up to using a partition of unity of Ω) for instance that $\partial_{x_n}\Phi\neq 0$ on the whole Ω. We integrate by parts

$$\langle h_f',\chi\rangle_{\mathcal{E}'(\mathbb{R}),C^\infty(\mathbb{R})}=-\int_\Omega f(z)v(z)\frac{1}{\partial_{x_n}\Phi(z)}\partial_{x_n}\big(\chi\circ\Phi(z)\big)dz=\int_\Omega\partial_{x_n}\left(f(z)v(z)\frac{1}{\partial_{x_n}\Phi(z)}\right)\chi\circ\Phi(z)dz,$$

and, since $v\in C_c^\infty(\Omega)$, we obtain $|\langle h_f',\chi\rangle|\leq C\|\chi\|_\infty$, and h_f' is also a distribution of order zero. Iterating this procedure (using the assumption $v\in C_c^\infty(\Omega)$) implies that $h_f^{(m)}$ is of order zero for all $m\in\mathbb{N}$, hence h_f is a C^∞ function on $\mathbb{R}$. Since moreover, $h_f\in\mathcal{E}'(\mathbb{R})$, we finally have $h_f\in C_c^\infty(\mathbb{R})$.

♣ `new version` We may now apply the following version of the Paley-Wiener theorem to h_f.

Theorem 3.2.7 (Paley-Wiener-Schwartz)**.** *Suppose $g\in\mathcal{E}'(\mathbb{R})$, of order zero. Then the following two statements are equivalent:*

- *$\mathrm{supp}(g)\subset(-\infty,0]$,*
- *$\widehat{g}$ can be extended continuously as an entire function which is uniformly bounded in the closed upper half-plane*
$$\mathbb{C}^+=\{x+iy; x\in\mathbb{R}, y\geq 0\}.$$

This is a particular case of the general Paley-Wiener-Schwartz theorem, see e.g. [Hör90, Theorem 7.3.1].

Proof of Theorem 3.2.8. The direct implication is simpler. Notice first that $\widehat{g}(\zeta)=\langle g,e^{-is\zeta}\rangle_{\mathcal{E}'(\mathbb{R}),C^\infty(\mathbb{R})}$ holds for $\zeta\in\mathbb{R}$ but also for $\zeta\in\mathbb{C}$, and $\overline{\partial}_\zeta\widehat{g}(\zeta)=\langle g,\overline{\partial}_\zeta e^{-is\zeta}\rangle_{\mathcal{E}'(\mathbb{R}),C^\infty(\mathbb{R})}=0$. Hence, $\widehat{g}(\zeta)$ is an entire function. Moreover, using the assumption $\mathrm{supp}(g)\subset(-\infty,0]$, we let $\chi_\delta\in C_c^\infty(\mathbb{R})$ such that $\chi_\delta=1$ on $\mathrm{supp}(g)$ and $\mathrm{supp}(\chi_\delta)\subset(-\infty,\delta]$. We have $g=g\chi_\delta$ and thus, for $\zeta=x+iy\in\mathbb{C}$,

$$\widehat{g}(\zeta)=\langle g,e^{-is\zeta}\rangle_{\mathcal{E}'(\mathbb{R}),C^\infty(\mathbb{R})}=\langle g,e^{-isx}e^{sy}\chi_\delta(s)\rangle_{\mathcal{E}'(\mathbb{R}),C^\infty(\mathbb{R})}.$$

Using that g is of order zero implies existence of $C > 0$ such that for $y \geq 0$,

$$|\widehat{g}(\zeta)| = \left|\langle g, e^{-isx} e^{sy} \chi_\delta(s)\rangle_{\mathcal{E}'(\mathbb{R}),C^\infty(\mathbb{R})}\right| \leq C \sup_{s\in\mathbb{R}} \left|e^{-isx} e^{sy} \chi_\delta(s)\right| \leq C e^{\delta y}.$$

Since this is true for all $\delta > 0$, we deduce that $|\widehat{g}(\zeta)| \leq C$ uniformly for $\mathrm{Im}(\zeta) \geq 0$.

Let us now prove the converse implication (that used in the proof of Lemma 3.2.5), and assume $\hat{g}$ is holomorphic in the interior of $\mathbb{C}^+$, and uniformly bounded, say, by C_0 on the whole $\mathbb{C}^+$. We define, for $\delta, \varepsilon > 0$ two small parameters (aimed at tending to zero), the function

$$G_{\varepsilon,\delta}(z) = \frac{\widehat{g}(z + i\delta)}{(1 - i\varepsilon z)^2},$$

which is a shift (by $-i\delta$) and a regularization of $\widehat{g}$. Note that $G_{\varepsilon,\delta}$ has a pole at the point $-\frac{i}{\varepsilon}$, and is holomorphic in $\{z \in \mathbb{C}, \mathrm{Im}(z) > \max(-\delta, -\frac{1}{\varepsilon})\}$, and in particular in a neighborhood of $\mathbb{C}^+$. Also, according to the uniform boundedness of $\widehat{g}$ on $\mathbb{C}^+$, the function $G_{\varepsilon,\delta}$ satisfies the bound:

$$|G_{\varepsilon,\delta}(z)| = \frac{|\widehat{g}(z + i\delta)|}{|1 - i\varepsilon z|^2} \leq \frac{C_0}{(1 + \varepsilon y)^2 + (\varepsilon x)^2}, \quad \text{with } z = x + iy, y \geq 0. \tag{3.12}$$

In particular, for any $y \geq 0$, the function $x \mapsto G_{\varepsilon,\delta}(x + iy)$ is in $L^1(\mathbb{R})$. Now, we are interested in its inverse Fourier transform (on the real line)

$$g_{\varepsilon,\delta}(s) = \frac{1}{2\pi} \int_{x\in\mathbb{R}} e^{isx} G_{\varepsilon,\delta}(x)\, dx,$$

which belongs to $L^\infty(\mathbb{R})$. Notice first that we have

$$g = \lim_{\varepsilon\to 0^+} \lim_{\delta\to 0^+} g_{\varepsilon,\delta}, \quad \text{in } \mathcal{S}'(\mathbb{R}), \tag{3.13}$$

since, for all $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\int_\mathbb{R} g_{\varepsilon,\delta}(s)\varphi(s)ds = \int_\mathbb{R} \widehat{g}_{\varepsilon,\delta}(x)\hat{\varphi}(x)dx = \int_\mathbb{R} G_{\varepsilon,\delta}(x)\hat{\varphi}(x)dx \to \int_\mathbb{R} \hat{g}(x)\hat{\varphi}(x)dx = \int_\mathbb{R} g(s)\varphi(s)ds,$$

as $\delta \to 0^+$ and then $\varepsilon \to 0^+$. This follows from two dominated convergence arguments, using that $\widehat{g} \in L^\infty(\mathbb{R})$, thus $\widehat{g}\hat{\varphi} \in L^\infty(\mathbb{R})$. According to (3.15), we are only left to proving that $g_{\varepsilon,\delta}(s) = 0$ for all $s \geq 0$.

To this aim, we write (using again the dominated convergence arguments with $G_{\varepsilon,\delta} \in L^1(\mathbb{R})$)

$$g_{\varepsilon,\delta}(s) = \lim_{N\to+\infty} \frac{1}{2\pi} \int_{-N}^{N} e^{is\zeta} G_{\varepsilon,\delta}(\zeta)\, d\zeta.$$

Using that $G_{\varepsilon,\delta}$ is holomorphic in a neighborhood of $\mathbb{C}^+$, we change the integration contour from $[-N, N]$ to the oriented rectangular contour $\gamma_N = [-N, -N + iN] \cup [-N + iN, N + iN] \cup [N + iN, N]$:

$$g_{\varepsilon,\delta}(s) = \lim_{N\to+\infty} \frac{1}{2\pi} \int_{\gamma_N} e^{isz} G_{\varepsilon,\delta}(z)dz.$$

We now estimate the integrand for $s \geq 0$, using (3.14):

- On $[-N, -N + iN]$, we have $|e^{isz} G_{\varepsilon,\delta}(z)| \leq \frac{C_0}{1+(\varepsilon N)^2}$, using that $\mathrm{Im}(z) \geq 0$ and $s \geq 0$.
- On $[-N + iN, N + iN]$, we have $|e^{isz} G_{\varepsilon,\delta}(z)| \leq \frac{C_0 e^{-sN}}{(1+\varepsilon N)^2}$, using that $\mathrm{Im}(z) = N$ and $s \geq 0$.
- On $[N + iN, N]$ we have as in the first case $|e^{isz} G_{\varepsilon,\delta}(z)| \leq \frac{C_0}{1+(\varepsilon N)^2}$.

Taking into account that the length of the path are of order N, we obtain for $s \geq 0$

$$\left|\int_{\gamma_N} e^{isz} G_{\varepsilon,\delta}(z) dz\right| \leq N \frac{C_0}{1+(\varepsilon N)^2} + 2N \frac{C_0 e^{-sN}}{(1+\varepsilon N)^2} + N \frac{C_0}{1+(\varepsilon N)^2} \to 0 \quad \text{as } N \to +\infty.$$

This yields $g_{\varepsilon,\delta}(s) = 0$ for all $s \geq 0, \varepsilon > 0, \delta > 0$, and thus (3.15) concludes that $\text{supp}(g) \subset (-\infty, 0]$, and hence the proof of the theorem. □

♣ `old version` We may now apply the following version of the Paley-Wiener theorem to h_f.

Theorem 3.2.8 (Paley-Wiener)**.** *Suppose $g \in \mathcal{S}(\mathbb{R})$. Then the following two statements are equivalent:*

- *$g(s) = 0$ for all $s > 0$,*
- *$\widehat{g}$ can be extended continuously as a bounded function in the closed upper half-plane*

$$\mathbb{C}^+ = \{x + iy; x \in \mathbb{R}, y \geq 0\},$$

with $\widehat{g}$ holomorphic in the interior.

Applying this result to the function h_f gives $\text{supp}(h_f) \subset]-\infty, 0]$. Therefore, we have proved that for $\chi \in C^\infty(\mathbb{R})$,

$$\text{supp}(\chi) \subset [0, +\infty) \quad \Longrightarrow \quad 0 = \langle h_f, \chi\rangle = \langle fv, \chi(\Phi)\rangle = \langle f, \chi(\Phi)v\rangle.$$

Since this is true for a subset of function f dense in $\mathcal{S}$ (those having compactly supported Fourier transform), this means that the function $\chi(\Phi)v$ is identically zero on $\mathbb{R}^{1+n}$ as soon as $\text{supp}(\chi) \subset [0, +\infty)$. That is to say $v = 0$ for $\Phi > 0$ or $\text{supp}(v) \subset \{\Phi \leq 0\}$, which concludes the proof of the lemma. □

It now remains to give a proof of the Phragmén-Lindelöf Theorem (Lemma 3.2.6) and the Paley-Wiener Theorem (Theorem 3.2.8).

Proof of Lemma 3.2.6. First note that the sector Q_1 can be rotated, say to quadrant

$$Q = \{z \in \mathbb{C}, \arg(z) \in [-\frac{\pi}{4}, \frac{\pi}{4}]\}.$$

We use the principal determination of the logarithm that is if $z = re^{i\theta}$ with $-\pi < \theta < \pi$. We define

$$g_\delta(z) = g(z) e^{-\delta z^{\frac{3}{2}}},$$

(where $z^{3/2} = r^{3/2} e^{3i\theta/2}$), which is holomorphic in the quadrant Q.

Also, we have the bound

$$|e^{-\delta z^{\frac{3}{2}}}| = e^{-\delta r^{3/2} \cos(3\theta/2)}.$$

On Q, we have $|\theta| \leq \pi/4$ and therefore $|3\theta/2| \leq 3\pi/8 < \pi/2$ and $\cos(3\theta/2) \geq \cos(3\pi/8) =: \eta > 0$.

So, the first assumption on g gives $|g_\delta(re^{i\theta})| \leq Ce^{cr} e^{-\delta r^{3/2}\eta}$. This implies $\lim_{z \in Q, |z| \to \infty} |g_\delta(z)| = 0$. As a consequence, there exists $R > 0$ such that $|g_\delta(z)| < 1/2$ on $\{|z| \geq R\} \cap Q$. Now, on the bounded set $Q^R = Q \cap \{|z| \leq R\}$, we apply the maximum principle ($\max_{\overline{Q^R}} |g_\delta| = \max_{\partial Q^R} |g_\delta|$) to the function g_δ. According to the second assumption we have $|g_\delta| \leq 1$ on ∂Q^R. This yields $|g_\delta| \leq 1$ on Q^R and hence $|g_\delta| \leq 1$ on Q. Finally letting δ tend to zero, we have $|g_\delta(z)| \to |g(z)|$ for all $z \in \overline{Q}$, which yields the sought result. □

Note that the a priori subexponential growth at infinity in the first assumption of the theorem could be replaced by the weaker assumption $|g(z)| \leq Ce^{c|z|^{2-\varepsilon}}$ for some $\varepsilon > 0$. However, the result is false for $\varepsilon = 0$. The classical counterexample on the quarter plane Q is the holomorphic function $z \mapsto e^{z^2}$. for $z = re^{\pm i\frac{\pi}{4}}$, it satisfies indeed $e^{z^2} = e^{r^2 e^{\pm i\frac{2\pi}{4}}} = e^{\pm ir^2}$ and hence $|e^{z^2}| = 1$ on ∂Q (however, e^{z^2} is clearly not bounded on the real axis).

Proof of Theorem 3.2.8. The direct implication is simpler. Assume $g(s) = 0$ for $s \geq 0$, then $\widehat{g}(\zeta) = \int_{s\leq 0} e^{-is\zeta} g(s)ds$ for $\zeta \in \mathbb{R}$. This formula can be continuously extended to $\mathbb{C}^+$ with the estimate

$$|\widehat{g}(x+iy)| \leq \int_{s\leq 0} e^{ys}|g(s)|ds \leq \int_{s\leq 0} |g(s)|ds = \|g\|_{L^1(\mathbb{R})}, \quad \text{for all } y \geq 0.$$

Its derivatives satisfy similar bounds (still on $\mathbb{C}^+$), and we may derivate under the integral. The holomorphicity of the integrand with respect to ζ implies that $\widehat{g}(\zeta)$ satisfies the Cauchy Riemann equations and is thus holomorphic in the interior of $\mathbb{C}^+$, with the estimate $|\widehat{g}(\zeta)| \leq \|g\|_{L^1(\mathbb{R})}$.

Let us now prove the converse implication (that used in the proof of Lemma 3.2.5), and assume $\hat{g}$ is holomorphic in the interior of $\mathbb{C}^+$, and uniformly bounded, say, by C_0 on the whole $\mathbb{C}^+$. We define, for $\delta, \varepsilon > 0$ two small parameters (aimed at tending to zero), the function

$$G_{\varepsilon,\delta}(z) = \frac{\widehat{g}(z+i\delta)}{(1-i\varepsilon z)^2},$$

which is a shift (by $-i\delta$) and a regularization of $\widehat{g}$. Note that $G_{\varepsilon,\delta}$ has a pole at the point $-\frac{i}{\varepsilon}$, and is holomorphic in $\{z \in \mathbb{C}, \mathrm{Im}(z) > \max(-\delta, -\frac{1}{\varepsilon})\}$, and in particular in a neighborhood of $\mathbb{C}^+$. Also, according to the uniform boundedness of $\widehat{g}$ on $\mathbb{C}^+$, the function $G_{\varepsilon,\delta}$ satisfies the bound:

$$|G_{\varepsilon,\delta}(z)| = \frac{|\widehat{g}(z+i\delta)|}{|1-i\varepsilon z|^2} \leq \frac{C_0}{(1+\varepsilon y)^2 + (\varepsilon x)^2}, \quad \text{with } z = x+iy. \tag{3.14}$$

Now, we are interested in its inverse Fourier transform

$$g_{\varepsilon,\delta}(s) = \frac{1}{2\pi}\int_{\xi\in\mathbb{R}} e^{is\zeta} G_{\varepsilon,\delta}(\zeta)\ d\zeta,$$

since we have

$$g(s) = \lim_{\varepsilon\to 0^+}\lim_{\delta\to 0^+} g_{\varepsilon,\delta}(s), \quad \text{for all } s \in \mathbb{R}. \tag{3.15}$$

This follows from two dominated convergence arguments, using that $\widehat{g} \in L^1(\mathbb{R})$. Therefore, we are only left to proving that $g_{\varepsilon,\delta}(s) = 0$ for all $s \geq 0$.

To this aim, we write (using again the dominated convergence arguments with $G_{\varepsilon,\delta} \in L^1(\mathbb{R})$)

$$g_{\varepsilon,\delta}(s) = \lim_{N\to+\infty}\frac{1}{2\pi}\int_{-N}^{N} e^{is\zeta} G_{\varepsilon,\delta}(\zeta)\ d\zeta.$$

Using that $G_{\varepsilon,\delta}$ is holomorphic in a neighborhood of $\mathbb{C}^+$, we change the integration contour from $[-N,N]$ to the oriented rectangular contour $\gamma_N = [-N, -N+iN] \cup [-N+iN, N+iN] \cup [N+iN, N]$:

$$g_{\varepsilon,\delta}(s) = \lim_{N\to+\infty}\frac{1}{2\pi}\int_{\gamma_N} e^{isz} G_{\varepsilon,\delta}(z)dz.$$

We now estimate the integrand for $s \geq 0$, using (3.14):

- On $[-N, -N+iN]$, we have $|e^{isz}G_{\varepsilon,\delta}(z)| \leq \frac{C_0}{1+(\varepsilon N)^2}$, using that $\mathrm{Im}(z) \geq 0$ and $s \geq 0$.
- On $[-N+iN, N+iN]$, we have $|e^{isz}G_{\varepsilon,\delta}(z)| \leq \frac{C_0 e^{-sN}}{(1+\varepsilon N)^2}$, using that $\mathrm{Im}(z) = N$ and $s \geq 0$.
- On $[N+iN, N]$ we have as in the first case $|e^{isz}G_{\varepsilon,\delta}(z)| \leq \frac{C_0}{1+(\varepsilon N)^2}$.

Taking into account that the length of the path are of order N, we obtain for $s \geq 0$

$$\left|\int_{\gamma_N} e^{isz} G_{\varepsilon,\delta}(z)dz\right| \leq N\frac{C_0}{1+(\varepsilon N)^2} + 2N\frac{C_0 e^{-sN}}{(1+\varepsilon N)^2} + N\frac{C_0}{1+(\varepsilon N)^2} \to 0 \quad \text{as } N \to +\infty.$$

This yields $g_{\varepsilon,\delta}(s) = 0$ for all $s \geq 0, \varepsilon > 0, \delta > 0$, and thus (3.15) concludes that $g(s) = 0$ for all $s \geq 0$, and hence the proof of the theorem. □

3.3 The Carleman estimate

3.3.1 The "conjugated operator"

As in the classical case, we need to check the effect of the conjugated operator. Yet, we have to be a little careful, first because $e^{\varepsilon\frac{|D_t|^2}{2\tau}}$ is not well defined on any Sobolev space and even not on $\mathcal{S}$.

As before, we make the change of variable $v = e^{\tau\Phi}u$ and for getting (3.4), we are left to prove

$$\tau\|e^{-\varepsilon\frac{|D_t|^2}{2\tau}}v\|^2_{H^1_\tau} \leq C\left\|e^{-\varepsilon\frac{|D_t|^2}{2\tau}}P_\Phi v\right\|^2_{L^2} + Ce^{-\mathsf{d}\tau}\|v\|^2_{H^1_\tau}$$

Our operator P commutes with $e^{-\varepsilon\frac{|D_t|^2}{2\tau}}$ since its coefficients are independent on t. Yet, the operator $P_\Phi = e^{\tau\Phi}Pe^{-\tau\Phi}$ may depend on t because Φ depends on t. We will take advantage of the fact that since Φ is quadratic, the principal symbol of P_Φ only involves derivative of Φ of order at least 1 and is therefore linear. We first prove the following simple lemma.

Lemma 3.3.1. *Let $u \in \mathcal{S}(\mathbb{R}^{n+1})$, then*

$$e^{-\varepsilon\frac{|D_t|^2}{2\tau}}(tu) = \left(t + i\varepsilon\frac{D_t}{\tau}\right)e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u.$$

Proof. We first recall the classical formula

$$\widehat{tv}(\xi) = \int_{\mathbb{R}^{n+1}} e^{-it\xi_t}e^{-ix\cdot\xi_x}tv(t,x)dxdt = \int_{\mathbb{R}^{n+1}} i\partial_{\xi_t}e^{-it\xi_t}e^{-ix\cdot\xi_x}v(t,x)dxdt = i\partial_{\xi_t}\widehat{v}(\xi).$$

$$\begin{aligned}\left(\widehat{e^{-\varepsilon\frac{|D_t|^2}{2\tau}}(tu)}\right)(\xi) &= e^{-\varepsilon\frac{|\xi_t|^2}{2\tau}}\widehat{(tu)}(\xi) = e^{-\varepsilon\frac{|\xi_t|^2}{2\tau}}i\partial_{\xi_t}\widehat{u}(\xi) = i\partial_{\xi_t}\left[e^{-\varepsilon\frac{|\xi_t|^2}{2\tau}}\widehat{u}(\xi)\right] + i\frac{\varepsilon\xi_t}{\tau}e^{-\varepsilon\frac{|\xi_t|^2}{2\tau}}\widehat{u}(\xi)\\ &= \left[t\widehat{e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u}\right](\xi) + i\left[\frac{\varepsilon D_t}{\tau}\widehat{e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u}\right](\xi).\end{aligned}$$

This can be rewritten as

$$e^{-\varepsilon\frac{|D_t|^2}{2\tau}}(tu) = te^{-\varepsilon\frac{|D_t|^2}{2\tau}}u + i\varepsilon\frac{D_t}{\tau}e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u = \left(t + i\varepsilon\frac{D_t}{\tau}\right)e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u,$$

which proves the lemma. □

Remark 3.3.2. Lemma 3.3.1 could easily be iterated to get the formula

$$e^{-\varepsilon\frac{|D_t|^2}{2\tau}}(t^k u) = \left(t + i\varepsilon\frac{D_t}{\tau}\right)^k e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u,$$

where the exponent k is meant in the sense of composition. For f polynomial in t, we would get

$$e^{-\varepsilon\frac{|D_t|^2}{2\tau}}(f(t)u) = f\left(t + i\varepsilon\frac{D_t}{\tau}\right)e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u.$$

This means that the "formal" conjugated operator of $f(t)$ by $e^{-\varepsilon\frac{|D_t|^2}{2\tau}}$ is a differential operator, whose order is given by the degree of the polynomial f.

For a general (even smooth) function $f(t)$, it seems therefore very hard to give a precise meaning to $f\left(t + i\varepsilon\frac{D_t}{\tau}\right)$.

Even in the analytic case, $f\left(t + i\varepsilon\frac{D_t}{\tau}\right)$ would be an infinite sum of differential operators, that means an operator of "infinite order". This is not clear how to define this in an exact way. Yet, some authors managed to give some meaning of an approximation of this formula. Namely, the idea is to replace $t + i\varepsilon\frac{D_t}{\tau}$ by some approximate operator $\chi\left(\frac{t}{\kappa}\right)t + i\chi\left(\frac{\varepsilon D_t}{\kappa\tau}\right)\varepsilon\frac{D_t}{\tau}$ for κ small. These operators have the advantage to be bounded and we can consider some infinite series. We refer to Hörmander [Hör97]. Similarly, it is possible to replace the holomorphic function f with a cutoff near small x and ξ_t, see Tataru [Tat95, Tat99].

We now want to understand how $Q^{\Phi}_{\varepsilon,\tau}$ "commutes" with an operator P. To this aim, let us first consider the simplest case in which $P = D_j$. We have the following key lemma.

Lemma 3.3.3. *Assume Φ is a real polynomial of degree two in the variable t. For all $k \in \{0, \cdots, n\}$ (with the convention $t = z_0$, $D_0 = D_t$)*

$$Q^{\Phi}_{\varepsilon,\tau} D_k = (D_k)_{\Phi,\varepsilon} Q^{\Phi}_{\varepsilon,\tau},$$

where (denoting $\Phi''_{t,z_k} = \partial_t \partial_{z_k} \Phi$)

$$(D_k)_{\Phi,\varepsilon} = D_k + i\tau \partial_k \Phi(z) - \varepsilon \Phi''_{t,z_k} D_t.$$

Note that since Φ is quadratic in the variable t, the quantity Φ''_{t,z_j} is actually constant in t! In particular, the principal symbol of $(D_k)_{\Phi,\varepsilon}$ is $\xi_k + i\tau\partial_k\Phi - \varepsilon\Phi''_{t,z_k}\xi_t$. ♣ `on suppose quadratique partout ou juste en` t

Proof. Since Φ is quadratic in the variable t, $\partial_k\Phi$ is a polynomial of degree 1 in t and can be written as

$$\partial_k \Phi = f_1(x) + t f_0.$$

where $f_1(x)$ (resp. f_0) is polynomial in x of order 1 (resp. a constant). In particular, Lemma 3.3.1 gives

$$\begin{aligned} e^{-\varepsilon \frac{|D_t|^2}{2\tau}} [(D_k + i\tau\partial_k\Phi)u] &= e^{-\varepsilon \frac{|D_t|^2}{2\tau}} [(D_k + i\tau(f_1(x) + t f_0))u] \\ &= \left[D_k + i\tau \left(f_1(x) + \left(t + i\varepsilon \frac{D_t}{\tau} \right) f_0 \right) \right] e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u \\ &= (D_k + i\tau\partial_k\Phi - \varepsilon f_0 D_t) e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u. \end{aligned}$$

To get an intrinsic expression, we notice that $f_0 = \partial_t\partial_k\Phi$, so $f_0 D_t$ can be written $\partial_t\partial_k\Phi D_t$. This concludes the proof of the lemma. □

This lemma allows to compute the principal symbol of the "conjugated operator" of general differential operators (with coefficients independent of t).

Corollary 3.3.4 (The "conjugated operator")**.** *Let $\Omega \subset \mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}_x$ and $P \in \mathrm{Diff}^m(\Omega)$ be a (classical) differential operator with principal symbol p_m. Assume also that* ***all its coefficients are independent on*** *t (that is $p_\alpha(z) = p_\alpha(x)$ for all $|\alpha| \leq m$). Let Φ be a real-valued function being* ***quadratic*** *in t. Then, for any $\varepsilon > 0$, there exists a unique $P_{\Phi,\varepsilon} v \in \mathrm{Diff}^m_\tau(\Omega)$ so that we have*

$$Q^{\Phi}_{\varepsilon,\tau} P = P_{\Phi,\varepsilon} Q^{\Phi}_{\varepsilon,\tau}.$$

Moreover, the principal symbol of $P_{\Phi,\varepsilon}$ is

$$p_{\Phi,\varepsilon}(z,\xi,\tau) = p_m(z, \xi + i\tau d\Phi(z) - \varepsilon\Phi''_{t,z}\xi_t),$$

where we use the notation $\Phi''_{t,z}\xi_t = \mathrm{Hess}(\Phi)((\xi_t, 0, \cdots, 0); \cdot) = \xi_t V$ with V the constant vector with coefficients $V_k = (\partial_t\partial_k\Phi)$.

We stress the fact that all coefficients of p should be independent of t: this is not an assumption on the principal part of the operator only.

Remark 3.3.5. The expression "conjugated operator" is a bit abusive since $e^{\varepsilon \frac{|D_t|^2}{2\tau}}$ is not well defined as an operator. Yet, we would like to write formally

$$P_{\Phi,\varepsilon} v = Q^{\Phi}_{\varepsilon,\tau} P \left(Q^{\Phi}_{\varepsilon,\tau} \right)^{-1} v = e^{-\varepsilon \frac{|D_t|^2}{2\tau}} e^{\tau\Phi} P e^{-\tau\Phi} e^{\varepsilon \frac{|D_t|^2}{2\tau}} v = e^{-\varepsilon \frac{|D_t|^2}{2\tau}} P_\Phi e^{\varepsilon \frac{|D_t|^2}{2\tau}} v.$$

Proof. The proof is similar to that of Lemma 1.3.10. We first prove the result for $D^\alpha = D_0^{\alpha_0} D_1^{\alpha_1} \cdots D_j^{\alpha_j} \cdots D_n^{\alpha_n}$. Using composition formula of Proposition 1.3.6 together with Lemma 3.3.3, we obtain that the "conjugated operator" $(D^\alpha)_{\Phi,\varepsilon}$ of D^α is in $\mathrm{Diff}^m_\tau(\Omega)$ and has principal symbol

$$\prod_{k=0}^{n} (\xi_k + i\tau\partial_k\Phi - \varepsilon(\partial_t\partial_k\Phi)\xi_t)^{|\alpha_k|} = (\xi + i\tau\nabla\Phi - \varepsilon\Phi''_{t,z}\xi_t)^\alpha,$$

where we use the notation $\Phi''_{t,z}\xi_t = \text{Hess}(\Phi)((\xi_t, 0, \cdots, 0); \cdot)$.

Since all the functions $p_\alpha(x)$ do not depend on t, they commute with $Q^\Phi_{\varepsilon,\tau}$. So, we get the conclusion of the corollary.

□

Remark 3.3.6. In the case of a second order operator P (with coefficients independent of t), with real symbol p, we have (denoting by $\tilde{p}$ the polar symmetric bilinear form of p),

$$\begin{aligned} p_{\Phi,\varepsilon}(z,\xi,\tau) &= p(z, \xi + i\tau d\Phi(z) - \varepsilon\Phi''_{t,z}\xi_t) \\ &= p(z, \xi - \varepsilon\Phi''_{t,z}\xi_t) - \tau^2 p(z, d\Phi(z)) + 2i\tau\tilde{p}(z, \xi - \varepsilon\Phi''_{t,z}\xi_t, d\Phi(z)). \end{aligned}$$

As in the classical case, an important point here is that $\text{Im}(p_{\Phi,\varepsilon}(z,\xi,\tau)) = \tau 2\tilde{p}(z, \xi - \varepsilon\Phi''_{t,z}\xi_t, d\Phi(z))$ and may be divided by τ.

A very important feature of the previous formula is that the principal symbol of $P_{\Phi,\varepsilon}$ is actually close to the principal symbol of P_Φ if ε is small. So, we can expect that it satisfies the same subelliptic estimates.

3.3.2 A first subelliptic estimate

We first write the following Lemma on p_Φ, that we have actually already used and proved in Proposition 3.2.3, using Lemma 2.1.8 and homogeneity, so we skip the proof.

Lemma 3.3.7. *Let Ω, P satisfy the assumptions of Theorem 3.1.1. Assume that the function Φ is pseudoconvex with respect to P at z_0 in $\xi_t = 0$, in the sense of Definition 3.1.2. Then there exist $C_1, C_2 > 0$ such that for any $(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}^+$, we have*

$$\frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\}(z_0,\xi,\tau) + C_1\left[\frac{|p_\Phi(z_0,\xi,\tau)|^2}{|\xi|^2+\tau^2} + |\xi_t|^2\right] \geq C_2(|\xi|^2+\tau^2).$$

where we have extended $\frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\}(z_0,\xi,\tau)$ by continuity at $\tau = 0$ with the value $2\{p,\{p,\Phi\}\}(z_0,\xi)$.

By perturbation, we can get a similar conclusion for the perturbated operator.

Lemma 3.3.8. *Let Ω, P satisfy the assumptions of Theorem 3.1.1. Assume that the function Φ is pseudoconvex with respect to P at z_0 in $\xi_t = 0$, in the sense of Definition 3.1.2. Then there exists $\varepsilon_0 > 0$ so that for any $0 \leq \varepsilon < \varepsilon_0$, there exist $C_1, C_2 > 0$ such that for any $(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}^+$, we have*

$$\frac{1}{i\tau}\{\overline{p_{\Phi,\varepsilon}}, p_{\Phi,\varepsilon}\}(z_0,\xi,\tau) + C_1\left[\frac{|p_{\Phi,\varepsilon}(z_0,\xi,\tau)|^2}{|\xi|^2+\tau^2} + |\xi_t|^2\right] \geq C_2(|\xi|^2+\tau^2).$$

where the quantity $\frac{1}{i\tau}\{\overline{p_{\Phi,\varepsilon}}, p_{\Phi,\varepsilon}\}(z_0,\xi,\tau)$ is extended by continuity at $\tau = 0$.

Proof. The Lemma mainly follows by saying that $p_{\Phi,\varepsilon}$ is a perturbation of p_Φ and using Lemma 3.3.7. Yet, we have to be a little careful because of the factor $\frac{1}{\tau}$. Noticing as before that $\frac{1}{i\tau}\{\overline{p_{\Phi,\varepsilon}}, p_{\Phi,\varepsilon}\} = \frac{2}{\tau}\{\text{Re}\, p_{\Phi,\varepsilon}, \text{Im}\, p_{\Phi,\varepsilon}\}$. Then, using Remark 3.3.6 we can write $\text{Im}\, p_{\Phi,\varepsilon} = \tau\widetilde{p_{\Phi,\varepsilon}}^i$. Moreover, $\widetilde{p_{\Phi,\varepsilon}}^i$ and all its derivatives are all continuous in ε. Hence, we can write $\frac{1}{i\tau}\{\overline{p_{\Phi,\varepsilon}}, p_{\Phi,\varepsilon}\} = 2\{\text{Re}\, p_{\Phi,\varepsilon}, \widetilde{p_{\Phi,\varepsilon}}^i\}$. It can therefore be extended by continuity to $\tau = 0$ and the result follows by a perturbation of Lemma 3.3.7. □

We are now ready to prove a first subelliptic estimate that will be crucial for the final proof of Theorem 3.1.3.

Proposition 3.3.9. *Let Ω, P satisfy the assumptions of Theorem 3.1.1. Assume that the function Φ is pseudoconvex with respect to P at z_0 in $\xi_t = 0$, in the sense of Definition 3.1.2. Then, there exist $\varepsilon > 0$, $r > 0$, $C > 0$ and $\tau_0 > 0$ so that we have the estimate*

$$\tau\|v\|^2_{H^1_\tau} \leq C\|P_{\Phi,\varepsilon}v\|^2_{L^2} + C\tau\|D_t v\|^2_{L^2}, \tag{3.16}$$

for any $v \in C^\infty_c(B(z_0,r))$ and $\tau \geq \tau_0$.

Note that the parameter $\varepsilon > 0$ is fixed by this proposition (in fact, by Lemma 3.3.8). Note that this estimate is extremely close to the usual Carleman estimate (2.2) of Theorem 2.1.1. The only difference is the last term $\tau \|D_t v\|_{L^2}^2$ in the right hand-side. This term comes from the fact that the pseudoconvexity assumption (and hence the symbolic estimate of Lemma 3.3.8) is made on $\xi_t = 0$ only, i.e. on $D_t = 0$ only. Also, remark that this additional term has precisely the same strength as the term $\tau \|v\|_{H^1_\tau}^2$ on the left handside of the estimate.

Proof. The proof is as well very similar to that of Theorem 2.1.1. A little care is needed to factorize the skew-adjoint part of the operator. Note first that the form of Estimate (3.16) remains unchanged under addition to P of a (classical) differential operator in $\mathrm{Diff}^1(\Omega)$, the coefficients of which do not depend on the variable t. Indeed, after conjugation, the latter perturbation will yield a perturbation of $P_{\Phi,\varepsilon}$ being in $\mathrm{Diff}^1_\tau(\Omega)$, which, applied to v, is bounded by $\|v\|_{H^1_\tau}^2$ and thus can be absorbed in the left handside for $\tau \geq \tau_0$ with τ_0 large enough.

We then notice that, with P satisfying the assumptions of Theorem 3.1.1, we have

$$P = \tilde{P} + R_1, \quad \text{with} \quad \tilde{P} = -D_t^2 + \sum_{1\leq i,j\leq n} D_i a^{ij}(x) D_j, \quad \text{and} \quad R_1 \in \mathrm{Diff}^1_\tau(\Omega),$$

where $a^{ij}(x) = a^{ji}(x)$. See also Example 1.3.12. The operator $\tilde{P}$ is chosen to be selfadjoint. According to the previous discussion, it is sufficient to prove Estimate (3.16) for P replaced by $\tilde{P}$. Applying Lemma 3.3.3 and the fact that $Q^\Phi_{\varepsilon,\tau}$ exactly commutes with $a^{ij}(x)$, we have the exact formula:

$$\tilde{P}_{\Phi,\varepsilon} = -(D_t + i\tau\partial_t\Phi - \varepsilon\Phi''_{t,t}D_t)^2 + \sum_{1\leq i,j\leq n} (D_i + i\tau\partial_i\Phi - \varepsilon\Phi''_{t,x_i}D_t) a^{ij}(x)(D_j + i\tau\partial_j\Phi - \varepsilon\Phi''_{t,x_j}D_t).$$

We now collect all terms being factored by τ to obtain, for some $M \in \mathrm{Diff}^1_\tau(\Omega)$,

$$\tilde{P}_{\Phi,\varepsilon} = -(D_t - \varepsilon\Phi''_{t,t}D_t)^2 + \sum_{1\leq i,j\leq n} (D_i - \varepsilon\Phi''_{t,x_i}D_t) a^{ij}(x)(D_j - \varepsilon\Phi''_{t,x_j}D_t) + \tau M,$$

and remark that $\tilde{P}_{\Phi,\varepsilon} - \tau M$ is a selfadjoint operator. As a consequence, when defining

$$P_{R,\varepsilon} = \frac{\tilde{P}_{\Phi,\varepsilon} + \tilde{P}^*_{\Phi,\varepsilon}}{2}, \qquad P_{I,\varepsilon} = \frac{\tilde{P}_{\Phi,\varepsilon} - \tilde{P}^*_{\Phi,\varepsilon}}{2i},$$

we notice that we have, as in the proof of Theorem 2.1.1, $P_{I,\varepsilon} = \frac{\tau M - \tau M^*}{2i} =: \tau\widetilde{P_{I,\varepsilon}}$ (that is, τ can be factorized in the skew-adjoint part of $P_{I,\varepsilon}$). With this decomposition, we have $P_{\Phi,\varepsilon} = P_{R,\varepsilon} + iP_{I,\varepsilon} = P_{R,\varepsilon} + i\tau\widetilde{P_{I,\varepsilon}}$, and may now proceed to the key computation, following the proof of Theorem 2.1.1. We obtain

$$\begin{aligned} \|P_{\Phi,\varepsilon}v\|_{L^2}^2 &= \|P_{R,\varepsilon}v\|_{L^2}^2 + \|P_{I,\varepsilon}v\|_{L^2}^2 + (i[P_{R,\varepsilon}, P_{I,\varepsilon}]v, v) \\ &= \|P_{R,\varepsilon}v\|_{L^2}^2 + \|P_{I,\varepsilon}v\|_{L^2}^2 + \tau\left(i[P_{R,\varepsilon}, \widetilde{P_{I,\varepsilon}}]v, v\right). \end{aligned}$$

The same computations lead to

$$\frac{1}{\tau}\|P_{\Phi,\varepsilon}v\|_{L^2}^2 \geq (Lv, v),$$

with

$$\begin{aligned} L &= C_1 P_{R,\varepsilon}(-\Delta + \tau^2)^{-1}P_{R,\varepsilon} + C_1 P_{I,\varepsilon}(-\Delta + \tau^2)^{-1}P_{I,\varepsilon} + \frac{i}{\tau}[P_{R,\varepsilon}, P_{I,\varepsilon}] \\ &= C_1 P_{R,\varepsilon}(-\Delta + \tau^2)^{-1}P_{R,\varepsilon} + C_1 P_{I,\varepsilon}(-\Delta + \tau^2)^{-1}P_{I,\varepsilon} + i[P_{R,\varepsilon}, \widetilde{P_{I,\varepsilon}}], \end{aligned}$$

for $\tau \geq \tau_0$, τ_0 large enough, and C_1 being taken as in the conclusion of Lemma 3.3.8. We thus obtain

$$\frac{1}{\tau}\|P_{\Phi,\varepsilon}v\|_{L^2}^2 + C_1\|D_t v\|_{L^2}^2 \geq \left((L + C_1 D_t^2)v, v\right). \tag{3.17}$$

The principal symbol of $L + C_1 D_t^2$ is

$$\frac{1}{i\tau}\{\overline{p_{\Phi,\varepsilon}}, p_{\Phi,\varepsilon}\}(z,\xi,\tau) + C_1\left[\frac{|p_{\Phi,\varepsilon}(z,\xi,\tau)|^2}{|\xi|^2+\tau^2} + |\xi_t|^2\right].$$

We conclude as before using the symbolic estimate of Lemma 3.3.8 at the point z_0, together with the Gårding inequality of Proposition 1.3.14. □

3.3.3 End of the proof of the Carleman estimate

Equipped with the subelliptic estimate (3.16), we may now proceed to the proof.

Setting $v = Q^{\Phi}_{\varepsilon,\tau}u = e^{-\frac{\varepsilon}{2\tau}|D_t|^2}(e^{\tau\Phi}u)$, we need to prove the estimate

$$\tau \|v\|^2_{H^1_\tau} \leq C\|P_{\Phi,\varepsilon}v\|^2_{L^2} + Ce^{-\mathsf{d}\tau}\left\|e^{\tau\Phi}u\right\|^2_{H^1_\tau}.$$

The latter is very close to (3.16), except for the last term, and it is very tempting to apply (3.16) to our function $v = Q^{\Phi}_{\varepsilon,\tau}u$. The hope is then that the term $\tau\|D_t v\|^2_{L^2}$ is estimated by using that the multiplier $e^{-\varepsilon\frac{|D_t|^2}{2\tau}}$ "localizes" where D_t is small. This will indeed be done at the end of the proof. However, the first problem we have to face is that, even if u is compactly supported, the function $v = Q^{\Phi}_{\varepsilon,\tau}u$ is not compactly supported in the variable t. Indeed, the operator $e^{-\frac{\varepsilon}{2\tau}|D_t|^2}$ is not local. We thus need to introduce an additional cutoff in time, and estimate all commutators it produces.

Proof of Theorem 3.1.3. We assume for simplicity that the point z_0 involved is $z_0 = (0, x_0)$, i.e. $t_0 = 0$.

We let $\varepsilon > 0$ and $r > 0$ be fixed by Proposition 3.3.9. We choose $r_0 > 0$ with $2r_0 = r$, and, all along the proof, we consider functions $u \in C^\infty_c(B(z_0, r_0/4))$. Let $\chi \in C^\infty_c(]-r_0, r_0[)$ such that $\chi = 1$ on $]-r_0/2, r_0/2[$.

Since v is not compactly supported in the variable t, we set $f = \chi(t)v(t,x)$ and we have $\mathrm{supp}(f) \subset [-r_0, r_0] \times B(x_0, r_0/4) \subset B(z_0, 2r_0) = B(z_0, r)$, so that we shall be able to apply Proposition 3.3.9 to the function f. Our goal is to estimate v, so that we write

$$\|v\|_{H^1_\tau} \leq \|f\|_{H^1_\tau} + \|v - f\|_{H^1_\tau}$$

where

$$v - f = (1-\chi)Q^{\Phi}_{\varepsilon,\tau}u = (1-\chi)e^{-\frac{\varepsilon}{2\tau}|D_t|^2}(\check{\chi}e^{\tau\Phi}u),$$

for some additional cutoff function $\check{\chi} \in C^\infty_c(]-r_0/3, r_0/3[)$ with $\check{\chi} = 1$ in a neighborhood of $[-r_0/4, r_0/4]$ so that $\check{\chi}u = u$. We are in position to apply the following lemma to estimate the remainder $v - f$.

Lemma 3.3.10. *Let $\chi_1 \in C^\infty(\mathbb{R}^{n+1})$, $\chi_2 \in C^\infty(\mathbb{R}^{n+1})$ with all derivatives bounded such that*

$$\mathrm{dist}(\mathrm{supp}(\chi_1), \mathrm{supp}(\chi_2)) > 0.$$

Then there exist $C, c > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^n)$ and all $\lambda \geq 0$, we have

$$\left\|\chi_1 e^{-\frac{|D_t|^2}{\lambda}}(\chi_2 u)\right\|_{L^2} \leq Ce^{-c\lambda}\|u\|_{L^2}, \qquad \left\|\chi_1 e^{-\frac{|D_t|^2}{\lambda}}(\chi_2 u)\right\|_{H^1_\tau} \leq Ce^{-c\lambda}\|u\|_{H^1_\tau}.$$

As a consequence of Lemma 3.3.10, we obtain, for $\tau \geq \tau_0$

$$\|v\|_{H^1_\tau} \leq \|f\|_{H^1_\tau} + Ce^{-c\frac{\tau}{\varepsilon}}\|e^{\tau\Phi}u\|_{H^1_\tau}. \tag{3.18}$$

The subelliptic estimate (3.16) applied to f gives

$$\tau\|f\|^2_{H^1_\tau} \leq C\|P_{\Phi,\varepsilon}f\|^2_{L^2} + C\tau\|D_t f\|^2_{L^2}, \tag{3.19}$$

and we need to estimate the two terms on the right handside in terms of v.

First, we estimate the term $\|P_{\Phi,\varepsilon}f\|_{L^2} = \|P_{\Phi,\varepsilon}\chi v\|_{L^2} \leq \|\chi P_{\Phi,\varepsilon}v\|_{L^2} + \|[P_{\Phi,\varepsilon},\chi]v\|_{L^2}$. For the commutator, we write $[P_{\Phi,\varepsilon},\chi]v = [P_{\Phi,\varepsilon},\chi]e^{-\frac{\varepsilon}{2\tau}|D_t|^2}\check{\chi}e^{\tau\Phi}u$. We notice that $[P_{\Phi,\varepsilon},\chi]$ is a differential operator of order 1 in (D,τ) with some coefficients supported on $\mathrm{supp}(\chi_t')$ that is, away from $\mathrm{supp}(\check{\chi})$. In particular, Lemma 3.3.10 implies $\|[P_{\Phi,\varepsilon},\chi]v\|_{L^2} \leq Ce^{-c\frac{\tau}{\varepsilon}}\left\|e^{\tau\Phi}u\right\|_{H^1_\tau}$. This yields

$$\|P_{\Phi,\varepsilon}f\|_{L^2} \leq \|P_{\Phi,\varepsilon}v\|_{L^2} + Ce^{-c\frac{\tau}{\varepsilon}}\left\|e^{\tau\Phi}u\right\|_{H^1_\tau}. \tag{3.20}$$

Second, we estimate the term $\|D_t f\|_{L^2}$. We obtain in a similar fashion

$$\|D_t f\|_{L^2} = \|D_t(\chi v)\|_{L^2} \leq \|\chi D_t v\|_{L^2} + \|\chi'(t)e^{-\frac{\varepsilon}{2\tau}|D_t|^2}\check{\chi}e^{\tau\Phi}u\|_{L^2} \leq \|D_t v\|_{L^2} + Ce^{-c\frac{\tau}{\varepsilon}}\left\|e^{\tau\Phi}u\right\|_{L^2} \tag{3.21}$$

where we have used again Lemma 3.3.10 in the last inequality.

Let ς a small constant to be fixed later on. We distinguish between frequencies of size smaller and bigger than $\varsigma\tau$. We obtain

$$\begin{aligned}\|D_t v\|_{L^2} = \|D_t e^{-\frac{\varepsilon}{2\tau}|D_t|^2}e^{\tau\Phi}u\|_{L^2} &\leq \|D_t\mathbb{1}_{|D_t|\leq\varsigma\tau}v\|_{L^2} + \|D_t\mathbb{1}_{|D_t|\geq\varsigma\tau}e^{-\frac{\varepsilon}{2\tau}|D_t|^2}e^{\tau\Phi}u\|_{L^2}\\ &\leq \|D_t\mathbb{1}_{|D_t|\leq\varsigma\tau}v\|_{L^2} + \max_{\xi_t\in[\varsigma\tau,+\infty)}(\xi_t e^{-\frac{\varepsilon}{2\tau}|\xi_t|^2})\|e^{\tau\Phi}u\|_{L^2}.\end{aligned}$$

Now, on $\mathbb{R}^+$, the function $s\mapsto se^{-\frac{\varepsilon}{2\tau}s^2}$ reaches its maximum at $s=\sqrt{\frac{\tau}{\varepsilon}}$, and is decreasing on $[\sqrt{\frac{\tau}{\varepsilon}},+\infty)$. Hence, if $\tau\geq\frac{1}{\varsigma^2\varepsilon}$, then $\sqrt{\frac{\tau}{\varepsilon}}\leq\varsigma\tau$, the function $s\mapsto se^{-\frac{\varepsilon}{2\tau}s^2}$ is decreasing on the interval $[\varsigma\tau,+\infty)$, and thus bounded by its value at $\varsigma\tau$. This yields, for all $\tau\geq\max(\tau_0,\frac{1}{\varsigma^2\varepsilon})$, the estimate

$$\|D_t v\|_{L^2} \leq \varsigma\tau\|v\|_{L^2} + \varsigma\tau e^{-\frac{\tau\varsigma^2\varepsilon}{2}}\|e^{\tau\Phi}u\|_{L^2}. \tag{3.22}$$

Combining all estimates so far, namely (3.18)-(3.19)-(3.20)-(3.21)-(3.22), we have proved that there are some constants $c>0$ (depending on ε) and $C>0$ so that for any $\varsigma>0$, we have for $\tau\geq\max(\tau_0,\frac{1}{\varsigma^2\varepsilon})$,

$$\tau\left\|v\right\|^2_{H^1_\tau} \leq C\left\|P_{\Phi,\varepsilon}v\right\|^2_{L^2} + C\varsigma^2\tau^3\|v\|^2_{L^2} + C\left(e^{-c\tau}+\varsigma^2\tau^3e^{-\tau\varsigma^2\varepsilon}\right)\left\|e^{\tau\Phi}u\right\|^2_{H^1_\tau}.$$

We now fix the constant ς small enough so that the term $C\varsigma^2\tau^3\|v\|^2_{L^2}\leq C\varsigma^2\tau\|v\|^2_{H^1_\tau}$ can be absorbed in the left handside of the estimate. This yields the sought estimate for $\tau\geq\max(\tau_0,\frac{1}{\varsigma^2\varepsilon})$, and concludes the proof of the theorem. ☐

Proof of Lemma 3.3.10. Using the Fourier transform of the Gaussian (classical computation), we have

$$(e^{-\frac{|D_t|^2}{\lambda}}f)(t) = \left(\frac{\lambda}{4\pi}\right)^{\frac12}\int_{\mathbb{R}_s} e^{-\frac{\lambda}{4}|s-t|^2}f(s)\,ds.$$

We have

$$\begin{aligned}\chi_1 e^{-\frac{|D_t|^2}{\lambda}}(\chi_2 u)(t,x) &= \left(\frac{\lambda}{4\pi}\right)^{\frac12}\int_{\mathbb{R}_s}\chi_1(t,x)e^{-\frac{\lambda}{4}|s-t|^2}(\chi_2 u)(s,x)\,ds\\ &= \left(\frac{\lambda}{4\pi}\right)^{\frac12}\chi_1(t,x)\int_{s,|t-s|\geq d}e^{-\frac{\lambda}{4}|s-t|^2}(\chi_2 u)(s,x)\,ds\end{aligned}$$

where we have used the properties of support for the second equality, so that

$$\begin{aligned}|\chi_1 e^{-\frac{|D_t|^2}{\lambda}}(\chi_2 u)|(t,x) &\leq \|\chi_1\|_{L^\infty}\left(\frac{\lambda}{4\pi}\right)^{\frac12}\int_{s,|t-s|\geq d}e^{-\frac{\lambda}{4}|s-t|^2}|\chi_2 u|(s,x)\,ds\\ &\leq \|\chi_1\|_{L^\infty}\left(\frac{\lambda}{4\pi}\right)^{\frac12}\left(\mathbb{1}_{|\cdot|\geq d}e^{-\frac{\lambda}{4}|\cdot|^2}*_{\mathbb{R}_s}|\chi_2 u|(\cdot,x)\right)(t).\end{aligned}$$

As a consequence, using the Young inequality, we have

$$\|\chi_1 e^{-\frac{|D_t|^2}{\lambda}}(\chi_2 u)\|_{L^2} \leq \|\chi_1\|_{L^\infty}\left(\frac{\lambda}{4\pi}\right)^{\frac12}\left\|\mathbb{1}_{|\cdot|\geq d}e^{-\frac{\lambda}{4}|\cdot|^2}\right\|_{L^1(\mathbb{R})}\|\chi_2 u\|_{L^2(\mathbb{R}^{n+1})}. \tag{3.23}$$

Next, using that

$$\frac{1}{2}\left\|\mathbb{1}_{|\cdot|\geq d}e^{-\frac{\lambda}{4}|\cdot|^2}\right\|_{L^1(\mathbb{R})} = \int_d^{+\infty} e^{-\frac{\lambda}{4}s^2}ds = \frac{2}{\sqrt{\lambda}}\int_{d\sqrt{\lambda}/2}^{+\infty} e^{-y^2}dy = \frac{2}{\sqrt{\lambda}}\int_{d\sqrt{\lambda}/2}^{+\infty} e^{-\frac{y^2}{2}}e^{-\frac{y^2}{2}}dy$$
$$\leq \frac{2}{\sqrt{\lambda}}e^{-\frac{d^2}{8}\lambda}\int_{d\sqrt{\lambda}/2}^{+\infty} e^{-\frac{y^2}{2}}dy \leq \frac{2}{\sqrt{\lambda}}e^{-\frac{d^2}{8}\lambda}\int_0^{+\infty} e^{-\frac{y^2}{2}}dy = \frac{2}{\sqrt{\lambda}}e^{-\frac{d^2}{8}\lambda}\sqrt{\frac{\pi}{2}}.$$

Coming back to (3.23), we have obtained the existence of a constant $C > 0$ such that for all $\lambda > 0$,

$$\|\chi_1 e^{-\frac{|D_t|^2}{\lambda}}(\chi_2 u)\|_{L^2} \leq C\,\|\chi_1\|_{L^\infty}\,\|\chi_2\|_{L^\infty}\, e^{-\frac{d^2}{8}\lambda}\|u\|_{L^2(\mathbb{R}^{n+1})},$$

which implies the result in L^2. The proof in H^1 or in H^1_τ follows the same. □

3.4 Lowering regularity

In Theorem 3.1.1, we have assumed the solution u to $Pu = 0$ to be C^∞ near (t_0, x_0). Looking carefully at the proof, one can notice that local H^1 regularity suffices to develop the arguments of the proof (up to using an appropriate generalization of the Paley-Wiener theorem ♣). For such H^1 solutions, we first notice that the required regularity of the coefficients of the principal part of $Q(x, D_x)$ in the x-variable is only C^1, as in the classical case, see **??**.♣ In case of

3.4.1 Unique continuation for C^1 coefficents

3.4.2 Unique continuation for distributional solutions

In the present section, we explain how, in case the coefficients $Q(x, D_x)$ are C^∞, the regularity of u solution to $Pu = 0$ is only $u \in \mathcal{D}'(\Omega)$. See [Hör97, Remark p205]

3.5 Global unique continuation and non characteristic hypersurfaces

3.5.1 Distance and metric

Let Ω_x, P be as in Theorem 3.1.1. Assume Ω_x connected. We are going to define the Riemannian distance related to the operator Q.

We can assume that $a^{ij}(x)$ is symmetric without changing the operator P. The ellipticity and positivity assumption shows that for any $x \in \Omega_x$, we can define the matrix $(g_{ij}) = (a^{ij})^{-1}$ which is still positive.

For any $x \in \Omega_x$ and $v \in \mathbb{R}^n$ ($\simeq T_x\mathbb{R}^n$), we define

$$|v|_{g(x)} := \sqrt{\sum_{i,j=1}^n g_{ij}(x)v_i v_j},$$

the Riemannian norm of the tangent vector v at the point x. Moreover, if $\gamma \in C^1([0,1];\Omega_x)$ (or even $\gamma \in W^{1,\infty}([0,1];\Omega_x)$ or $\gamma \in W^{1,1}([0,1];\Omega_x)$) is a smooth path, we define its length as

$$\text{length}(\gamma) = \int_0^1 |\dot{\gamma}(t)|_{g(\gamma(t))}dt.$$

This allows to define the Riemannian distance associated to g as

$$\text{dist}(x_1, x_2) = \inf\left\{\text{length}(\gamma), \gamma \in C^1([0,1];\Omega_x), \gamma(0) = x_1; \gamma(1) = x_2\right\}.$$

3.5.2 The global theorem

Theorem 3.5.1 (Semi-global unique continuation for waves)**.** *Let Ω_x, P be as in Theorem 3.1.1. Let x_0, $x_1 \in \Omega_x$. Let ω_0 be neighborhood of x_0 in Ω_x. Then, for any $T > \operatorname{dist}(x_0, x_1)$, there exist $\varepsilon > 0$ and V_{x_1} one neighborhood of x_1 so that for any $u \in C^\infty(\Omega)$,*

$$\begin{cases} Pu &= 0 \text{ in }]-T,T[\times\Omega_x, \\ u &= 0 \text{ in }]-T,T[\times\omega_0 \end{cases} \Longrightarrow u = 0 \text{ in }]-\varepsilon,\varepsilon[\times V_{x_1}. \tag{3.24}$$

As a corollary, we deduce the following result. Given an open set $\operatorname{Int}(\mathcal{M}) \subset \mathbb{R}^n$ endowed with a metric g and for $E \subset \mathcal{M}$, we introduce the largest distance of E to a point of $\mathcal{M}$:

$$\mathcal{L}(\mathcal{M}, E) := \sup_{x\in\mathcal{M}} \operatorname{dist}(x, E), \quad \operatorname{dist}(x, E) = \inf_{y\in E} \operatorname{dist}(x, y). \tag{3.25}$$

Corollary 3.5.2 (Global unique continuation for waves)**.** *Let $\operatorname{Int}(\mathcal{M}) \subset \mathbb{R}^n$ be a bounded domain and let Δ_g be an elliptic operator on $\mathcal{M}$. For any nonempty open subset ω of $\mathcal{M}$, if u satisfies*

$$u \in C^\infty((-T,T)\times \operatorname{Int}(\mathcal{M})), \quad \partial_t^2 u - \Delta_g u = 0 \text{ in } (-T,T)\times\operatorname{Int}(\mathcal{M}), \quad T > \mathcal{L}(\mathcal{M},\omega),$$

then u vanishes identically.

Proof of Theorem 3.5.1. According to Lemma 3.5.3 below, we can find local coordinates (w, x_n) near γ in which the path γ by $\gamma(s) = (0, s\ell_0)$ and the metric is given by the matrix $m(w, x_n) \in M_n(\mathbb{R})$ with

$$m(w, x_n) = \begin{pmatrix} m'(x_n) & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}_{M_n(\mathbb{R})}(|w|), \quad \text{for } w \in B_{\mathbb{R}^{n-1}}(0,\delta), \delta > 0, \tag{3.26}$$

with $m'(x_n) \in M_{n-1}(\mathbb{R})$ (uniformly) definite symmetric. With these coordinates in the space variable, and still using the straight time variable, the symbol of the wave operator is given by

$$p(t, w, x_n, \xi_t, \xi_w, \xi_n) = p(w, x_n, \xi_t, \xi_w, \xi_n) = -\xi_t^2 + \langle m(w, x_n)\xi, \xi\rangle, \quad \xi = (\xi_w, \xi_n), \tag{3.27}$$

where we have used ξ_t for the dual of the time variable and ξ_w, ξ_n for the dual to $w \in B_{\mathbb{R}^{n-1}}(0,\delta)$ and $x_n \in [0, \ell_0]$.

We now aim to apply Theorem 3.1.1 and we need to construct appropriate non characteristic hypersurfaces.

Pick again t_0 with $\ell_0 < t_0 < T$. For $b < \delta$ small, to be fixed later on, we define

$$x_n = l, \quad x' = (t, w), \quad D = \left\{(t,w) \,\middle|\, \left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2 \le 1\right\}$$

$$G(t, w, \varepsilon) = \varepsilon\ell_0\psi\left(\sqrt{\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2}\right), \quad \phi_\varepsilon(t, w, x_n) := G(t, w, \varepsilon) - x_n, \quad \varepsilon \in [0, 1]$$

where ψ is such that

$$\psi \text{ even}, \quad \psi(\pm 1) = 0, \quad \psi(0) = 1,$$
$$\psi(s) \ge 0, \quad |\psi'(s)| \le \alpha, \text{ for } s \in [-1, 1],$$

with $1 < \alpha < \frac{t_0}{\ell_0}$. This is possible since $\frac{t_0}{\ell_0} > 1$.

Note also that the fact that ψ is even gives that $G(t, w, \varepsilon)$ is actually smooth.

Note also that the point $(t = 0, w = 0, x_n = \ell_0)$ corresponding in the local coordinates to x^1 belongs to $\{\phi_1 = 0\}$. We have

$$d\phi_\varepsilon(t, w, x_n) = \varepsilon\ell_0\left(\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2\right)^{-1/2}\psi'\left(\sqrt{\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2}\right)\left(\frac{t\,dt}{t_0^2} + \frac{w\,dw}{b^2}\right) - dx_n.$$

Given the form of the principal symbol of the wave operator in these coordinates (see (3.26)-(3.27)), we obtain

$$\begin{aligned} p(w,x_n,d\phi_\varepsilon(t,w,x_n)) &= -\varepsilon^2\ell_0^2\frac{t^2}{t_0^4}\left(\left(\frac{w}{b}\right)^2+\left(\frac{t}{t_0}\right)^2\right)^{-1}|\psi'|^2 \\ &\quad +\ell_0^2\frac{\varepsilon^2}{b^4}\langle m'(x_n)w,w\rangle\left(\left(\frac{w}{b}\right)^2+\left(\frac{t}{t_0}\right)^2\right)^{-1}|\psi'|^2+1 \\ &\quad +O(|w|^2)\left(1+\frac{\varepsilon^2\ell_0^2}{b^4}|w|^2\left(\left(\frac{w}{b}\right)^2+\left(\frac{t}{t_0}\right)^2\right)^{-1}|\psi'|^2\right), \end{aligned}$$

where $|\psi'|^2$ is taken at the point $\left(\sqrt{\left(\frac{w}{b}\right)^2+\left(\frac{t}{t_0}\right)^2}\right)$. Now, since $\alpha<\frac{t_0}{\ell_0}$ and $m'(x_n)$ is uniformly (for $x_n\in[0,\ell_0]$) definite positive, there is $\eta>0$ so that for $|w|\leq b$ small enough, we have

$$\begin{aligned} 1+O(|w|^2) &\geq \alpha^2\frac{\ell_0^2}{t_0^2}+\eta \\ \langle m'(x_n)w,w\rangle+O(|w|^2)|w|^2 &\geq \frac{1}{2}\langle m'(x_n)w,w\rangle\geq 0. \end{aligned}$$

Hence, there is a sufficiently small neighborhood (taking again b small enough) of the path (i.e. of $w=0$), in which we have (for any $\varepsilon\in[0,1]$), and any $(t,w,x_n)\in\overline{D}\times[0,\ell_0]$

$$\begin{aligned} p(w,x_n,d\phi_\varepsilon(t,w,x_n)) &\geq -\frac{\varepsilon^2}{t_0^2}\ell_0^2\left(\frac{t}{t_0}\right)^2\left(\left(\frac{w}{b}\right)^2+\left(\frac{t}{t_0}\right)^2\right)^{-1}|\psi'|^2+\alpha^2\frac{\ell_0^2}{t_0^2}+\eta \\ &\geq -\frac{\ell_0^2}{t_0^2}|\psi'|^2+\alpha^2\frac{\ell_0^2}{t_0^2}+\eta\geq\eta. \end{aligned}$$

So, the surface $\{\phi_\varepsilon=0\}$ is noncharacteristic for any $\varepsilon\in[0,1]$ and, therefore, strictly pseudoconvex with respect to the wave operator.

Now, define $K_\varepsilon=\{x_n\leq G(t,w,\varepsilon)\}\cap\{x_n\geq 0\}$.

Consider $\varepsilon_0=sup\{\varepsilon;u=0\text{ on }K_\varepsilon\}$. A continuity argument yields that that $u=0$ on K_{ε_0}. A compactness argument on the compact set (taking into account the "corners") and the successive application of Theorem 3.1.1 gives the result. ♣ `un peu rapide...` □

Lemma 3.5.3. *Let $\gamma:[0,1]\to\Omega_x$ be a smooth path without self intersection (that is to say, γ is injective) of length ℓ_0 so that $\gamma(0)=x_0$ and $\gamma(1)=x_1$.*

Then, there are some coordinates $(w,l)\in B_{\mathbb{R}^{n-1}}(0,\varepsilon)\times[0,\ell_0]$ in an open neighborhood U near $\gamma([0,1])$ so that

- $\gamma([0,1])=\{w=0\}\times[0,\ell_0]$,
- *the metric g is of the form* $m(l,w)=\begin{pmatrix}1&0\\0&m'(l)\end{pmatrix}+O_{M_n(\mathbb{R})}(|w|)$,

Proof. The path γ is of length ℓ_0 so, we can reparametrize it by $\gamma:[0,\ell_0]\to\Omega_x$ such that γ is unitary (that is $|\dot\gamma(s)|_{\gamma(s)}=1$ for all $s\in(0,\ell_0)$). Moreover, since γ does not have self intersection, there exist U a neighborhood in Ω_x of γ and a diffeomorphism ψ such that

- $\psi(U)\subset\{(x,y)\in\mathbb{R}^n\,|x\in[-\varepsilon,\ell_0+\varepsilon],|y|\leq\varepsilon\}$,
- $\psi(\gamma(s))=(s,0)$,
- $\psi(U)=\{(x,y)\in\mathbb{R}^n,f_1(y)\leq x\leq f_2(y)\,|x\in[-\varepsilon,\ell_0+\varepsilon],|y|\leq\varepsilon\}$ for some smooth functions f_i locally defined

Then, we make some change of variable to diagonalize the metric on γ. By unitarity of the coordinates, the metric on γ has the form

$$m(x,0) = \begin{pmatrix} 1 & l(x) \\ {}^t l(x) & g(x) \end{pmatrix},$$

where l is a line vector and g is a positive definite matrix. We perform the change of variable $\Phi : (x,y) \mapsto (\widetilde{x},\widetilde{y}) = (x - a_x \cdot y, y)$. In $y = 0$, we have $D\Phi(x,0) = \begin{pmatrix} 1 & -a_x \\ 0 & Id \end{pmatrix}$ with ${}^t D\Phi(x,0) = \begin{pmatrix} 1 & 0 \\ -{}^t a_x & Id \end{pmatrix}$ (in particular, the change of variable is valid for small y) and $D\Phi(x,0)^{-1} = \begin{pmatrix} 1 & a_x \\ 0 & Id \end{pmatrix}$ with ${}^t D\Phi(x,0)^{-1} = \begin{pmatrix} 1 & 0 \\ {}^t a_x & Id \end{pmatrix}$. Moreover, in the new coordinates, the set in $\{\widetilde{y} = 0\}$ and the metric there is given by

$${}^t D\Phi(x,0)^{-1} m(x,0) D\Phi(x,0)^{-1} = \begin{pmatrix} 1 & l(x) + a(x) \\ {}^t l(x) +^t a(x) & * \end{pmatrix}$$

So, we choose $a(x) = -l(x)$ so that in this new coordinates $m(x,0)$ is of the form

$$m(x,0) = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}. \tag{3.28}$$

The expected property of m is then obtained by the mean value theorem using the diagonal form (3.28) on γ. □

3.6 Approximate controllability

3.7 Further remarks

3.7.1 The general theorem

of Tataru Robbiano-Zuily, Hörmander

3.7.2 Quantitative estimates

boundary Carleman estimates

3.8 Exercises on Chapter 3

♣ `Mieux preciser la regularite des solutions dans les 2 exos`

Exercise 16 (The Gaussian multiplier).

1. Compute the Fourier transform of $s \mapsto e^{-s^2}$ on $\mathbb{R}$. *Hint: differentiate the Fourier transform and solve a differential equation.*
2. Deduce the Fourier transform of the Gaussian $x \mapsto e^{-|x|^2}$ on $\mathbb{R}^n$.
3. Deduce the Fourier transform of $x \mapsto e^{-\frac{|x|^2}{\lambda}}$ for $\lambda > 0$ on $\mathbb{R}^n$. *Hint: use a scaling argument.*
4. For $\lambda > 0$, we define the Fourier multiplier $S_\lambda = e^{-\frac{|D_x|^2}{\lambda}}$ on $\mathcal{S}(\mathbb{R}^n)$ by $\mathcal{F}(S_\lambda u)(\xi) = e^{-\frac{|\xi|^2}{\lambda}}\mathcal{F}(u)(\xi)$. Prove that $S_\lambda u = f_\lambda * u$ with
$$f_\lambda = \left(\frac{\lambda}{4\pi}\right)^{n/2} e^{-\lambda\frac{|x|^2}{4}}.$$
5. Relate $S_\lambda u$ to the solution of the heat equation on $\mathbb{R}^n$ with initial datum $u \in \mathcal{S}(\mathbb{R}^n)$, namely
$$\begin{cases} \partial_t v - \Delta v &= 0 \\ v(0,x) &= u(x). \end{cases} \tag{3.29}$$
Deduce that the solution v of (3.29) writes $v(t,\cdot) = K_t * u$ with $K_t(x) = \left(\frac{1}{4\pi t}\right)^{n/2} e^{-\frac{|x|^2}{4t}}$.
6. Let $u \in \mathcal{S}(\mathbb{R}^n)$ be nonnegative and not identically vanishing. Prove that for any $\lambda > 0$, and $x \in \mathbb{R}^n$, $(S_\lambda u)(x) > 0$. Deduce that S_λ is not local: the property $\operatorname{supp}(S_\lambda u) \subset \operatorname{supp}(u)$ does not hold.
7. (a) Prove that for $z \geq 0$, we have
$$\int_z^{+\infty} e^{-s^2} ds = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{e^{-z^2(1+s^2)}}{1+s^2} ds \leq \frac{\sqrt{\pi}}{2} e^{-z^2}.$$
   (b) Prove that for all $m \in \mathbb{N}$, there is $C_m > 0$ such that for all $r \geq 0, t \in (0,4]$,
$$\int_r^{+\infty} s^m e^{-\frac{s^2}{t}} ds \leq C_m \langle r \rangle^{m-1} t e^{-\frac{r^2}{t}}$$
   (c) Deduce that there is $C_n > 0$ such that for all closed set $E \subset \mathbb{R}^n$ all $x \in \mathbb{R}^n$ and all $t \in (0,4]$, we have
$$\int_E e^{-\frac{1}{t}|x-y|^2}\, dy \leq C_n \langle \operatorname{dist}(x,E)\rangle^{n-2} \sqrt{t} e^{-\frac{\operatorname{dist}(x,E)^2}{t}}, \tag{3.30}$$
   where dist denotes the Euclidean distance in $\mathbb{R}^n$.

   (d) Conclude that there is $C > 0$ such that for all $u \in C_b^0(\mathbb{R}^n)$, all $x \in \mathbb{R}^n$ and all $\lambda \geq 1$, we have
$$|S_\lambda u(x)| \leq C \langle \operatorname{dist}(x, \operatorname{supp}(u))\rangle^{n-2} \lambda^{\frac{n-1}{2}} e^{-\frac{\lambda}{4}\operatorname{dist}(x,\operatorname{supp}(u))^2} \|u\|_{L^\infty}.$$
   (Note that this might be interpreted as a refinement of the result of Lemma 3.3.10: here, the coefficient in the exponential decay is made explicit – and is optimal – in terms of the Euclidean distance).

Correction 16. 1. We denote $f(s) = e^{-s^2}$ differentiate

$$\begin{aligned}\frac{d\widehat{f}(\xi)}{d\xi} &= \frac{d}{d\xi}\int_{\mathbb{R}^n} e^{-is\xi}e^{-s^2}ds = -i\int_{\mathbb{R}^n} se^{-is\xi}e^{-s^2}ds = \frac{i}{2}\int_{\mathbb{R}^n} e^{-is\xi}\partial_s(e^{-s^2})ds \\ &= -\frac{\xi}{2}\int_{\mathbb{R}^n} e^{-is\xi}e^{-s^2}ds = -\frac{\xi}{2}\widehat{f}(\xi).\end{aligned}$$

So, this equation can be explicitely solved by $\widehat{f}(\xi) = \widehat{f}(0)e^{-\frac{\xi^2}{4}}$. It is still a Gaussian, but we have to find the normalization constant $\widehat{f}(0)$. To compute $\widehat{f}(0) = \int_{\mathbb{R}} e^{-s^2}ds$, notice that $\left(\int_{\mathbb{R}} e^{-s^2}ds\right)^2 = \int_{\mathbb{R}^2} e^{-(x^2+y^2)}dxdy = \int_{r>0} e^{-r^2}(2\pi r)dr = -\pi\int_{r>0}\partial_r(e^{-r^2})dr = \pi$. We thus have $\int_{\mathbb{R}} e^{-s^2}ds = \sqrt{\pi}$ and $\widehat{f}(\xi) = \sqrt{\pi}e^{-\frac{\xi^2}{4}}$.

2. In higher dimension, we compute

$$\int_{\mathbb{R}^n} e^{-ix\cdot\xi}e^{-|x|^2} = \int_{\mathbb{R}_{x_1}}\cdots\int_{\mathbb{R}_{x_n}} e^{-ix_1\xi_1}\cdots e^{-ix_n\xi_n}e^{-x_1^2}\cdots e^{-x_n^2}dx_1\cdots dx_n$$
$$= \widehat{f}(\xi_1)\cdots\widehat{f}(\xi_n) = \pi^{n/2}e^{-\frac{|\xi|^2}{4}}.$$

3. Now, we want to compute $\widehat{g_\lambda}(\xi)$ where $g_\lambda = e^{-\frac{|x|^2}{\lambda}}$. By scaling, we have

$$\widehat{g_\lambda}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}e^{-\frac{|x|^2}{\lambda}}dx = \lambda^{n/2}\int_{\mathbb{R}^n} e^{-i\sqrt{\lambda}y\cdot\xi}e^{-|y|^2}dy = \lambda^{n/2}g_1(\sqrt{\lambda}\xi) = (\pi\lambda)^{n/2}e^{-\lambda\frac{|\xi|^2}{4}}.$$

That is $\widehat{e^{-\frac{|x|^2}{\lambda}}}(\xi) = (\pi\lambda)^{n/2}e^{-\lambda\frac{|\xi|^2}{4}}$.

4. Now, we want to give a convolution formulation for S_λ. We have for $u \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{S_\lambda u}(\xi) = e^{-\frac{|\xi|^2}{\lambda}}\widehat{u}(\xi)$. So, using (1.24), we have

$$S_\lambda u = \mathcal{F}^{-1}\left(e^{-\frac{|\xi|^2}{\lambda}}\widehat{u}\right) = \mathcal{F}^{-1}\left(e^{-\frac{|\xi|^2}{\lambda}}\right) * u = f_\lambda * u,$$

with

$$f_\lambda = \mathcal{F}^{-1}\left(e^{-\frac{|\xi|^2}{\lambda}}\right) = \frac{1}{(2\pi)^n}\widehat{e^{-\frac{|\cdot|^2}{\lambda}}} = \frac{(\pi\lambda)^{n/2}}{(2\pi)^n}e^{-\lambda\frac{|x|^2}{4}} = \left(\frac{\lambda}{4\pi}\right)^{n/2}e^{-\lambda\frac{|x|^2}{4}}. \tag{3.31}$$

5. Assume $u \in \mathcal{S}(\mathbb{R}^n)$ and v is a smooth solution of the heat equation (3.29) with $v(t,\cdot) \in \mathcal{S}(\mathbb{R}^n)$ for $t \geq 0$. Then, applying the Fourier transform in x yields

$$\begin{cases} \partial_t\widehat{v}(t,\xi) + |\xi|^2\widehat{v}(t,\xi) &= 0 \\ \widehat{v}(0,\xi) &= \widehat{u}(\xi). \end{cases}$$

This leads to $\widehat{v}(t,\xi) = e^{-t|\xi|^2}\widehat{u}(\xi)$, i.e. $v(t,\cdot) = e^{-t|D_x|^2}u = S_{1/t}u$. Together with the expression of $S_{1/t}$ as a convolution operator in the previous question, we deduce $v(t,\cdot) = e^{-t|D_x|^2}u = S_{1/t}u = \left(\frac{1}{4\pi t}\right)^{n/2}e^{-\frac{|x|^2}{4t}} * u$.

6. We deduce that

$$S_\lambda u(x) = f_\lambda * u(x) = \left(\frac{\lambda}{4\pi}\right)^{n/2}\int_{\mathbb{R}^n} e^{-\lambda\frac{|x-y|^2}{4}}u(y)dy, \quad x \in \mathbb{R}^n, \lambda > 0.$$

This is nonnegative for all $x \in \mathbb{R}^n$ since the integrand is nonnegative. Moreover, having $S_\lambda u(x) = 0$ would yield $e^{-\lambda\frac{|x-y|^2}{4}}u(y) = 0$ for all $y \in \mathbb{R}^n$, which is not the case except if u vanishes identically. In particular, we have fr $\lambda > 0$

$$\Big(u \in L^1(\mathbb{R}^n), u \geq 0, u \text{ not identically vanishing}\Big) \implies \text{supp}(S_\lambda u) = \mathbb{R}^n.$$

The conclusion holds even if $\text{supp}(u) = \overline{B}(0,1)$, and thus S_λ is not local.

7. (a) To prove the equality, differentiate both terms with respect to $z > 0$:

$$\frac{d}{dz}\left(\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\frac{e^{-z^2(1+s^2)}}{1+s^2}ds\right) = \frac{1}{\sqrt{\pi}}\int_0^{+\infty} -2ze^{-z^2(1+s^2)}ds = \frac{2e^{-z^2}}{\sqrt{\pi}}\int_0^{+\infty} -ze^{-z^2s^2}ds$$
$$= \frac{2e^{-z^2}}{\sqrt{\pi}}\int_0^{+\infty} -e^{-y^2}dy = -e^{-z^2} = \frac{d}{dz}\left(\int_z^{+\infty} e^{-s^2}ds\right).$$

Moreover, for $z = 0$ both integrals coincide, thus the formula for all $z \geq 0$. Then, the inequality follows from

$$\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\frac{e^{-z^2(1+s^2)}}{1+s^2}ds = \frac{e^{-z^2}}{\sqrt{\pi}}\int_0^{+\infty}\frac{e^{-z^2s^2}}{1+s^2}ds \leq \frac{e^{-z^2}}{\sqrt{\pi}}\int_0^{+\infty}\frac{1}{1+s^2}ds = \frac{\sqrt{\pi}}{2}e^{-z^2}$$

(b) As a consequence, we first have

$$\int_r^{+\infty} s^m e^{-\frac{s^2}{t}}ds = t^{\frac{m+1}{2}}\int_r^{+\infty}(s/\sqrt{t})^m e^{-\frac{s^2}{t}}ds/\sqrt{t} = t^{\frac{m+1}{2}}\int_{r/\sqrt{t}}^{\infty} y^m e^{-y^2}dy$$

Integrating by parts once, we obtain

$$\int_\alpha^\infty y^m e^{-y^2}dy = \frac{1}{2}\alpha^{m-1}e^{-\alpha^2} + \frac{m-1}{2}\int_\alpha^\infty y^{m-2}e^{-y^2}dy.$$

Iterating integration by parts, we see that for all $0 < t \leq 4$, $r \geq 0$, we have

$$t^{\frac{m+1}{2}}\int_{r/\sqrt{t}}^\infty y^m e^{-y^2}dy \leq C_m t^{\frac{m+1}{2}}\langle r/\sqrt{t}\rangle^{m-1}e^{-\frac{r^2}{t}} \leq C_m\langle r\rangle^{m-1}t^1 e^{-\frac{r^2}{t}},$$

which yields the sought result.

(c) As a consequence, there exists $C_n > 0$ such that for all $r \geq 0$, $t \in (0, 4]$,

$$\int_{|y|\geq r} e^{-\frac{|y|^2}{t}}dy = |\mathbb{S}^{n-1}|\int_{\rho\geq r} e^{-\frac{\rho^2}{t}}\rho^{n-1}d\rho \leq C_n\langle r\rangle^{n-2}te^{-\frac{r^2}{t}}.$$

Given a closed set $E \subset \mathbb{R}^n$, if the point x does not belong to E, we have $E \cap B(x, \operatorname{dist}(x, E)) = \emptyset$, and hence $E \subset B(x, \operatorname{dist}(x, E))^c$. As a consequence, if $x \notin E$, then for any $t \in (0, 4]$, we may apply the last inequality to deduce

$$\int_E e^{-\frac{1}{t}|x-y|^2}\,dy \leq \int_{B(x,\operatorname{dist}(x,E))^c} e^{-\frac{1}{t}|x-y|^2}\,dy \leq C_n\langle\operatorname{dist}(x, E)\rangle^{n-2}te^{-\frac{\operatorname{dist}(x,E)^2}{t}}.$$

If $x \in E$, we simply notice that for any $t > 0$

$$\int_E e^{-\frac{1}{t}|x-y|^2}\,dy \leq \int_{\mathbb{R}^n} e^{-\frac{1}{t}|x-y|^2}\,dy = (\pi t)^{\frac{n}{2}}.$$

(this inequality actually holds for all $x \in \mathbb{R}^n$). The last two inequalities imply the sought estimate valid for all $x \in \mathbb{R}^n$.

(d) We recall that $S_\lambda u = f_\lambda * u$ with $f_\lambda = \left(\frac{\lambda}{4\pi}\right)^{n/2}e^{-\lambda\frac{|x|^2}{4}}$. Taking $t = 4/\lambda \in (0, 4]$ in the above estimate and writing $|u| \leq \|u\|_{L^\infty}\mathbb{1}_{\operatorname{supp}(u)}$ yields, for $\lambda \geq 1$,

$$|S_\lambda u(x)| \leq \left(\frac{\lambda}{4\pi}\right)^{n/2}\int_{\mathbb{R}^n} e^{-\lambda\frac{|x-y|^2}{4}}|u(y)|dy \leq \left(\frac{\lambda}{4\pi}\right)^{n/2}\|u\|_{L^\infty}\int_{\operatorname{supp}(u)} e^{-\lambda\frac{|x-y|^2}{4}}dy$$
$$\leq C\lambda^{n/2}\|u\|_{L^\infty}\langle\operatorname{dist}(x, \operatorname{supp}(u))\rangle^{n-2}\lambda^{-1/2}e^{-\frac{\lambda}{4}\operatorname{dist}(x,\operatorname{supp}(u))^2},$$

whence the sought estimate.

Exercise 17 (Schrödinger equation)**.** In this exercise, we consider solutions to the Schrödinger equation in $\mathbb{R}^n$, namely

$$\left\{ \begin{array}{rcl} i\partial_t u - \Delta u & = & 0, \\ u(0,x) & = & u_0(x). \end{array} \right. \tag{3.32}$$

1. Compute the Fourier transform in the space variable of a solution to (3.32) in terms of $\hat{u}_0$. Prove that if u solves (3.32), then we have for all $s \in \mathbb{R}$ and all $t \in \mathbb{R}$, $\|u(t,\cdot)\|_{H^s(\mathbb{R}^n)} = \|u_0\|_{H^s(\mathbb{R}^n)}$.
2. Prove that
$$\mathcal{F}\left(\left(\frac{1}{-4\pi i t}\right)^{n/2} e^{\frac{|x|^2}{4it}}\right)(\xi) = e^{it|\xi|^2}, \quad \text{for all } t \neq 0$$
(the Fourier transform is understood as an element of $\mathcal{S}'(\mathbb{R}^n)$). *Hint: Use an analytic continuation argument together with Exercise 16.*
3. Prove that there is $C > 0$ such that
$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{C}{|t|^{n/2}} \|u_0\|_{L^1}, \quad \text{for all } t \neq 0,$$
for any smooth solution u with $u(t) \in \mathcal{S}(\mathbb{R}^n)$ of the Schrödinger equation 3.32.
4. Assume $n = 1$ for simplicity, and let $C, \gamma > 0$ be two parameters. Prove that if $u_0 \in L^1(\mathbb{R}^n)$ is such that
$$|u_0(x)| \leq Ce^{-\gamma|x|}, \quad \text{for all } x \in \mathbb{R}, \tag{3.33}$$
then for any $t \neq 0$ the solution $x \mapsto u(t,x)$ can be extended as a holomorphic function in a neighborhood $\Gamma_t \subset \mathbb{C}$ of the real axis, to be determined.
5. Still in the case $n = 1$, deduce that if $u_0 \in L^\infty_{\text{comp}}(\mathbb{R})$ does not vanish identically, then, for all $t \neq 0$, the associated solution $u(t,\cdot)$ to (3.32) cannot vanish on a nonempty open set $\omega \subset \mathbb{R}$.

   We now consider the Schrödinger equation (3.32) set on the torus $x \in \mathbb{T}^n := \mathbb{R}^n/(2\pi\mathbb{Z})^n$. We denote by $e^{-it\Delta}u_0$ the solution $u(t,\cdot)$ of (3.32).
6. Prove that there exists $T > 0$ such that $e^{-i(t+T)\Delta}u_0 = e^{-it\Delta}u_0$ for all $u_0 \in L^2(\mathbb{T}^n)$ and all $t \in \mathbb{R}$.
7. Deduce that the analogues of Questions 3 (decay of the L^∞ norm for data in L^1) and 5 (solutions arising from compactly supported data "fill the whole space" for all times $t \neq 0$) are false on $\mathbb{T}^n$.

Correction 17. 1. If u solves (3.32), then we have $\hat{u}(t,\xi) = e^{it|\xi|^2}\hat{u}_0(\xi)$ for all $t \in \mathbb{R}$. In particular,
$$\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(t,\xi)|^2 d\xi = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}_0(\xi)|^2 d\xi,$$
and hence $\|u(t,\cdot)\|_{H^s(\mathbb{R}^n)} = \|u_0\|_{H^s(\mathbb{R}^n)}$ for all $t \in \mathbb{R}$.

2. We have seen in Exercise 16 (see (3.31)) that
$$\mathcal{F}\left(\left(\frac{1}{4\pi t}\right)^{n/2} e^{-\frac{|x|^2}{4t}}\right)(\xi) = e^{-t|\xi|^2}, \quad \text{for all } t > 0.$$
We now wish to formally replace t by $-it$ in this formula using an analytic continuation argument. To this aim, take a text function $\varphi \in \mathcal{S}$. We write $\mathbb{C}^+ = \{z \in \mathbb{C}, \text{Re}(z) > 0\}$ and define the function
$$z \mapsto f_\varphi(z) = \left\langle \mathcal{F}\left(\left(\frac{1}{4\pi z}\right)^{n/2} e^{-\frac{|\cdot|^2}{4z}}\right), \varphi \right\rangle_{\mathcal{S}',\mathcal{S}} = \left(\frac{1}{4\pi z}\right)^{n/2} \left\langle e^{-\frac{|\cdot|^2}{4z}}, \hat{\varphi} \right\rangle_{\mathcal{S}',\mathcal{S}}, \quad z \in \mathbb{C}^+.$$

in which we choose the principal determination of the logarithm (that is, taking arguments in $(-\pi,\pi)$), which is holomorphic on $\mathbb{C}^+$. The function f_φ defines a holomorphic function on the half plane $\mathbb{C}^+ = \{z \in \mathbb{C}, \mathrm{Re}(z) > 0\}$ (use derivation under the integral). Moreover, according to Formula (3.31), it coincides with the function

$$g_\varphi(z) = \left\langle e^{-z|\cdot|^2}, \varphi \right\rangle_{\mathcal{S}',\mathcal{S}},$$

for $z = t \in \mathbb{R}_+^*$. So, since f_φ and g_φ are two holomorphic functions on $\mathbb{C}^+$ that are equal on $\mathbb{R}_+^*$, we have $f_\varphi = g_\varphi$ on $\mathbb{C}^+$ by analytic continuation.

Moreover, the formula defining f_φ extends by continuity (using a dominated convergence argument) to the set

$$\overline{\mathbb{C}^+} \setminus \{0\} = \{z \in \mathbb{C}, \mathrm{Re}(z) \geq 0, z \neq 0\}.$$

It is also clear that g_φ extends continuously to that set; hence both functions coincide on that set. Taking $z = -it$ for $t \neq 0$ finally yields the sought formula.

3. From the above two questions, we have obtained, say, for $u_0 \in \mathcal{S}(\mathbb{R}^n)$, $t \neq 0$,

$$\begin{aligned} u(t,x) &= \mathcal{F}^{-1}_{\xi\to x}\left(e^{it|\xi|^2}\hat{u}_0(\xi)\right) = \mathcal{F}^{-1}\left(e^{it|\cdot|^2}\right) * \mathcal{F}^{-1}(\hat{u}_0)(x) \\ &= \left(\frac{1}{(-4\pi it)^{n/2}} e^{\frac{|\cdot|^2}{4it}} * u_0\right)(x) = \frac{1}{(-4\pi it)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{|x-y|^2}{4it}} u_0(y)dy. \end{aligned}$$

As a consequence, we have for the following estimate for $t \neq 0$,

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{n/2}} \int_{\mathbb{R}^n} \left| e^{\frac{|x-y|^2}{4it}} \right| |u_0(y)|dy = \frac{1}{(4\pi|t|)^{n/2}} \|u_0\|_{L^1(\mathbb{R}^n)}.$$

This is a so-called *dispersive estimate*: although the L^2 norm of the solution is preserved along time, its maximal value decays.

4. We set

$$u(t,z) := \frac{1}{(-4\pi it)^{1/2}} \int_{\mathbb{R}} e^{\frac{(z-y)^2}{4it}} u_0(y)dy, \tag{3.34}$$

and want to understand for which $z \in \mathbb{C}$ the integral converges absolutely. Note that here, it is key that we write $(z-y)^2$ (usual square of a complex number) and not $|z-y|^2$ (modulus of a complex number) in order to obtain in the end a holomorphic function in the variable z. Writing $z = a + ib$ with $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$, we have

$$\left| e^{\frac{(z-y)^2}{4it}} u_0(y) \right| = e^{\mathrm{Re}\left(\frac{(a-y+ib)^2}{4it}\right)} |u_0(y)| = e^{\mathrm{Re}\left(\frac{(a-y)^2 - b^2 + 2ib(a-y)}{4it}\right)} |u_0(y)| = e^{\frac{b(a-y)}{2t}} |u_0(y)|.$$

So the above formula for $u(t,z)$ is well-defined whenever $\int_{\mathbb{R}} e^{-\frac{by}{2t}} |u_0(y)|dy < +\infty$. Since we assume $|u_0(y)| \leq Ce^{-\gamma|y|}$, it is enough that $\int_{\mathbb{R}} e^{-\frac{by}{2t}} e^{-\gamma|y|}dy < +\infty$, that is to say:

- if $t > 0, b > 0$, $\gamma > \frac{b}{2t}$, i.e. $b < 2\gamma t$ (integrability condition at $-\infty$);
- if $t < 0, b < 0$, $\gamma > \frac{b}{2t}$, i.e. $|b| = -b < 2\gamma(-t) = 2\gamma|t|$ (integrability condition at $-\infty$);
- if $t < 0, b > 0$, $\gamma > -\frac{b}{2t}$, i.e. $b < -2t\gamma = 2\gamma|t|$ (integrability condition at $+\infty$);
- if $t > 0, b < 0$, $\gamma > -\frac{b}{2t}$, i.e. $|b| = -b < 2\gamma t$ (integrability condition at $+\infty$).

That is to say, $|b| < 2\gamma|t|$. As a consequence, if we set

$$\Gamma_t := \{z \in \mathbb{C}, |\mathrm{Im}(z)| < 2\gamma|t|\}, \qquad \text{for } t \neq 0,$$

we may differenciate under the integral (3.34) in any compact set of Γ_t; applying $\overline{\partial}$ then yields $\overline{\partial}_z u(t,z) = 0$ for all $z \in \Gamma_t$ and $t \neq 0$. Hence, we have extended $u(t,\cdot)$ as a holomorphic function in Γ_t, which is an open strip containing the real line.

Notice also that, for $t > 0$, say, the decay of u_0 near $+\infty$ dictates the width of the strip *under* the real axis, whereas the decay of u_0 near $-\infty$ dictates the width of the strip *above* the real axis.

5. If $u_0 \in L^\infty_{\text{comp}}(\mathbb{R})$, then it satisfies (3.33) for all $\gamma > 0$; hence, for all $t \neq 0$, we have $\Gamma_t = \mathbb{C}$. As a consequence, for all $t \neq 0$, the function $u(t,\cdot)$ is an entire function: if it vanishes on an open set, it then has to vanish identically, that is $u(t,\cdot) = 0$ on $\mathbb{R}$. Since the Schrödinger equation (3.32) is well-posed backward, this then implies that u_0 vanishes identically on $\mathbb{R}$, which prove the statement.

6. We use Fourier series on the torus: the family $(e_k)_{k\in\mathbb{Z}^n}$ with $e_k(x) = (2\pi)^{n/2}e^{ik\cdot x}$ forms a Hilbert basis of $L^2(\mathbb{T}^n)$. Therefore, $v \in L^2(\mathbb{T}^n)$ writes:

$$v(x) = \sum_{k\in\mathbb{Z}^n} (v, e_k)_{L^2(\mathbb{T}^n)} e_k(x) = \sum_{k\in\mathbb{Z}^n} v_k e^{ik\cdot x}, \quad v_k = \frac{1}{(2\pi)^{n/2}}(v, e_k)_{L^2(\mathbb{T}^n)} = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} v(x)e^{-ik\cdot x}dx.$$

Moreover, We have, for $j \in \{1,\cdots,n\}$, $D_j e_k(x) = D_j e^{ik\cdot x} = k_i e^{ik\cdot x}$ so that

$$-\Delta e_k = \sum_{j=1}^n D_j^2 e_k = \Big(\sum_{j=1}^n k_j^2\Big)e_k = |k|^2 e_k.$$

Therefore, the solution to the Schrödinger equation (3.32) writes:

$$e^{-it\Delta}u_0 = \sum_{k\in\mathbb{Z}^n} e^{it|k|^2}(u_0, e_k)_{L^2(\mathbb{T}^n)} e_k.$$

We next remark that for any $k \in \mathbb{Z}^n$, we have $|k|^2 \in \mathbb{N}$ and thus $e^{i(t+2\pi)|k|^2} = e^{it|k|^2}$. That is to say $e^{-i(t+2\pi)\Delta}u_0 = e^{-it\Delta}u_0$ for all $u_0 \in L^2(\mathbb{T}^n)$: the Schrödinger flow is 2π-periodic on the torus.

7. From this periodicity property, we deduce in particular that for any $u_0 \in L^\infty(\mathbb{T}^n) \subset L^2(\mathbb{T}^n)$, we have

$$\|u(2\pi\ell,\cdot)\|_{L^\infty(\mathbb{T}^n)} = \|u_0\|_{L^\infty(\mathbb{T}^n)}, \qquad \text{for all } \ell \in \mathbb{Z},$$

preventing any decay to zero of the L^∞ norm. Note that we have conservation of the L^2 norm, so that

$$\|u_0\|_{L^2(\mathbb{T}^n)} = \|u(t,\cdot)\|_{L^2(\mathbb{T}^n)} \leq (2\pi)^{n/2}\|u(t,\cdot)\|_{L^\infty(\mathbb{T}^n)}$$

which already prevents decay of the L^∞-norm (or any L^p norm). Note that this remark holds in any domain with finite Lebesgue measure.

The property of Question 5 is slightly more subtle. Here, if $u_0 \in L^\infty(\mathbb{T}^n)$ vanishes in some open set $\omega \subset \mathbb{T}^n$, then we have

$$u(2\pi\ell,\cdot) = u_0 = 0 \text{ a.e. on } \omega, \qquad \text{for all } \ell \in \mathbb{Z}.$$

This disprove the property of Question 5 for the Schrödinger flow on the torus.

Exercise 18 (Carleman estimate, part of the exam of May, 2016)**.** ♣ `uniformiser notation avec le reste :` $\psi \to \Phi \ldots$ For $\varepsilon > 0$ $\tau > 0$, define the Fourier multiplier $e^{-\frac{\varepsilon|D|^2}{2\tau}}$ defined by

$$\Big(\widehat{e^{-\frac{\varepsilon|D|^2}{2\tau}}u}\Big)(\xi) = e^{-\frac{\varepsilon|\xi|^2}{2\tau}}\widehat{u}(\xi) \quad \forall \xi \in \mathbb{R}^n$$

for $u \in \mathcal{S}(\mathbb{R}^n)$ (notice the slight difference with the one used in the main part of Chapter 3). Using the Fourier transform of the Gaussian, we also have the formula

$$e^{-\frac{\varepsilon|D|^2}{2\tau}}u(x) = \Big(\frac{\tau}{2\pi\varepsilon}\Big)^{n/2}\int_{\mathbb{R}^n} e^{-\frac{\tau}{2\varepsilon}|x-y|^2}u(y)dy.$$

Here, we denote by $Q^{\psi}_{\varepsilon,\tau}$ the operator defined by $Q^{\psi}_{\varepsilon,\tau}u = e^{-\frac{\varepsilon|D|^2}{2\tau}}(e^{\tau\psi}u)$ for $u \in \mathcal{S}(\mathbb{R}^n)$ (as opposed to the main part of Chapter 3)).

We also let $P \in \mathrm{Diff}^m(\mathbb{R}^n)$ with principal symbol p and $x_0 \in \mathbb{R}^n$.

Let ψ be a real-valued quadratic function defined on $\mathbb{R}^n$. More precisely,

$$\psi(x) = \sum_{k,l=1}^{n} a_{k,l}x_k x_l + \sum_{k=1}^{n} b_k x_k + c$$

where $a_{k,l}$, b_k and $c \in \mathbb{R}$ are some constants.

Assume $p(x_0, \nabla\psi(x_0)) \neq 0$.

1. Denote $p_\psi(x,\xi,\tau) = p(x, \xi + i\tau\nabla\psi)$. Check that for $\xi = 0$, $p_\psi(x,\xi,\tau) = (i\tau)^m p(x,\nabla\psi)$.

   Prove that

   $$\xi = 0, \tau > 0 \Rightarrow |p_\psi(x_0,\xi,\tau)|^2 > 0.$$

2. Prove that there exist some constants $C_1 > 0$, $C_2 > 0$ so that

   $$|p_\psi(x_0,\xi,\tau)|^2 + C_1|\xi|^{2m} \geq C_2(\xi^2+\tau^2)^m. \tag{3.35}$$

   for all $\xi \in \mathbb{R}^n$, $\tau \geq 0$.

3. Let $l(x) = \sum_{j=1}^n l_j x_j$ a linear function with $l_j \in \mathbb{C}$ constants. Compute the operator $l_{\varepsilon,\tau}$ so that $e^{-\frac{\varepsilon|D|^2}{2\tau}}(lu) = l_{\varepsilon,\tau}e^{-\frac{\varepsilon|D|^2}{2\tau}}u$.

4. Compute the operator $D^{\psi}_{j,\varepsilon,\tau}$ so that $Q^{\psi}_{\varepsilon,\tau}(D_j u) = D^{\psi}_{j,\varepsilon,\tau}Q^{\psi}_{\varepsilon,\tau}u$. Prove that $D^{\psi}_{j,\varepsilon,\tau} = D_j + i\tau\nabla\psi - \varepsilon\sum_{k=1}^n(a_{j,k}+a_{k,j})D_k$.

   Denote A the constant matrix with coefficients $A_{k,j} = a_{j,k} + a_{k,j}$. Compute the principal symbol of $D^{\psi}_{j,\varepsilon,\tau} \in \mathrm{Diff}^1_\tau$.

5. From now on, assume that P can be written $P = \sum_{|\alpha|\leq m} p_\alpha D^\alpha \in \mathrm{Diff}^m$ with $p_\alpha \in \mathbb{C}$, that is an operator with **constant coefficients**.

   Denote $P_{\psi,\varepsilon}$ the operator so that $Q^{\psi}_{\varepsilon,\tau}(Pu) = P_{\psi,\varepsilon}Q^{\psi}_{\varepsilon,\tau}u$. Prove that $P_{\psi,\varepsilon} \in \mathrm{Diff}^m_\tau$ is a differential operator depending on τ of order m. Prove that its principal symbol denoted $p_{\psi,\varepsilon}$ is

   $$p_{\psi,\varepsilon}(x,\xi,\tau) = \sum_{|\alpha|=m} p_\alpha(\xi + i\tau\nabla\psi - \varepsilon A\xi)^\alpha.$$

6. Prove that there exists $\varepsilon > 0$, C_3, $C_4 > 0$ so that we have

   $$|p_{\psi,\varepsilon}(x_0,\xi,\tau)|^2 + C_3|\xi|^{2m} \geq C_4(\xi^2+\tau^2)^m. \tag{3.36}$$

   for all $\xi \in \mathbb{R}^n$, $\tau \geq 0$.

7. Now, we fix $\varepsilon > 0$ so that (3.36) holds.

   We admit the Gårding type Lemma 3.8.1, written below, for operators of order $2m$.

   Now, prove that there exists C_5, C_6 τ_0, r_0 so that

   $$\|P_{\psi,\varepsilon}f\|^2_{L^2} + C_5\,\||D|^m f\|^2_{L^2} \geq C_6\,\|f\|^2_{H^m_\tau} \tag{3.37}$$

   for all $\tau \geq \tau_0$, $f \in C^\infty_0(B(x_0,r_0))$.

   Here $|D|^m$ is the Fourier multiplier of symbol $|\xi|^m$.

8. Let $u \in C_0^\infty(B(x_0, r_0/4))$. Denote $v = Q^{\psi}_{\varepsilon,\tau} u$. Let $\chi \in C_0^\infty(B(x_0, r_0)$ so that $\chi = 1$ on $B(x_0, r_0/2)$. Denote $f = \chi v$.

   Prove that there exists $C > 0$, $c > 0$ so that

$$\begin{aligned} \|(1-\chi)v\|_{H^m_\tau} &\leq Ce^{-c\tau} \left\|e^{\tau\psi}u\right\|_{H^m_\tau} \\ \|[P_{\psi,\varepsilon}, \chi]v\|_{L^2} &\leq Ce^{-c\tau} \left\|e^{\tau\psi}u\right\|_{H^m_\tau} \\ \||D|^m f\|_{L^2} &\leq C \||D|^m v\|_{L^2} + Ce^{-c\tau} \left\|e^{\tau\psi}u\right\|_{H^m_\tau} \end{aligned}$$

   For the last one, we could consider (or not) to simplify that $m = 2k$ is even so that $|D|^m = (-\Delta)^k$.

9. Conclude using (3.37) that (for some different constants $C > 0$), uniformly for $\tau \geq \tau_0$

$$\|v\|^2_{H^m_\tau} \leq C \|P_{\psi,\varepsilon} v\|^2_{L^2} + C \||D|^m v\|^2_{L^2} + Ce^{-c\tau} \left\|e^{\tau\psi}u\right\|^2_{H^m_\tau} \tag{3.38}$$

10. Prove the estimate (for some different constants $C > 0$), uniformly for $\tau \geq \tau_0$

$$\left\|Q^{\psi}_{\varepsilon,\tau} u\right\|^2_{H^m_\tau} \leq C \left\|Q^{\psi}_{\varepsilon,\tau} Pu\right\|^2_{L^2} + Ce^{-c\tau} \left\|e^{\tau\psi}u\right\|^2_{H^m_\tau} \tag{3.39}$$

Lemma 3.8.1. *Let $N \in \mathrm{Diff}^{2m}_\tau$ with principal symbol $n(x, \xi, \tau)$ real-valued. Assume moreover that for $x_0 \in \mathbb{R}^n$, there is the inequality*

$$n(x_0, \xi, \tau) \geq C(\xi^2 + \tau^2)^m$$

for all $\xi \in \mathbb{R}^n$, $\tau \geq 0$. Then, there exists $r_0 > 0$, $\tau_0 \geq 0$ so that

$$\mathrm{Re}\,(Nf, f) \geq C \|f\|_{H^m_\tau}$$

for all $\tau \geq \tau_0$, $f \in C_0^\infty(B(x_0, r_0))$.

Exercise 19 (Unique continuation, part of the exam of May, 2016)**.** Let P be an operator as in the introduction of the previous exercice. We assume to have proved the following statement: If ψ is quadratic real and satisfies $p(x_0, \nabla\psi(x_0)) \neq 0$, then we have the estimates (3.39) for any $u \in C_0^\infty(B(x_0, r_0))$ and $\tau \geq \tau_0$.

Now, let $\varphi \in C^\infty(\mathbb{R}^n)$ real-valued so that $p(x_0, \nabla\varphi(x_0)) \neq 0$.
Let Ω be an open neighborhood of x_0. Let $u \in C^\infty(\Omega)$ so that $Pu = 0$ on Ω and $u = 0$ on $\Omega \cap \{\varphi(x) \geq 0\}$.

1. Define $\psi(x) = \nabla\varphi(x_0)\cdot(x - x_0) - \lambda|x - x_0|^2$ for some $\lambda > 0$ to be chosen later on. Compute $\nabla\psi(x_0)$.

2. Prove that there exists $\lambda > 0$ and $r_1 > 0$ so that

$$\varphi(x) \leq 0 \text{ and } |x - x_0| \leq r_1 \Longrightarrow \psi(x) \leq -|x - x_0|^2 \tag{3.40}$$

3. Now, λ and r_0, r_1 are fixed so that (3.41) and (3.39) are true. Prove that there exists $r_2 > 0$ so that

$$|x - x_0| \leq r_2 \Longrightarrow \psi(x) \leq c/4 \tag{3.41}$$

   where c is the constant in (3.39).

4. Let $r = \min(r_0, r_1, r_2)$. Let $\chi \in C_0^\infty(B(x_0, r)$ so that $\chi = 1$ on $B(x_0, r/2)$. Denote $w = \chi u$. Prove that there exist some $\eta > 0$ and some new constant $C > 0$ so that

$$\begin{gathered} \left\|Q^{\psi}_{\varepsilon,\tau} Pw\right\|_{L^2} \leq Ce^{-\tau\eta} \\ e^{-c\tau} \left\|e^{\tau\psi}w\right\|_{H^m_\tau} \leq Ce^{-\tau\eta} \end{gathered}$$

   for all $\tau \geq \tau_0$.

5. Conclude a unique continuation property and formulate the Theorem that we proved.

Appendix A

Appendix

A.1 The Dirichlet problem for some second order elliptic operators

♣`No taught in class. Moreover, faut changer le poids de carleman en` Φ `pour coherence des notations`

In this section, we shall consider a particular class of operators as described in Remark **??**, that is, with symbols the form $p_2(x,\xi) = Q_x(\xi)$ where Q_x is a smooth family of real quadratic forms. Assuming that the variables x_a are tangent to the boundary, and that the functions satisfy Dirichlet boundary conditions, we prove a counterpart of the local estimate of Theorem **??** for this boundary value problem. For this, the main goal to achieve is to prove a Carleman estimate adapted to this boundary value problem. All local, semiglobal and global results shall then follow.

This situation is of particular interest for the wave equation for which x_a is the time variable, which is always tangent to the boundary of cylindrical domains.

For the sake of simplicity, we shall further assume that the operator principal symbol of P is independent of the x_a variable (we would otherwise need to assume the coefficients of P to be analytic with respect to x_a). This allows to avoid some additional technicalities in the (already rather technical) proofs.

A.1.1 Some notation

Here, we shall always assume that the analytic variables are tangential to the boundary, that is

$$x = (x_a, x_b) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}_+, \quad \text{with } \mathbb{R}^{n_b}_+ = \mathbb{R}^{n_b - 1} \times \mathbb{R}_+, \quad \text{and } x_b = (x_b', x_b^n).$$

When the distinction between analytic and non-analytic variables is not essential, we shall split the variables according to

$$x = (x', x_n) \in \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+, \quad \text{with } x' = (x_a, x_b') \in \mathbb{R}^{n_a + n_b - 1}, \quad \text{and } x_n = x_b^n \in \mathbb{R}^+.$$

We also denote by $\xi' \in \mathbb{R}^{n-1}$ the cotangential variables and ξ_n the conormal variable, by $D' = \frac{1}{i}(\partial_{x'})$ the associated tangential derivations and $D_n = \frac{1}{i}\partial_{x_n}$ the normal derivation.

For any $r_0 > 0$, we define

$$K_{r_0} = \left\{x \in \mathbb{R}^n_+ ; |x| \leq r_0\right\} = \overline{B}_{\mathbb{R}^n}(0, r_0) \cap \{x_n \geq 0\}. \tag{A.1}$$

We denote by $C^\infty_c(\mathbb{R}^n_+)$ the space of restrictions to $\mathbb{R}^n_+$ of functions in $C^\infty_c(\mathbb{R}^n)$, and by $C^\infty_c(K_{r_0})$ the space of functions $C^\infty_c(\mathbb{R}^n_+)$ supported in K_{r_0}. the trace of a function $f \in C^\infty_c(\mathbb{R}^n_+)$ at $x_n = 0$ is denoted by $f_{|x_n=0}$.

We denote by $(f,g) = \int_{\mathbb{R}^n_+} f\overline{g}$, $\|f\|^2_{0,+} = (f,f)$ the $L^2(\mathbb{R}^n_+)$ inner product and norm. For $k \in \mathbb{N}$, the norm $\|\cdot\|_{k,+}$ will denote the classical Sobolev norm on $\mathbb{R}^n_+$ and $\|\cdot\|_{k,+,\tau}$ the associated weighted norms, that is,

$$\|f\|^2_{k,+,\tau} = \sum_{j+|\alpha|\leq k} \tau^{2j} \|\partial^\alpha f\|^2_{0,+}, \quad \tau \geq 1.$$

We also define the tangential Sobolev norms, given by

$$|f|^2_{k,\tau} = \left\| (|D'| + \tau)^k f \right\|^2_{0,+} \sim \sum_{j+|\alpha|\le k} \tau^{2j} \left\| \partial^\alpha_{x'} f \right\|^2_{0,+}, \quad \tau \ge 1.$$

We shall also use, for $f, g \in C^\infty_c(\mathbb{R}^n_+)$, the notation $(f,g)_0 = \int_{\mathbb{R}^{n-1}} f_{|x_n=0}(x') g_{|x_n=0}(x') dx'$.

Finally, for $j \in \mathbb{N}$, we denote by $\mathcal{D}^k_\tau$, the space of *tangential* differential operators, i.e. operators of the form

$$P(x, D', \tau) = \sum_{j+|\alpha|\le k} a_{j,\alpha}(x) \tau^j D'^\alpha,$$

and by

$$\sigma(P) = p(x, \xi', \tau) = \sum_{j+|\alpha|=k} a_{j,\alpha}(x) \tau^j \xi'^\alpha$$

their principal symbol.

A.1.2 The Carleman estimate

In this section, we state and prove the Carleman estimate of Theorem (2.5.2) asociated to the Dirichlet problem . Note that it applies also to elliptic operator, but also to wave type operators.

To prove Theorem 2.5.2, we define the conjugated operator $P_\psi = e^{\tau\psi} P e^{-\tau\psi} = P(x, D + i\tau\psi')$.

When proving the theorem, we shall drop the index + in the norms to lighten the notation; of course, all inner norms and integrals are meant on $\mathbb{R}^n_+$. We first need the following proposition.

Theorem A.1.1. *Under the assumptions of Theorem 2.5.2, there exist $C > 0$, $\tau_0 > 0$ such that for any $\tau > \tau_0$ and $f \in C^\infty_c(K_{r_0})$, we have*

$$\tau \|f\|^2_{1,\tau} \le C \left\| P_\psi f \right\|^2_0 + \tau^3 |f_{|x_n=0}|^2_0 + \tau |Df_{|x_n=0}|^2_0. \tag{A.2}$$

If moreover $\partial_{x_n}\psi > 0$ for $(x', x_n = 0) \in K_{r_0}$, then

$$\tau \|f\|^2_{1,\tau} \le C \left\| P_\psi f \right\|^2_0, \quad \text{for all } f \in C^\infty_c(K_{r_0}) \text{ such that } f_{|x_n=0} = 0. \tag{A.3}$$

Proof. Defining $\tilde{Q}_2 = \frac{1}{2}(P_\psi + P^*_\psi)$ and $\tilde{Q}_1 = \frac{1}{2i\tau}(P_\psi - P^*_\psi)$, we have

$$P_\psi = \tilde{Q}_2 + i\tau \tilde{Q}_1,$$

and denote by $\tilde{q}_j$ the principal symbol of $\tilde{Q}_j$, $j = 1, 2$. We have

$$\left\{ \begin{array}{lcl} \tilde{Q}_2 & = & D^2_n + Q_2 \\ \tilde{Q}_1 & = & D_n \psi'_{x_n} + \psi'_{x_n} D_n + 2Q_1, \end{array} \right. \tag{A.4}$$

where $Q_2 \in \mathcal{D}^2_\tau$ and $Q_1 \in \mathcal{D}^1_\tau$ with principal symbols

$$\begin{array}{lcl} q_2 & = & -\tau^2 (\psi'_{x_n})^2 + r(x, \xi') - \tau^2 r(x, \psi'_{x'}) \\ q_1 & = & \tilde{r}(x_b, \xi', \psi'_{x'}), \end{array}$$

where $\tilde{r}$ is the bilinear form associated with the quadratic form r. Note that, even if it does not appear in the notation, all these operators depend upon the parameter τ.

With this notation, we hence have $p_\psi = \tilde{q}^0_2 + i\tau\tilde{q}^0_1$, so that $\frac{1}{i\tau}\{\overline{p}_\psi, p_\psi\} = 2\{\tilde{q}^0_2, \tilde{q}^0_1\}$. Assumptions (**??**) and (**??**) then translate respectively into

$$\{\tilde{q}^0_2, \tilde{q}^0_1\}(x, \xi) > 0, \quad \text{if } p(x, \xi) = 0, \quad x \in K_{r_0}, \tau = 0; \tag{A.5}$$

$$\{\tilde{q}^0_2, \tilde{q}^0_1\}(x, \xi) > 0, \quad \text{if } p_\psi(x, \xi) = 0, \quad x \in K_{r_0}, \tau > 0, \tag{A.6}$$

where the second assertion is a direct consequence of (**??**), and the first one follows from (**??**) together with the fact that, using that p is real, we have

$$\lim_{\tau\to 0^+} \frac{1}{i\tau}\{\overline{p}_\psi, p_\psi\} = \left. \frac{\partial}{\partial\tau} \frac{1}{i}\{\overline{p}_\psi, p_\psi\} \right|_{\tau=0} = 2\{p, \{p, \psi\}\}.$$

Next, we have the integration by parts formulæ:

$$\begin{cases} (g, \tilde{Q}_2 f) & = & (\tilde{Q}_2 g, f) - i\left[(g, D_n f)_0 + (D_n g, f)_0)_0\right], \\ (g, \tilde{Q}_1 f) & = & (\tilde{Q}_1 g, f) - 2i\left(\psi'_{x_n} g, f\right)_0. \end{cases} \tag{A.7}$$

So, we have for $f \in C_c^\infty(K_{r_0})$

$$\|P_\psi f\|_0 = \left\|\tilde{Q}_2 f\right\|_0^2 + \tau^2\left\|\tilde{Q}_1 f\right\|_0^2 + i\tau\left[\left(\tilde{Q}_1 f, \tilde{Q}_2 f\right) - \left(\tilde{Q}_2 f, \tilde{Q}_1 f\right)\right]. \tag{A.8}$$

So, we get, using the integration by parts formulæ (A.7)

$$\|P_\psi f\|_0 = \left\|\tilde{Q}_2 f\right\|_0^2 + \tau^2\left\|\tilde{Q}_1 f\right\|_0^2 + i\tau\left([\tilde{Q}_2, \tilde{Q}_1] f, f\right) + \tau\mathcal{B}(f), \tag{A.9}$$

with the boundary term

$$\begin{aligned} \mathcal{B}(f) &= \left[(\tilde{Q}_1 f, D_n f)_0 + (D_n \tilde{Q}_1 f, f)_0\right] - 2\left(\psi'_{x_n}\tilde{Q}_2 f, f\right)_0 \\ &= 2(\psi'_{x_n} D_n f, D_n f)_0 + (M_1 f, D_n f)_0 + (M_1' D_n f, f)_0 + (M_2 f, f)_0, \end{aligned} \tag{A.10}$$

for some tangential operator M_1 of order 1 (in ξ', τ) (note that terms of order two in D_n cancel).

Now that we have made the exact computations, we will make some estimates on the symbols of the interior part of the commutator. The idea is to tranfer the positivity assumption of the full symbol to some positivity of a tangential symbol, which will then allow to apply the tangential Gårding.

The first step is to perform a factorisation of $[\tilde{Q}_2, \tilde{Q}_1]$ with respect to $\tilde{Q}_1$ and $\tilde{Q}_2$ to have a tangential reminder. Since $[\tilde{Q}_2, \tilde{Q}_1]$ is of order 2, it can be written $i[\tilde{Q}_2, \tilde{Q}_1] = C_2 + C_1 D_n + C_0 D_n^2$ where $C_i \in \mathcal{D}_\tau^i$. But using (A.4), and $\psi'_{x_n} \neq 0$ on K_{r_0}, we can replace $D_n = \frac{1}{2\psi'_{x_n}}\tilde{Q}_1 + \mathcal{D}_\tau^1$ and $D_n^2 = \tilde{Q}_2 - Q_2$. So, in particular, we can write

$$i[\tilde{Q}_2, \tilde{Q}_1] = B_0\tilde{Q}_2 + B_1\tilde{Q}_1 + B_2. \tag{A.11}$$

where $B_i \in \mathcal{D}_\tau^i$ with real symbol b_i. Now, we need to

- use the assumption to get some positivity of the symbol $\{\overline{p}_\psi, p_\psi\}$, this is Lemma A.1.2;
- transfer this information to a tangential information on the symbol, this is Lemma A.1.3.

Lemma A.1.2. *There exist $C_1, C_2 > 0$ such that for all $(x, \xi) \in K_{r_0} \times \mathbb{R}^n$ and $\tau > 0$, we have*

$$(|\xi|^2 + \tau^2) \leq C_1\{\tilde{q}_2^0, \tilde{q}_1^0\}(x, \xi) + C_2\left[\frac{|p_\psi(x,\xi)|^2}{|\xi|^2 + \tau^2}\right].$$

Proof. All the terms are homogeneous of degree 2 in (ξ, τ) and continuous on the compact $(x, \xi, \tau) \in K_{r_0} \times \{(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}^+, |\xi|^2 + \tau^2 = 1\}$. Thus, on this set, the result is a consequence of (A.5), (A.6) and Lemma 2.1.8 applied to $f = \frac{|p_\psi(x,\xi)|^2}{|\xi|^2+\tau^2} \geq 0$, $g = \{\tilde{q}_2^0, \tilde{q}_1^0\}$ and $h = 0$. The result on the whole $K_{r_0} \times \mathbb{R}^n \times \mathbb{R}^+$ follows by homogeneity. □

Now, we set

$$\mu(x, \xi') = (q_1)^2 + (\psi'_{x_n})^2 q_2.$$

The symbol $\mu(x, \xi')$ satisfies the property that $\mu(x, \xi') = 0$ if and only if there exists ξ_n real such that $p_\psi(x, \xi', \xi_n) = 0$. This is easily seen by noticing that the zero of q_1 can only be with $\xi_n = -\frac{q_1}{\psi'_{x_n}}$.

Notice also that $\mu(x, \xi')$ is a tangential symbol of order 2.

Lemma A.1.3. *There exist $C_1, C_2 > 0$ such that for all $(x, \xi') \in K_{r_0} \times \mathbb{R}^{n-1}$ and $\tau > 0$, we have*

$$(|\xi'|^2 + \tau^2) \leq C_1 b_2 + C_2\left[\frac{[\mu(x, \xi')]^2}{|\xi'|^2 + \tau^2}\right]. \tag{A.12}$$

Proof. Note first that for any (x, ξ', ξ_n) with $\xi_n = -\frac{q_1(x,\xi')}{\psi'_{x_n}}$, we have $\tilde{q}_1(x, \xi', \xi_n) = 0$ and

$$p_\psi(x, \xi', \xi_n) = \tilde{q}_2(x, \xi', \xi_n) = (\psi'_{x_n})^{-2}\mu(x, \xi').$$

Now, assume $\mu(x, \xi') = 0$ and $\xi_a = 0$. Setting $\xi_n = -\frac{q_1(x,\xi')}{\psi'_{x_n}}$, we have $p_\psi(x, \xi', \xi_n) = 0$. Using Lemma **??**, we have $\{\tilde{q}_2, \tilde{q}_1\}(x, \xi', \xi_n) > 0$. According to the definition of B_2 in (A.11), we have $b_2(x, \xi', \xi_n) > 0$. As a consequence, we have

$$\mu(x, \xi') = 0 \Longrightarrow b_2(x, \xi', \xi_n) > 0.$$

Moreover, all terms in (A.12) are homogeneous of degree 2 in the variables (ξ', τ) and continuous on $(\xi', \tau) \neq (0, 0)$. Hence, applying Lemma 2.1.8 below on the compact set $K_{r_0} \times \{(\xi', \tau) \in \mathbb{R}^{n-1} \times \mathbb{R}^+, |\xi'|^2 + \tau^2 = 1\}$ yields (A.12) on that set. The conclusion follows by homogeneity. □

Taking the real part of (A.9) and using (A.11), we obtain

$$\|P_\psi f\|_0 - \tau \operatorname{Re}(\mathcal{B}(f)) = \left\|\tilde{Q}_2 f\right\|_0^2 + \tau^2 \left\|\tilde{Q}_1 f\right\|_0^2 + \tau \operatorname{Re}(B_2 f, f) + \tau \operatorname{Re}\left((B_0\tilde{Q}_2 + B_1\tilde{Q}_1)f, f\right). \quad \text{(A.13)}$$

Concerning the remainder term, we have

$$\begin{aligned} \tau\left|\operatorname{Re}\left((B_0\tilde{Q}_2 + B_1\tilde{Q}_1)f, f\right)\right| &\leq \tau\|f\|_0\|\tilde{Q}_2 f\|_0 + \tau|f|_1\|\tilde{Q}_1 f\|_0 \\ &\leq \tau^{-1/2}\left(\tau|f|_{1,\tau}^2 + \|\tilde{Q}_2 f\|_0^2 + \tau^2\|\tilde{Q}_1 f\|_0^2\right). \quad \text{(A.14)} \end{aligned}$$

Defining now

$$\Sigma = (Q_1)^2 + (\psi'_{x_n})^2 Q_2,$$

with principal symbol μ, and for an operator G with principal symbol $\frac{\mu(x,\xi')}{|\xi'|^2+\tau^2}$, the tangential Gårding inequality (that means with some derivatives only in the variable x'), in which symbols are allowed to depend smoothly upon the variable x_n yields, for τ sufficiently large,

$$|f|_{1,\tau}^2 \leq C \operatorname{Re}(B_2 f, f) + \operatorname{Re}(\Sigma f, Gf). \quad \text{(A.15)}$$

Writing $\psi'_{x_n} D_n = \frac{1}{2}(\tilde{Q}_1 - [D_n, \psi'_{x_n}]) - Q_1$ (where ψ'_{x_n} does not vanish), this allows to estimate the full norm $\|f\|_{1,\tau}$ according to

$$\|f\|_{1,\tau} \leq C(\|\tilde{Q}_1 f\|_0 + |f|_{1,\tau}). \quad \text{(A.16)}$$

Recalling the definitions of $\tilde{Q}_i$ in (A.4), we also have

$$\begin{aligned} \Sigma &= \left(\frac{1}{2}(\tilde{Q}_1 - [D_n, \psi'_{x_n}]) - \psi'_{x_n} D_n\right)^2 \\ &\quad +(\psi'_{x_n})^2\left(\tilde{Q}_2 - D_n^2\right) \\ &= \left(\frac{1}{2}(\tilde{Q}_1 - [D_n, \psi'_{x_n}]) - \psi'_{x_n} D_n\right)\frac{1}{2}(\tilde{Q}_1 - [D_n, \psi'_{x_n}]) \\ &\quad +(\psi'_{x_n})^2\left(\tilde{Q}_2\right), \quad \text{(A.17)} \end{aligned}$$

and hence

$$\Sigma \in (\psi'_{x_n})^2\tilde{Q}_2 - \frac{1}{2}\psi'_{x_n} D_n \tilde{Q}_1 + \mathcal{D}_\tau^1 \tilde{Q}_1 + \mathcal{D}_\tau^1 + \mathcal{D}_\tau^0 D_n.$$

We now want to estimate the term $\operatorname{Re}(\Sigma f, Gf)$ in (A.15). For this, integrating by parts in the tangential direction x_a, we have

$$\left|\left(\psi''_{x_n,x_a}\left((\psi'_{x_n})^2 D_n + Q_1\psi'_{x_n}; D_a\right) f, Gf\right)\right| \leq C\|\langle D_a\rangle f\|\|f\|_{1,\tau}.$$

This yields

$$
\begin{aligned}
|(\Sigma f, Gf)| &\leq C\|\tilde{Q}_2 f\|_0 \|f\|_0 + \left| \left(\frac{1}{2i} \psi'_{x_n} \tilde{Q}_1 f, Gf \right)_0 \right| \\
&\quad + \|\tilde{Q}_1 f\|_0 \|f\|_{1,\tau} + \|f\|_0 \|f\|_{1,\tau} + C\| \langle D_a \rangle f\| \|f\|_{1,\tau} \\
&\leq \left| \left(\frac{1}{2i} \psi'_{x_n} \tilde{Q}_1 f, Gf \right)_0 \right| + C\|f\|_{1,\tau} \left(\tau^{-1} \|\tilde{Q}_2 f\|_0 + + \|\tilde{Q}_1 f\|_0 + \tau^{-1} \|f\|_{1,\tau} + \|D_a f\|_0 \right) \quad \text{(A.18)}
\end{aligned}
$$

According to (A.15) and (A.16) and (A.18), this now implies

$$
\|f\|_{1,\tau}^2 \lesssim \operatorname{Re}(B_2 f, f) + \|\tilde{Q}_1 f\|_0^2 + \left| \left(\frac{1}{2i} \psi'_{x_n} \tilde{Q}_1 f, Gf \right)_0 \right| + \tau^{-2} \|\tilde{Q}_2 f\|_0^2 + \|D_a f\|_0^2.
$$

Coming back to (A.13), we obtain, for τ large enough,

$$
\begin{aligned}
\tau \|f\|_{1,\tau}^2 &\lesssim \|P_\psi f\|_0^2 - \tau \operatorname{Re}(\mathcal{B}(f)) - \left\| \tilde{Q}_2 f \right\|_0^2 - \tau^2 \left\| \tilde{Q}_1 f \right\|_0^2 + \tau \left| \left(\frac{1}{2i} \psi'_{x_n} \tilde{Q}_1 f, Gf \right)_0 \right| \\
&\lesssim \|P_\psi f\|_0^2 - \tau \operatorname{Re}(\mathcal{B}(f)) + \tau \left| \left(\frac{1}{2i} \psi'_{x_n} \tilde{Q}_1 f, Gf \right)_0 \right|.
\end{aligned}
$$

Recalling te definition of $\tilde{Q}_1$, we have $\psi'_{x_n} \tilde{Q}_1 = D_n + G_1$, where $G_1 \in \mathcal{D}_\tau^1$ is a differential operator of order 1 (in (τ, D')), we finally have

$$
\tau \|f\|_{1,\tau}^2 \lesssim \|P_\psi f\|_0^2 - \tau \operatorname{Re}(\mathcal{B}(f)) + \tau \left| (D_n f + G_1 f, Gf)_0 \right|, \tag{A.19}
$$

where G a tangential pseudodifferential operator of order zero, Recalling the form of $\mathcal{B}(f)$ in (A.10) gives the bound $|\mathcal{B}(f)| \leq \tau^2 |f_{|x_n=0}|_0^2 + |Df_{|x_n=0}|_0^2$, which concludes the proof of (A.2).

Now if $f_{|x_n=0} = 0$, all tangential derivatives vanish. With (A.19) and the form of $\mathcal{B}(f)$ in (A.10), this yields

$$
\tau \|f\|_{1,\tau}^2 \lesssim \|P_\psi f\|_0^2 - 2\tau (\psi'_{x_n} D_n f, D_n f)_0,
$$

which proves (A.3) since $\psi'_{x_n} > 0$ for $(x', x_n = 0) \in K$. This concludes the proof of Proposition A.1.1. □

Appendix B

Correction of (some) exercises

B.1 Exercises on Chapter 1

B.1.1 Exercice 1

Let $A = a_{\alpha,\beta}(x)D^\alpha\tau^\beta$. $B = b_{\alpha',\beta'}(x)D^{\alpha'}\tau^{\beta'}$ of respective order m_1 and m_2 and full symbol a and b.

$$A \circ Bu = a_{\alpha,\beta}(x)D^\alpha\tau^\beta\left[b_{\alpha',\beta'}(x)D^{\alpha'}\tau^{\beta'}u\right] = a_{\alpha,\beta}(x)\tau^{\beta+\beta'}D^\alpha\left[b_{\alpha',\beta'}(x)D^{\alpha'}u\right]$$

Using Leibniz formula

$$\partial^\alpha(fg) = \sum_{\gamma+\delta=\alpha}\binom{\alpha}{\gamma}(\partial^\gamma f)(\partial^\delta g),$$

we get

$$D^\alpha\left[b_{\alpha',\beta'}(x)D^{\alpha'}u\right] = \frac{1}{i^{|\alpha|}}\sum_{\gamma+\delta=\alpha}\binom{\alpha}{\gamma}(\partial^\gamma b_{\alpha',\beta'})(\partial^\delta D^{\alpha'}u)$$

So, we get

$$A \circ Bu = \frac{1}{i^{|\alpha|}}\sum_{\gamma+\delta=\alpha}a_{\alpha,\beta}(x)\tau^{\beta+\beta'}\binom{\alpha}{\gamma}(\partial^\gamma b_{\alpha',\beta'})(\partial^\delta D^{\alpha'}u)$$

Each term in the sum is a differential operator of order $\beta+\beta'+|\delta|+|\alpha'| \le \beta+\beta'+|\alpha|+|\alpha'| = m_1+m_2$.

This maximum is reached only for the term $\delta=\alpha$, $\gamma=0$, $\binom{\alpha}{\gamma}=1$ where we have the term

$$\frac{1}{i^{|\alpha|}}a_{\alpha,\beta}(x)\tau^{\beta+\beta'}b_{\alpha',\beta'}(x)(\partial^\alpha D^{\alpha'}u) = a_{\alpha,\beta}(x)\tau^{\beta+\beta'}b_{\alpha',\beta'}(x)(D^\alpha D^{\alpha'}u) = (ab)(x,D,\tau).$$

Let us now see the terms of order m_1+m_2-1. They are so that $\beta+\beta'+|\delta|+|\alpha'| = m_1+m_2-1$, that is $|\delta| = m_1-1$ and $|\gamma|=1$. Moreover, $\gamma=(1,0,0\cdots,0)$ or $\gamma=(0,1,0\cdots,0)$, etc... We denote these vectors e_j. The sum is amongst terms so that $\alpha_j\ge 1$. In each of these cases, $\binom{\alpha}{e_j} = \binom{\alpha_1}{0}\cdots\binom{\alpha_j}{1}\cdots\binom{\alpha_n}{0} = \alpha_j$.

$$\begin{aligned}&\frac{1}{i^{|\alpha|}}\sum_{j=1,\alpha_j\ge1}^{n}a_{\alpha,\beta}(x)\tau^{\beta+\beta'}\alpha_j(\partial^j b_{\alpha',\beta'})(\partial^{\alpha-e_j}D^{\alpha'}u)\\ =\ &\frac{1}{i}\sum_{j=1,\alpha_j\ge1}^{n}\alpha_j a_{\alpha,\beta}(x)\tau^{\beta+\beta'}(\partial_j b_{\alpha',\beta'})(D^{\alpha-e_j}D^{\alpha'}u).\end{aligned}$$

Its symbol is

$$\frac{1}{i}\sum_{j=1,\alpha_j\ge1}^{n}a_{\alpha,\beta}(x)\tau^{\beta+\beta'}(\partial_j b_{\alpha',\beta'})\alpha_j\xi^{\alpha-e_j}\xi^{\alpha'}.$$

We ony recognize $\alpha_j \xi^{\alpha - e_j} = \partial_{\xi_j} \xi^\alpha$. And the formula is still true and equal to zero if $\alpha_j = 0$. So, the term of order $m_1 + m_2 - 1$ is therefore

$$\frac{1}{i} \sum_{j=1}^{n} a_{\alpha,\beta}(x) \tau^{\beta+\beta'} (\partial_j b_{\alpha',\beta'}) (\partial_{\xi_j} \xi^\alpha) \xi^{\alpha'} = \frac{1}{i} \sum_{j=1}^{n} (\partial_{\xi_j} a)(\partial_{x_j} b).$$

Now, take $A = \sum_{|\alpha|+\beta \leq m_1} a_{\alpha,\beta}(x) D^\alpha \tau^\beta$ and $B = b_{\alpha',\beta'}(x) D^{\alpha'} \tau^{\beta'}$. Decompose $A = a_{m_1}(x, D, \tau) + a_{m_1-1}(x, D, \tau) + r(x, D, \tau)$ with $a_{m_1}(x, D, \tau)$ homogeneous of degree m_1, $a_{m_1-1}(x, D, \tau)$ homogeneous of degree $m_1 - 1$ and $r(x, D, \tau)$ of order at most $m_1 - 2$.

The previous calculation shows

$$\begin{aligned} A \circ B &= a_{m_1}(x, D, \tau) \circ B + a_{m_1-1}(x, D, \tau) \circ B + r(x, D, \tau) \circ B \\ &= (a_{m_1} b)(x, D, \tau) + \frac{1}{i} \sum_{j=1}^{n} \left[(\partial_{\xi_j} a_{m_1})(\partial_{x_j} b) \right] (x, D, \tau) + (a_{m_1-1} b)(x, D, \tau) + r(x, D, \tau) \circ B \\ &= (ab)(x, D, \tau) + \frac{1}{i} \sum_{j=1}^{n} \left[(\partial_{\xi_j} a_{m_1})(\partial_{x_j} b) \right] (x, D, \tau) - (rb)(x, D, \tau) + r(x, D, \tau) \circ B \end{aligned}$$

where a and b are the full symbol of A and B (actually the coefficients greater than $m_1 - 1$ and $m_1 - 1$ are enough). So, we can write the formula in this case

$$A \circ B = (ab)(x, D, \tau) + \frac{1}{i} \sum_{j=1}^{n} \left[(\partial_{\xi_j} a_{m_1})(\partial_{x_j} b) \right] (x, D, \tau) + C(x, D, \tau) \tag{B.1}$$

where C is of order at most $m_1 + m_2 - 2$.

Let us now finally get to the general case, take $B = b_{m_2}(x, D, \tau) + b_{m_2-1}(x, D, \tau) + s(x, D, \tau)$ with $b_{m_1}(x, D, \tau)$ homogeneous of degree m_2, $b_{m_2-1}(x, D, \tau)$ homogeneous of degree $m_2 - 1$ and $s(x, D, \tau)$ of order at most $m_2 - 2$. Applying Formula (B.1) to B equal to $b_{m_2}(x, D, \tau)$ and $b_{m_2-1}(x, D, \tau)$, we get

$$A \circ B = (ab_{m_2})(x, D, \tau) + \frac{1}{i} \sum_{j=1}^{n} \left[(\partial_{\xi_j} a_{m_1})(\partial_{x_j} b_{m_2}) \right] (x, D, \tau) + C_1(x, D, \tau) \tag{B.2}$$

$$+ (ab_{m_2-1})(x, D, \tau) + \frac{1}{i} \sum_{j=1}^{n} \left[(\partial_{\xi_j} a_{m_1})(\partial_{x_j} b_{m_2-1}) \right] (x, D, \tau) + C_2(x, D, \tau) \tag{B.3}$$

where C_1 is of order at most $m_2 - 2$ and C_2 $m_2 - 3$.

In particular, since $(ab_{m_2})(x, D, \tau) + (ab_{m_2-1})(x, D, \tau) = (ab)(x, D, \tau) + C_3(x, D, \tau)$ where C_3 is of order at most $m_1 + m_2 - 2$ and $(\partial_{\xi_j} a_{m_1})(\partial_{x_j} b_{m_2-1})$ is of order at most $m_1 + m_2 - 2$, we have the equivalent of Formula (B.1) in the general case. This also proves Proposition 1.3.6.

Note that it means that

- the symbol of order $m_1 + m_2$ is $a_{m_1} b_{m_2}$.
- the symbol of order $m_1 + m_2 - 1$ is

$$a_{m_1} b_{m_2-1} + a_{m_1-1} b_{m_2-1} + \frac{1}{i} \sum_{j=1}^{n} \left[(\partial_{\xi_j} a_{m_1})(\partial_{x_j} b_{m_2-1}) \right]$$

This directly gives that $[A, B]$ is of order at most $m_1 + m_2 - 1$ with principal symbol of order $m_1 + m_2 - 1$

$$\frac{1}{i} \sum_{j=1}^{n} \left[(\partial_{\xi_j} a_{m_1})(\partial_{x_j} b_{m_2}) \right] - \frac{1}{i} \sum_{j=1}^{n} \left[(\partial_{x_j} a_{m_1})(\partial_{\xi_j} b_{m_2}) \right] = \frac{1}{i} \left\{ a_{m_1}, b_{m_2} \right\}.$$

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