ON THE SET OF NON RADIATIVE SOLUTIONS FOR THE ENERGY CRITICAL WAVE EQUATION

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Abstract. Non radiative solutions of the energy critical non linear wave equation are global solutions u that furthermore have vanishing asymptotic energy outside the lightcone at both $t \to \pm\infty$:

$$\lim_{t\to\pm\infty} \|\nabla_{t,x} u(t)\|_{L^2(|x|\geqslant|t|+R)} = 0,$$

for some $R > 0$. They were shown to play an important role in the analysis of long time dynamics of solutions, in particular regarding the soliton resolution: we refer to the seminal works of Duyckaerts, Kenig and Merle, see [8] and the references therein.

We show that the set of non radiative solutions which are small in the energy space is a manifold whose tangent space at 0 is given by non radiative solutions to the linear equation (described in [3]). We also construct nonlinear solutions with an arbitrary prescribed radiation field.

1. Introduction

We consider solutions $u : I \times \mathbb{R}^d \to \mathbb{R}$ (I interval of $\mathbb{R}$) of the energy critical semilinear wave equation in dimension $3 \leqslant d \leqslant 6$:

$$\Box u = f(u), \tag{1}$$

with $f(x) = \pm|x|^{q-1}x$ or $f(x) = \pm x^q$ (if q is an integer), where $q = \frac{d+2}{d-2}$ is the $\dot H^1$-critical exponent. If u is a time dependent function, we denote $\vec u = (u, \partial_t u)$. Denote $\mathcal{H} := \dot H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. For a time interval $I \subset \mathbb{R}$, we define the spaces

$$W(I) = L^q(I, L^{2q}(\mathbb{R}^d)) \quad \text{and} \quad N(I) = L^1(I, L^2(\mathbb{R}^d))$$

together with

$$X(I) = \{u \in \mathscr{C}(I, \dot H^1(\mathbb{R}^d)) \cap W(I) : \partial_t u \in \mathscr{C}(I, L^2(\mathbb{R}^d))\},$$

with the natural norm

$$\|u\|_{X(I)} = \|u\|_{\mathscr{C}(I,\dot H^1(\mathbb{R}^d))} + \|\partial_t u\|_{\mathscr{C}(I,L^2(\mathbb{R}^d))} + \|u\|_{W(I)}.$$

We now define the linear and nonlinear flows: if $(u_0, u_1) \in \mathcal{H}$, then $\vec u_L(t) = S_L(t)(u_0, u_1)$ is the solution of the linear wave equation

$$\begin{cases} \Box u_L = 0, \\ \vec u_L(0) = (u_0, u_1). \end{cases} \tag{2}$$

Similarly, concerning the nonlinear equation, the problem is locally well posed for data (u_0, u_1) in $\mathcal{H}$ and furthermore, if they are small in that space, the non linear solution is global and scatters linearly as $t \to \pm\infty$: see for example Strauss

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[20], Rauch [19], Pecher [18], Ginibre-Velo [10] or Lindblad-Sogge [17] among others. In that case, we will denote $\vec{u}(t) = \mathcal{S}(t)(u_0, u_1)$ the solution to the nonlinear wave equation (1) with initial data $\vec{u}(0) = (u_0, u_1)$. We may write $S_L(u_0, u_1)$ and $\mathcal{S}(u_0, u_1)$ to denote the space time function $\vec{u}_L$ and $\vec{u}$ respectively.

For a space time function $\vec{v} \in X(\mathbb{R})$, we define its radiation energy outside a light cone (of base $R \geqslant 0$) by

$$E_{\mathrm{ext},R}(\vec{v}) := \frac{1}{2}\Bigg(\lim_{t\to+\infty} (\|\nabla v\|^2_{L^2(|x|\geqslant t+R)} + \|\partial_t v\|^2_{L^2(|x|\geqslant t+R)}) + \lim_{t\to-\infty} (\|\nabla v\|^2_{L^2(|x|\geqslant |t|+R)} + \|\partial_t v\|^2_{L^2(|x|\geqslant |t|+R)}) \Bigg),$$

provided that the limits exist.
If u is a solution to the linear energy critical wave equation (2), due to finite speed of propagation, the energy outside a light cone

$$\|\nabla u\|^2_{L^2(|x|\geqslant t+R)} + \|\partial_t u\|^2_{L^2(|x|\geqslant t+R)}$$

is decreasing as a function of $t \geq 0$ and admits a limit as $t \to +\infty$, for any $R \geqslant 0$ (and also as $t \to -\infty$), and so its radiation energy is well defined for any $R \geqslant 0$. If $u \in X(\mathbb{R})$ is a global solution to the non linear energy critical wave equation (2), there is linear scattering, so that the radiative energy is well defined as well (this is always the case for small data). See also [6] for the case of large data, global solutions to (2) which enjoy an a priori $\dot{H}^1 \times L^2$ bound.

We say that a space time function $\vec{v} \in X(\mathbb{R})$ is *non radiative* if $E_{\mathrm{ext},R}(\vec{v}) = 0$ for some $R \geqslant 0$. Non radiative solutions play a crucial role as the main obstruction in the energy channel method: this machinery was developed with great success, by Duyckaerts, Kenig, Merle and collaborators, to understand the long time behavior of solution to the *radial* energy critical non linear wave equation, in relation with the soliton resolution conjecture. The classification of small *radial* non radiative solutions of (1) was addressed in [7, 1]; we also refer [5, 8] and the references therein. An important issue, raised in [1, Theorem 1.3], is to extend solutions which are non radiative for some $R > 0$ (defined on the exterior light cone $\{|x| \geqslant t + R\}$), to solutions which are non radiative for $R = 0$. All in all, we believe that a fine understanding of these particular solutions might constitute a useful step as well in the soliton resolution in the general case (without symmetry).

Our goal in this article is to give a description of an initial data which leads to non radiative solutions $\vec{u}$ to (1).

We described in [3], for odd dimensions, the linear space $P_L(R)$ of initial datum $(v_0, v_1) \in \mathcal{H}$ that give rise to a solution $\vec{v} = S_L(v_0, v_1)$ to the linear wave equation such that $E_{\mathrm{ext},R}(\vec{v}) = 0$, in terms of the the Radon transform of the initial data (v_0, v_1) and according to its decomposition in spherical harmonics: for the convenience of the reader, we give further details in the Appendix A, see in particular (20). This was first done for radial data in odd dimension by [12], and in even dimension in [14] (see also [2]), and it was extended to non radial data for odd dimensions in [3] and later in even dimension in [13].
Let us define the operator $\mathcal{T}$ as follows: for a function v defined on $\mathbb{R}^d$, $\mathcal{T}v$ is a function of two variables (s, ω), defined on $\mathbb{R} \times \mathbb{S}^{d-1}$ by its (partial) Fourier transform in the first variable s:

$$\mathcal{F}_{s\to\nu}(\mathcal{T}v)(\nu, \omega) = c_0 |\nu|^{\frac{d-1}{2}} (e^{i\tau} \mathbb{1}_{\nu<0} + e^{-i\tau} \mathbb{1}_{\nu\geqslant 0}) \hat{v}(\nu\omega), \tag{3}$$

$$\text{where} \quad \tau := \frac{d-1}{4}\pi, \qquad c_0 = \frac{1}{\sqrt{2(2\pi)^{d-1}}},$$

and $\hat{v}$ is the Fourier transform on $\mathbb{R}^d$ of v:

$$\forall \xi \in \mathbb{R}^d, \quad \hat{v}(\xi) = \int e^{-i\xi\cdot x} v(x)dx.$$

The previous formula can also be expressed in term of the Radon transform $\mathcal{R}$: it is defined for $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{R}f : \mathbb{R} \times \mathbb{S}^{d-1} \to \mathbb{R}, \quad (s,\omega) \mapsto \int_{\omega\cdot y = s} f(y)dy,$$

(with Lebesgue measure on the hyperplane $\{y : \omega \cdot y = s\}$) and it can be extended to $f \in L^2(\mathbb{R}^d)$. Then there holds

$$\mathcal{T} = m_d(D_s)\mathcal{R} \quad \text{where} \quad m_d(\nu) := c_0|\nu|^{\frac{d-1}{2}} \left(e^{i\tau}\mathbb{1}_{\nu<0} + e^{-i\tau}\mathbb{1}_{\nu\geqslant 0}\right).$$

In odd dimension, this relation simply writes:

$$\mathcal{T} = \frac{(-1)^{\frac{d-1}{2}}}{\sqrt{2(2\pi)^{d-1}}} \partial_s^{\frac{d-1}{2}} \mathcal{R}.$$

We refer to [3] for details.

Our statement regarding the radiation of linear wave solutions is as follows. It is closely related to the radiation field of Friedlander [9], see also Katayama [11] for a related result.

Proposition 1.1 (Radiation field and concentration of energy on the light cone, [3, Theorem 1.1]). *Let $(v_0,v_1) \in \mathcal{H}$, and $\vec{v} = S_L(v_0,v_1)$ be the linear solution to (2). Then as $t \to +\infty$, there holds the convergence in $L^2(\mathbb{R}^d)^{1+d}$*

$$\nabla_{t,x}v(t,x) - \frac{1}{\sqrt{2}|x|^{\frac{d-1}{2}}}(\partial_s\mathcal{T}v_0 - \mathcal{T}v_1)\left(|x|-t, \frac{x}{|x|}\right) \times \begin{pmatrix} -1 \\ x/|x| \end{pmatrix} \to 0. \tag{4}$$

Furthermore, one has

(5)

$$\lim_{t\to+\infty} \|\nabla v\|^2_{L^2(|x|\geqslant t+R)} = \lim_{t\to+\infty} \|\partial_t v\|^2_{L^2(|x|\geqslant t+R)} = \frac{1}{2}\|\partial_s\mathcal{T}v_0 - \mathcal{T}v_1\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}.$$

The function $\partial_s\mathcal{T}v_0 - \mathcal{T}v_1$ in (4) is called the radiation field (at $+\infty$) of $\vec{v}$. Note that changing v_1 to $-v_1$ and reversing time, we get the same result in negative time

(6)

$$\lim_{t\to-\infty} \|\nabla v\|^2_{L^2(|x|\geqslant t+R)} = \lim_{t\to-\infty} \|\partial_t v\|^2_{L^2(|x|\geqslant t+R)} = \frac{1}{2}\|\partial_s\mathcal{T}v_0 + \mathcal{T}v_1\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})},$$

so that

$$E_{\text{ext},R}(\vec{v}) = \|\partial_s\mathcal{T}v_0\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})} + \|\mathcal{T}v_1\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}.$$

We want to define $\mathcal{P}(R)$ the (nonlinear) space of initial datum giving rise to non-linear radiative solutions. More precisely, we denote

$$\mathcal{P}(R) = \{(u_0,u_1) : \mathcal{S}(u_0,u_1) \text{ is defined globally on } \mathbb{R} \text{ and } E_{\text{ext},R}(\mathcal{S}(u_0,u_1)) = 0\}.$$

Our first result states that around $0 \in \mathcal{H}$, $\mathcal{P}(R)$ is a submanifold of $\mathcal{H}$, whose tangent space at 0 is $P_L(R)$.

Theorem 1.2. *Let $d = 3$ or 5. Let $R > 0$, and denote π_R the orthogonal projection on $P_L(R)$ (in $\mathcal{H}$).*
There exists $\varepsilon > 0$ and a $\mathscr{C}^1$ map

$$\Phi : B_{\mathcal{H}}(0,\varepsilon) \to \mathcal{H}.$$

so that Φ is a diffeomorphism to its image $V = \Phi(B_{\mathcal{H}}(0,\varepsilon)) \subset B_{\mathcal{H}}(0,2\varepsilon)$ whose differential at zero is the identity and satisfies

$$\forall (u_0,u_1) \in B_{\mathcal{H}}(0,\varepsilon), \quad \|(u_0,u_1) - \Phi(u_0,u_1)\|_{\mathcal{H}} \leqslant \|(u_0,u_1)\|_{\mathcal{H}}^q.$$

$$\forall (u_0,u_1) \in B_{\mathcal{H}}(0,\varepsilon), \quad \pi_R \circ \Phi(u_0,u_1) = \pi_R(u_0,u_1).$$

Moreover, when restricted to $P_L(R)$, we have $\Phi\left(P_L(R) \cap B_{\mathcal{H}}(0,\varepsilon)\right) = \mathcal{P}(R) \cap V$. In particular, $\mathcal{P}(R) \cap V$ is a submanifold of $\mathcal{H}$ with tangent space at 0 equal to $P_L(R)$. Moreover, $(\pi_R)_{|\mathcal{P}(R)\cap V}$ is a chart from $\mathcal{P}(R) \cap V$ to $P_L(R) \cap B_{\mathcal{H}}(0,\varepsilon)$ with inverse Φ.

In particular, this result proves that there are a lot of nonlinear radiative solutions, at least as many as the linear set $P_L(R)$ which is actually a large space, see Appendix A. This description extends some results of [1] to the non radial case.

Simple non radiative solutions can be constructed as follows: it suffices to consider a static solutions $u(t,x) = u(x)$ for $|x| \geq |t| + R$, with $-\Delta u = f(u)$. Such solutions outside of a ball have been precisely described in our recent work [4] for analytic nonlinearity (which is useful for (1) in dimension 3). The set $\mathcal{P}(R)^{stat}$ of such small solutions is also a manifold whose tangent set at 0 is the set $P_L(R)^{stat}$ of linear solutions of $\Delta u_L = 0$; but $\mathcal{P}(R)^{stat}$ is actually a strict subset of $\mathcal{P}(R)$, by a substantial margin: see Remark A.1 for more precisions.
$P_L(R)^{stat}$ is also a subset of $P_L(R)$, so we recover the inclusion $\mathcal{P}(R)^{stat}$ into $\mathcal{P}(R)$ at the tangent space level. Yet, in [4], we give a more precise statement: the nonlinear static solutions of $\mathcal{P}(R)^{stat}$ "look" like the linear one $P_L(R)^{stat}$ at infinity. In a suitable space Z_r of analytic functions on $\mathbb{S}^{d-1}$ adapted to the operator Δ, there exists a unique $u_L \in P_L(R)^{stat}$ so that

$$\|(u - u_L)(r\cdot)\|_{Z_r} \underset{r\to+\infty}{\longrightarrow} 0.$$

Moreover, the application $u \mapsto u_L$, that appears as a kind of scattering operator is a (local) bijection. It would be very interesting to obtain such precise description for the nonlinear non radiative solutions.

Our second result is related to wave operator: it says that given any radiation field F (as in (4)), there exists a unique nonlinear solution of (1) with this prescribed radiation field. This result already appears in the literature, in a form or another, see below. The precise statement is as follows.

Theorem 1.3. *Let $3 \leqslant d \leqslant 6$ and $F \in L^2(\mathbb{R} \times \mathbb{S}^{d-1})$. Then, there exist $T \in \mathbb{R}$ and a unique $u \in X([T,+\infty)$ solution of the nonlinear equation (1) so that, as $t \to +\infty$,*

$$\nabla_{t,x} u(t,x) - \frac{1}{\sqrt{2}|x|^{\frac{d-1}{2}}} F\left(|x| - t, \frac{x}{|x|}\right) \times \begin{pmatrix} -1 \\ x/|x| \end{pmatrix} \to 0 \quad \text{in} \quad L^2(\mathbb{R}^d)^{1+d}.$$

Furthermore, if $\|F\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}$ is small enough, one can choose $T = 0$ and $u \in X(\mathbb{R})$ is defined globally.

We refer to [16, Theorem 1.1] for a result with a similar flavor, for wave type equations (with other nonlinearities) in dimension 3, but in different functional spaces; see also [15].
Theorem 1.3 is independent of Theorem 1.2, but relies on a linear scattering result in X and on Proposition 2.5 which ensures that the map giving the radiation of (from a linear solution) is onto: this last result goes back to Friedlander [9] (see also the appendix of [6]). Here, we provide a new proof based on formula (3), which actually gives an explicit expression of the inverse.

2. Proofs

The spaces $W(I)$, $X(I)$ and $N(I)$ were chosen to satisfy the following Strichartz and nonlinear estimates. For a constant independent of the interval I (or of $t \in \mathbb{R}$), we have

$$\begin{aligned} \left\|S_L(t)(u_0,u_1)\right\|_{X(\mathbb{R})} &\leqslant C\left\|(u_0,u_1)\right\|_{\mathcal{H}}, \\ \left\|(u(0),\partial_t u(0))\right\|_{\mathcal{H}} &\leqslant C\left\|u\right\|_{X(\mathbb{R})}, \\ \left\|\int_{-\infty}^{+\infty}\cos(\tau|D_x|)h(\tau)d\tau\right\|_{L^2(\mathbb{R}^d)} &\leqslant C\left\|h\right\|_{N(\mathbb{R})}, \\ \left\|\int_{-\infty}^{+\infty}\cos(\tau|D_x|)h(\tau)d\tau\right\|_{L^2(\mathbb{R}^d)} &\leqslant C\left\|h\right\|_{N(\mathbb{R})}, \\ \left\|\int_{-\infty}^{+\infty}\sin(\tau|D_x|)h(\tau)d\tau\right\|_{L^2(\mathbb{R}^d)} &\leqslant C\left\|h\right\|_{N(\mathbb{R})}, \\ \left\|v\right\|_{X([t,+\infty))}+\left\|\vec{v}(t)\right\|_{\mathcal{H}} &\leqslant C\left\|h\right\|_{N([t,+\infty))} \\ \text{where}\quad v(t)=\int_t^{+\infty}\frac{\sin((.-\tau)|D_x|)}{|D_x|}h(\tau)d\tau. \end{aligned} \tag{7}$$

The related Strichartz estimates can for example be found in [17, Theorem 3.1], see also [10]. Also notice that N is such that if $h \in N([A,+\infty))$ for some $A \in \mathbb{R}$, then

$$\|h\|_{N([t,+\infty))} \to 0 \text{ as } t \to +\infty, \tag{8}$$

(and similarly in a neighbourhood of $-\infty$). We will finally need the nonlinear estimate

$$\|f(u)-f(v)\|_{N(I)} \leqslant C\,\|u-v\|_{W(I)}\,(\|u\|_{W(I)}^{q-1}+\|v\|_{W(I)}^{q-1}). \tag{9}$$

It does hold in the cases considered for (1) since $|f'(s)| \leqslant C|s|^{q-1}$ and due to Hölder estimates. In fact, our proofs work in any functional setting that respects the above conditions (7)-(8)-(9).

Let us start by a a few observations related to the operator $\mathcal{T}$.

Definition 2.1. We denote:

$$\begin{aligned} L^2_{\text{odd}}(\mathbb{R}\times\mathbb{S}^{d-1}) &:= \left\{F \in L^2(\mathbb{R}\times\mathbb{S}^{d-1});\ F(s,\omega)=-F(-s,-\omega), a.e.\right\}, \\ L^2_{\text{even}}(\mathbb{R}\times\mathbb{S}^{d-1}) &:= \left\{F \in L^2(\mathbb{R}\times\mathbb{S}^{d-1});\ F(s,\omega)=F(-s,-\omega), a.e.\right\}. \end{aligned}$$

Lemma 2.2 ([3, Lemma 4.14])**.** *Let d be odd. $\mathcal{T}$ defines an isometry from $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}\times\mathbb{S}^{d-1})$ and is therefore an isomorphism to its range defined by*

$$Range(\mathcal{T}) = \begin{cases} L^2_{\text{even}}(\mathbb{R}\times\mathbb{S}^{d-1}) & \text{if } d \equiv 1[4], \\ L^2_{\text{odd}}(\mathbb{R}\times\mathbb{S}^{d-1}) & \text{if } d \equiv 3[4]. \end{cases}$$

Similarly, $\partial_s\mathcal{T} : \dot{H}^1(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1}\times\mathbb{R})$ is isometric and

$$Range(\partial_s\mathcal{T}) = \begin{cases} L^2_{\text{odd}}(\mathbb{R}\times\mathbb{S}^{d-1}) & \text{if } d \equiv 1[4], \\ L^2_{\text{even}}(\mathbb{R}\times\mathbb{S}^{d-1}) & \text{if } d \equiv 3[4]. \end{cases}.$$

We obtain the following corollary.

Corollary 2.3. *Let d be odd and $R > 0$. There exists a continuous linear map $G^1_R : L^2((R,+\infty)\times\mathbb{S}^{d-1}) \mapsto L^2(\mathbb{R}^d)$ so that for any $F \in L^2((R,+\infty)\times\mathbb{S}^{d-1})$, $\mathcal{T}G^1_R F = F$ a.e. on $(R,+\infty)$.*
Similarly, there exists a continuous linear map $G^0_R : L^2((R,+\infty)\times\mathbb{S}^{d-1}) \mapsto \dot{H}^1(\mathbb{R}^d)$ so that for any $F \in L^2((R,+\infty)\times\mathbb{S}^{d-1})$, $\partial_s\mathcal{T}G^0_R F = F$ a.e. on $(R,+\infty)$.

Proof. We just prove the result for $\mathcal{T}$ and $d \equiv 1[4]$, the other cases being similar. Since $Range(\mathcal{T}) = L^2_{\text{even}}(\mathbb{R} \times \mathbb{S}^{d-1})$ is a closed subsets of the Banach space $L^2(\mathbb{R} \times \mathbb{S}^{d-1})$, we can apply the open mapping Theorem of Banach to define a continuous inverse $\mathcal{T}^{-1}$ from $L^2_{\text{even}}(\mathbb{R} \times \mathbb{S}^{d-1})$ to $L^2(\mathbb{R}^d)$. Let $\widetilde{F}$ be the even extension of $F \in L^2((R, +\infty) \times \mathbb{S}^{d-1})$ that is equal to zero on $s \in [-R, R]$. More precisely

$$
\begin{aligned}
&\widetilde{F}(s,\omega) = F(s,\omega) \quad \text{ for } s > R\\
&\widetilde{F}(s,\omega) = F(-s,-\omega) \quad \text{ for } s < -R\\
&\widetilde{F}(s,\omega) = 0 \quad \text{ for } s \in [-R, R].
\end{aligned}
$$

It is clear that $\widetilde{F} \in L^2_{\text{even}}(\mathbb{R} \times \mathbb{S}^{d-1})$. Defining $G^1_R F = \mathcal{T}^{-1}\widetilde{F}$, we obtain $\mathcal{T} G^1_R F = \widetilde{F}$ which satisfies the expected result. $\square$

Given a source term f, we can now construct a solution to the linear equation with this source term, which is non radiative.

Proposition 2.4. *Let d odd and X, N functional spaces satisfying* (7) *and* (8). *Let $h \in N(\mathbb{R})$. There exists a* continuous *linear map $T : N(\mathbb{R}) \to X(\mathbb{R})$, such that for any $h \in N(\mathbb{R})$, $u = Th$ is the unique element $u \in X(\mathbb{R})$ satisfying*

(1) u is solution of $\Box u = h$,
(2) $E_{\text{ext},R}(\vec{u}) = 0$,
(3) $\pi_R(\vec{u}(0)) = 0$.

Proof. Step 1. We first look for $\widetilde{u}$ satisfying the hypothesis 1) and 2), but not necessarily 3) We decompose $\tilde{u} = v + w$ where

$$v(t) := \int_{-\infty}^{t} \frac{\sin((t-\tau)|D_x|)}{|D_x|} h(\tau) d\tau,$$

so that $\Box v = h$ with morally 0 data at $-\infty$ and w solution of $\Box w = 0$ is to be chosen later on. Notice that changing t to $-t$ in (7), we get

$$\|\vec{v}(t)\|_{\mathcal{H}} \leqslant \left\| \int_{-\infty}^{t} \frac{\sin((t-\tau)|D_x|)}{|D_x|} h(\tau) d\tau \right\|_{\mathcal{H}} \leqslant C \|h\|_{N((-\infty,t])}.$$

Using (8), this directly implies

$$\lim_{t \to -\infty} (\|\nabla v\|^2_{L^2(\mathbb{R}^d)} + \|\partial_t v\|^2_{L^2(\mathbb{R}^d)}) = 0. \tag{10}$$

Also, by (7), there hold

$$\|v\|_{X(\mathbb{R})} \leqslant C \|h\|_{N(\mathbb{R})}.$$

Let us now estimate the exterior energy (outside a truncated cone) of $\vec{v}$ as $t \to +\infty$. We write

$$
\begin{aligned}
v(t) &= \int_{-\infty}^{+\infty} \frac{\sin((t-\tau)|D_x|)}{|D_x|} h(\tau) d\tau - \int_{t}^{+\infty} \frac{\sin((t-\tau)|D_x|)}{|D_x|} h(\tau) d\tau \\
&= \frac{\sin(t|D_x|)}{|D_x|} \int_{-\infty}^{+\infty} \cos(\tau|D_x|) h(\tau) d\tau - \cos(t|D_x|) \int_{-\infty}^{+\infty} \frac{\sin(\tau|D_x|)}{|D_x|} h(\tau) d\tau \\
&\quad - \int_{t}^{+\infty} \frac{\sin((t-\tau)|D_x|)}{|D_x|} h(\tau) d\tau \\
&=: \frac{\sin(t|D_x|)}{|D_x|} v_{1+} + \cos(t|D_x|) v_{0+} + r(t). \qquad (11)
\end{aligned}
$$

In other words, $\vec{v} = S_L(v_{0+}, v_{1+}) + \vec{r}$. We estimate using (7)

$$\|v_{1+}\|_{L^2(\mathbb{R}^d)} = \left\| \int_{-\infty}^{+\infty} \cos(\tau|D_x|) h(\tau) d\tau \right\|_{L^2(\mathbb{R}^d)} \leqslant C \|h\|_{N(\mathbb{R})}, \tag{12}$$

$$\|v_{0+}\|_{\dot H^1(\mathbb{R}^d)}=\left\|\int_{-\infty}^{+\infty}\sin(\tau|D_x|)h(\tau)d\tau\right\|_{L^2(\mathbb{R}^d)}\leqslant C\,\|h\|_{N(\mathbb{R})}\,,$$
$$\|r\|_{X(\mathbb{R})}=\left\|\int_{\cdot}^{+\infty}\frac{\sin((\cdot-\tau)|D_x|)}{|D_x|}h(\tau)d\tau\right\|_{X(\mathbb{R})}\leqslant C\,\|h\|_{N(\mathbb{R})}\,,$$
$$\|\vec r(t)\|_{\mathcal{H}}=\left\|\int_{t}^{+\infty}\frac{\sin((t-\tau)|D_x|)}{|D_x|}h(\tau)d\tau\right\|_{\mathcal{H}}\leqslant C\,\|h\|_{N([t,+\infty))}\,.$$

In particular, due to (8), we have

$$\lim_{t\to+\infty}(\|\nabla r\|^2_{L^2(\mathbb{R}^d)}+\|\partial_t r\|^2_{L^2(\mathbb{R}^d)})=0. \tag{13}$$

We will select $(w_0,w_1)=(w(0),\partial_t w(0))$ the data at initial time for w, so that $w(t)=S_L(t)(w_0,w_1)$. We can now compute the radiation of u in terms of (w_0,w_1) and (v_{0+},v_{1+}). Indeed, for $t\to-\infty$, using (6) and (10), we have

$$\begin{aligned}&\lim_{t\to-\infty}(\|\nabla\widetilde u\|^2_{L^2(|x|\geqslant|t|+R)}+\|\partial_t\widetilde u\|^2_{L^2(|x|\geqslant|t|+R)})\\&\quad=\lim_{t\to-\infty}(\|\nabla w(t)\|^2_{L^2(|x|\geqslant|t|+R)}+\|\partial_t w(t)\|^2_{L^2(|x|\geqslant|t|+R)})\\&\quad=\|\partial_s\mathcal{T}w_0+\mathcal{T}w_1\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}.\end{aligned}$$

Similarly, for $t\to+\infty$, $\vec{\widetilde u}(t)=S_L(t)(w_0+v_{0+},w_1+v_{1+})+\vec r(t)$ so that using (13) and (5), we have

$$\begin{aligned}&\lim_{t\to+\infty}(\|\nabla\widetilde u\|^2_{L^2(|x|\geqslant|t|+R)}+\|\partial_t\widetilde u\|^2_{L^2(|x|\geqslant|t|+R)})\\&\quad=\lim_{t\to+\infty}(\|\nabla(v+w)(t)\|^2_{L^2(|x|\geqslant|t|+R)}+\|\partial_t(v+w)(t)\|^2_{L^2(|x|\geqslant|t|+R)})\\&\quad=\|\partial_s\mathcal{T}(w_0+v_{0+})-\mathcal{T}(w_1+v_{1+})\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}.\end{aligned}$$

Hence, summing up, we get:

$$\begin{aligned}E_{\mathrm{ext},R}(\vec{\widetilde u})=&\frac12\|\partial_s\mathcal{T}w_0+\mathcal{T}w_1\|_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}\\&+\frac12\|\partial_s\mathcal{T}(w_0+v_{0+})-\mathcal{T}(w_1+v_{1+})\|_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}.\end{aligned}\tag{14}$$

We therefore look for $(w_0,w_1)\in\dot H^1\times L^2$ such that

$$\begin{cases}\partial_s\mathcal{T}w_0+\mathcal{T}w_1=0\text{ on }[R,+\infty)\times\mathbb{S}^{d-1},a.e.\\\partial_s\mathcal{T}(w_0+v_{0+})-\mathcal{T}(w_1+v_{1+})=0\text{ on }[R,+\infty)\times\mathbb{S}^{d-1},a.e.\end{cases}\tag{15}$$

Equivalently:

$$\begin{cases}2\mathcal{T}w_1=-\mathcal{T}v_{1+}+\partial_s\mathcal{T}v_{0+}\text{ on }[R,+\infty)\times\mathbb{S}^{d-1},a.e.\\2\partial_s\mathcal{T}w_0=-\partial_s\mathcal{T}v_{0+}+\mathcal{T}v_{1+}\text{ on }[R,+\infty)\times\mathbb{S}^{d-1},a.e.\end{cases}$$

Due to Corollary 2.3, the previous equations can be solved with a continuous inverse. To summarize, we finally define

$$w_1=\frac12G^1_R(-\mathcal{T}v_{1+}+\partial_s\mathcal{T}v_{0+})\quad\text{and}\quad w_0=\frac12G^0_R(-\partial_s\mathcal{T}v_{0+}+\mathcal{T}v_{1+}).\tag{16}$$

Then (w_0,w_1) solve the system (15) and, thanks to Lemma 2.2 and (12), satisfy the estimates

$$\begin{aligned}\|(w_0,w_1)\|_{\mathcal{H}}&\leqslant C\,\|\mathcal{T}v_{1+}-\partial_s\mathcal{T}v_{0+}\|_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}\\&\leqslant C\,\|v_{1+}\|_{L^2(\mathbb{R}^d)}+C\,\|v_{0+}\|_{\dot H^1(\mathbb{R}^d)}\leqslant C\,\|h\|_{N(\mathbb{R})}\,.\end{aligned}$$

Then we let $\vec{\tilde{u}} = \vec{v} + S_L(w_0, w_1)$ where $\vec{v}$ is defined in (11) and (w_0, w_1) is defined in (16). Then $\Box \widetilde{u} = \Box v = h$ and, in view of (14), $E_{\mathrm{ext},R}(\vec{\tilde{u}}) = 0$. Also, we have the bound

$$\|\widetilde{u}\|_{X(\mathbb{R})} \leqslant \|v\|_{X(\mathbb{R})} + \|w\|_{X(\mathbb{R})} \leqslant C\,\|h\|_{N(\mathbb{R})} + \|(w_0, w_1)\|_{\mathcal{H}} \leqslant C\,\|h\|_{N(\mathbb{R})}\,.$$

Step 2. Now that $\tilde{u}$ is defined, we simply let $u = \widetilde{u} - u_R$ where $\vec{u}_R = S_L(\pi_R\,(\widetilde{u}(0), \partial_t \widetilde{u}(0)))$: indeed, u_R is a non radiative solution, and solves $\Box u_R = 0$. Also, regarding continuity of the map, we just need to write

$$\|u_R\|_{X(\mathbb{R})} \leqslant C\,\|\pi_R\,(\widetilde{u}(0), \partial_t \widetilde{u}(0))\|_{\mathcal{H}} \leqslant \|(\widetilde{u}(0), \partial_t \widetilde{u}(0))\|_{\mathcal{H}} \leqslant C\,\|\widetilde{u}\|_{X(\mathbb{R})} \leqslant C\,\|h\|_{N(\mathbb{R})}\,,$$

so that $\|u\|_{X(\mathbb{R})} \leqslant C\|h\|_{N(\mathbb{R})}$. This finishes the existence part.

Step 3. Concerning uniqueness: let u_1 and u_2 be two such solutions of the problem. In particular, $z = u_1 - u_2$ satisfy:

(1) z is solution of $\Box z = 0$,
(2) $E_{\mathrm{ext},R}(\vec{z}) = 0$,
(3) $\pi_R(\vec{z}(0)) = 0$.

In particular, the first and second assumptions imply $(z(0), \partial_t z(0)) \in P_L(R)$ and therefore $\vec{z}(0) = \pi_R(\vec{z}(0))$. Together with the third assumption, we infer $(z(0), \partial_t z(0)) = 0$ and therefore $z = 0$, and $u_1 = u_2$. $\Box$

With Proposition 2.4 in hand, we can now prove the theorem.

Proof of Theorem 1.2. For $(u_0, u_1) \in \mathcal{H}$, let $\vec{u}_L = S_L(u_0, u_1)$. We are looking for a solution u of

$$u = u_L + T(f(u)). \tag{17}$$

Indeed, if $u \in X(\mathbb{R})$ solves (17), then

$$\Box u = \Box(Tf(u)) = f(u),$$

so that u solves (1). To solve (17), given $(u_0, u_1) \in P_L(R)$ with $\|(u_0, u_1)\|_{\mathcal{H}} \leqslant \varepsilon$, we use a fixed point argument on small closed balls $B(0,\varepsilon)$ of $X(\mathbb{R})$ for the map

$$G : r \mapsto T(f(u_L + r)).$$

Due to the continuity of $T : N(\mathbb{R}) \to X(\mathbb{R})$ (provided by Proposition 2.4), and using (7) and (9), we get for $r, \tilde{r} \in X(\mathbb{R})$,

$$\begin{aligned}
&\|G(r)\|_{X(\mathbb{R})} \leqslant C\,\|f(u_L + r)\|_{N(\mathbb{R})} \leqslant C\,\|u_L + r\|^q_{X(\mathbb{R})} \leqslant C(\varepsilon^q + \|r\|^q_X),\\
&\|G(r) - G(r'))\|_{X(\mathbb{R})} \leqslant C\,\|f(u_L + r) - f(u_L + \widetilde{r})\|_{N(\mathbb{R})}\\
&\qquad \leqslant C\,\|r - \widetilde{r}\|_{W(\mathbb{R})}\,(\|u_L + r\|^{q-1}_{W(\mathbb{R})} + \|u_L + \widetilde{r}\|^{q-1}_{W(\mathbb{R})})\\
&\qquad \leqslant C\,\|r - \widetilde{r}\|_X\,(\varepsilon^{q-1} + \|r\|^{q-1}_X + \|\widetilde{r}\|^{q-1}_X).
\end{aligned}$$

So, for ε small enough, G admits a unique fixed point v in $\overline{B}_{X(\mathbb{R})}(0,\varepsilon)$, the closed ball of radius ε in $X(\mathbb{R})$. Furthermore

$$\|v\|_{X(\mathbb{R})} = \|G(v)\|_{X(\mathbb{R})} \leqslant C\|(u_0, u_1)\|^q_{\mathcal{H}}. \tag{18}$$

Then $u := u_L + v$ solves (17). Also, by regularity of the Banach fixed point with parameter, the map $(u_0, u_1) \mapsto v$ is $\mathscr{C}^1$ from $B_{\mathcal{H}}(0,\varepsilon)$ to $X(\mathbb{R})$ (notice that the nonlinearity is $\mathscr{C}^1$), with differential 0 at $0 \in \mathcal{H}$, due to (18). Finally,

$$\pi_R(u(0), \partial_t u(0)) = \pi_R(u_0, u_1) + \pi_R(T(f(u))(0), \partial_t T(f(u))(0)) = \pi_R(u_0, u_1).$$

Therefore, the map

$$\Phi : (u_0, u_1) \mapsto (u, \partial_t u)(0),$$

(where u is as above) satisfies the first part of Theorem 1.2, up to possibly diminishing ε.

Now, assuming $(u_0, u_1) \in P_L(R)$, we define $\vec{u} = \mathcal{S}\Phi(u_0, u_1)$, the associated nonlinear solution. We have $f(u) \in N(\mathbb{R})$ due to (9) and as $E_{\text{ext},R}(S_L(u_0, u_1)) = 0$, the radiation of u is well defined and

$$E_{\text{ext},R}(\vec{u}) = E_{\text{ext},R}(T(f(u)), \partial_t T(f(u))) = 0,$$

so that $u \in \mathcal{P}(R)$. So, we have proved $\Phi(P_L(R) \cap B_{\mathcal{H}}(0, \varepsilon)) \subset \mathcal{P}(R) \cap V$.

Reciprocally, let $(v_0, v_1) \in \mathcal{P}(R) \cap V$. By definition of V, it can be written $(v_0, v_1) = \Phi(u_0, u_1)$ with $(u_0, u_1) \in B_{\mathcal{H}}(0, \varepsilon)$. Denoting $\vec{u} = \mathcal{S}\Phi(u_0, u_1) = \mathcal{S}(v_0, v_1)$, the associated nonlinear solution, we have, by definition of Φ, $\vec{u} = S_L(u_0, u_1) + T(f(u))$. In particular, as $E_{\text{ext},R}(T(f(u)), \partial_t T(f(u))) = 0$, we have

$$E_{\text{ext},R}(\vec{u}) = E_{\text{ext},R}(S_L(u_0, u_1)).$$

Now we assumed $(v_0, v_1) \in \mathcal{P}(R)$, so that $E_{\text{ext},R}(\vec{u}) = E_{\text{ext},R}(\mathcal{S}(v_0, v_1)) = 0$, and

$$E_{\text{ext},R}(S_L(u_0, u_1)) = 0.$$

Thus, $(u_0, u_1) \in P_L(R)$ and $(v_0, v_1) \in \Phi(P_L(R) \cap B_{\mathcal{H}}(0, \varepsilon))$.

The last statement of the theorem is only a rephrasing of the previous results in terms of submanifolds in Banach spaces. □

Now, we turn to the proof of Theorem 1.3 and begin by a Proposition stating that the radiation operator is onto.

Proposition 2.5 (Friedlander [9]). *The application*

$$\begin{aligned} \mathcal{H} &\longrightarrow L^2(\mathbb{R} \times \mathbb{S}^{d-1}) \\ (v_0, v_1) &\longmapsto \partial_s \mathcal{T} v_0 - \mathcal{T} v_1 \end{aligned}$$

is a bijective isometry.

Proof. For the convenience of the reader, we provide a proof with an explicit inversion formula in terms of Fourier transform. Formula (3) gives

$$\mathcal{F}_{s\to\nu}(\partial_s \mathcal{T} v_0 - \mathcal{T} v_1)(\nu, \omega) = c_0 |\nu|^{\frac{d-1}{2}} (e^{i\tau} \mathbb{1}_{\nu<0} + e^{-i\tau} \mathbb{1}_{\nu\geqslant 0})(i\nu \hat{v}_0(\nu\omega) - \hat{v}_1(\nu\omega)).$$

For the injectivity, we could compute directly that the application is an isometry, see for instance [3, Lemma 2.1.] for a closely related computation. Here we can directly check that $\partial_s \mathcal{T} v_0 - \mathcal{T} v_1 = 0$ implies $i\nu\hat{v}_0(\nu\omega) = \hat{v}_1(\nu\omega)$ almost everywhere in $\mathbb{R} \times \mathbb{S}^{d-1}$. Applying at (ν, ω) and $(-\nu, -\omega)$, it gives $(v_0, v_1) = (0, 0)$.

For the surjectivity, given $F \in L^2(\mathbb{R} \times \mathbb{S}^{d-1})$, denote for simplicity $\hat{F} = \mathcal{F}_{s\to\nu} F$, and define v_0 and v_1 by their Fourier transform as follows: for $\xi \in \mathbb{R}^d \setminus \{0\}$, with $\xi = \rho\omega$ where $\rho > 0$ and $\omega \in \mathbb{S}^{d-1}$, we set

$$\begin{aligned} \hat{v}_0(\xi) &= \frac{1}{2ic_0 \rho^{\frac{d+1}{2}}} \left(e^{i\tau} \hat{F}(\rho, \omega) - e^{-i\tau} \hat{F}(-\rho, -\omega) \right), \\ \hat{v}_1(\xi) &= -\frac{1}{2c_0 \rho^{\frac{d-1}{2}}} \left(e^{i\tau} \hat{F}(\rho, \omega) + e^{-i\tau} \hat{F}(-\rho, -\omega) \right). \end{aligned}$$

Then for $\omega \in \mathbb{S}^{d-1}$, we have for $\nu > 0$

$$\begin{aligned} \mathcal{F}_{s\to\nu}(\partial_s \mathcal{T} v_0 - \mathcal{T} v_1)(\nu, \omega) &= c_0 \nu^{\frac{d-1}{2}} e^{-i\tau} (i\nu \hat{v}_0(\nu\omega) - \hat{v}_1(\nu\omega)) \\ &= c_0 \nu^{\frac{d-1}{2}} e^{-i\tau} \left(\frac{i\nu}{2ic_0 \nu^{\frac{d+1}{2}}} \left(e^{i\tau} \hat{F}(\nu, \omega) - e^{-i\tau} \hat{F}(-\nu, -\omega) \right) \right. \end{aligned}$$

$$+\frac{1}{2c_0\nu^{\frac{d-1}{2}}}\left(e^{i\tau}\hat{F}(\nu,\omega)+e^{-i\tau}\hat{F}(-\nu,-\omega)\right)\Bigg)$$
$$=\hat{F}(\nu,\omega),$$

and if $\nu<0$,

$$\begin{aligned}\mathcal{F}_{s\to\nu}(\partial_s\mathcal{T}v_0-\mathcal{T}v_1)(\nu,\omega)&=c_0|\nu|^{\frac{d-1}{2}}e^{i\tau}\left(-i|\nu|\hat{v}_0(|\nu|(-\omega))-\hat{v}_1(|\nu|(-\omega))\right)\\&=c_0|\nu|^{\frac{d-1}{2}}e^{i\tau}\Bigg(-\frac{i|\nu|}{2ic_0|\nu|^{\frac{d+1}{2}}}\left(e^{i\tau}\hat{F}(|\nu|,-\omega)-e^{-i\tau}\hat{F}(\nu,\omega)\right)\\&\qquad+\frac{1}{2c_0|\nu|^{\frac{d-1}{2}}}\left(e^{i\tau}\hat{F}(|\nu|,-\omega)+e^{-i\tau}\hat{F}(\nu,\omega)\right)\Bigg)\\&=\hat{F}(\nu,\omega).\end{aligned}$$

Hence there hold

$$(\partial_s\mathcal{T}v_0-\mathcal{T}v_1)=F.$$

We verify that (v_0,v_1) defined as above are indeed in $\mathcal{H}$.

$$\begin{aligned}\|v_0\|_{\dot{H}^1}^2&=\frac{1}{(2\pi)^d}\left\||\cdot|\hat{v}_0(\cdot)\right\|_{L^2}^2=\frac{1}{(2\pi)^d}\int_0^{+\infty}\rho^{d-1}\int_{\omega\in\mathbb{S}^{d-1}}\rho^2\left|\hat{v}_0(\rho\omega)\right|^2\,d\omega\,d\rho\\&=\frac{1}{4c_0^2(2\pi)^d}\int_0^{+\infty}\int_{\omega\in\mathbb{S}^{d-1}}\left|e^{i\tau}\hat{F}(\rho,\omega)-e^{-i\tau}\hat{F}(-\rho,-\omega)\right|^2\,d\omega\,d\rho.\end{aligned}$$

$$\begin{aligned}\|v_1\|_{L^2}^2&=\frac{1}{(2\pi)^d}\left\|\hat{v}_1(\cdot)\right\|_{L^2}^2=\frac{1}{(2\pi)^d}\int_0^{+\infty}\rho^{d-1}\int_{\omega\in\mathbb{S}^{d-1}}\left|\hat{v}_1(\rho\omega)\right|^2\,d\omega\,d\rho\\&=\frac{1}{4c_0^2(2\pi)^d}\int_0^{+\infty}\int_{\omega\in\mathbb{S}^{d-1}}\left|e^{i\tau}\hat{F}(\rho,\omega)+e^{-i\tau}\hat{F}(-\rho,-\omega)\right|^2\,d\omega\,d\rho.\end{aligned}$$

Finally, it is an isometry: indeed, $\frac{1}{4c_0^2(2\pi)^d}=\frac{1}{4\pi}$ and

$$\begin{aligned}&\left|e^{i\tau}\hat{F}(\rho,\omega)-e^{-i\tau}\hat{F}(-\rho,-\omega)\right|^2+\left|e^{i\tau}\hat{F}(\rho,\omega)+e^{-i\tau}\hat{F}(-\rho,-\omega)\right|^2\\&\qquad\qquad=2\left|\hat{F}(\rho,\omega)\right|^2+2\left|\hat{F}(-\rho,-\omega)\right|^2,\end{aligned}$$

so that

$$\begin{aligned}\|v_0\|_{\dot{H}^1}^2+\|v_1\|_{L^2}^2&=\frac{1}{2\pi}\int_0^{+\infty}\int_{\omega\in\mathbb{S}^{d-1}}\left|\hat{F}(\rho,\omega)\right|^2+\left|\hat{F}(-\rho,-\omega)\right|^2\,d\omega\,d\rho\\&=\frac{1}{2\pi}\int_0^{+\infty}\int_{\omega\in\mathbb{S}^{d-1}}\left|\hat{F}(\rho,\omega)\right|^2+\left|\hat{F}(-\rho,\omega)\right|^2\,d\omega\,d\rho\\&=\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\omega\in\mathbb{S}^{d-1}}\left|\hat{F}(\rho,\omega)\right|^2\,d\omega\,d\rho=\int_{\mathbb{R}}\int_{\omega\in\mathbb{S}^{d-1}}|F(s,\omega)|^2\,d\omega\,ds\\&=\|F\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}^2.\end{aligned}$$

□

Proof of Theorem 1.3. Step 1. We first construct the linear scattering state, that is find $(v_0,v_1)\in\mathcal{H}$ such that, denoting $\vec{v}_L=S_L(v_0,v_1)$, as $t\to+\infty$,

$$\nabla_{t,x}v_L(t,x)-\frac{1}{\sqrt{2}|x|^{\frac{d-1}{2}}}F\left(|x|-t,\frac{x}{|x|}\right)\times\begin{pmatrix}-1\\x/|x|\end{pmatrix}\to 0\quad\text{in}\quad L^2(\mathbb{R}^d)^{1+d}.\tag{19}$$

Due to Proposition 2.5, there exists $(v_0,v_1)\in\mathcal{H}$ so that

$$F=(\partial_s\mathcal{T}v_0-\mathcal{T}v_1).$$

In view of (4), we see that $\vec{v}_L=S_L(\cdot)(v_0,v_1)$ satisfies the expected asymptotic (19).

Step 2. We now construct $\vec{u}$, solution to (1) such that $\|\vec{u}-\vec{v}_L(t)\|_{\mathcal{H}} \to 0$ as $t \to +\infty$: this is simply the wave operator, and is standard. We provide some elements of proof for the sake of completeness. We decompose $\vec{u}(t) = \vec{v}_L(t) + \vec{w}(t)$ and write $\vec{w}$ as solution of a fixed point problem. Let $T \in \mathbb{R}$ to be chosen later: the Duhamel formula on $[t, \tau]$ (for $\tau \geqslant t$) gives

$$\vec{v}_L(\tau) + \vec{w}(\tau) = S_L(\tau - t)(\vec{v}_L(t) + \vec{w}(t)) + \int_t^\tau S_L(\tau - s) \begin{pmatrix} 0 \\ f(v_L(s) + w(s)) \end{pmatrix} ds.$$

Notice that $\vec{v}_L(t) = S_L(t-T)\vec{v}_L(T)$; compose by $S_L(t-\tau)$ and let $\tau \to +\infty$: as $\|S_L(t-\tau)\vec{w}(\tau)\|_{\mathcal{H}} = \|\vec{w}(\tau)\|_{\mathcal{H}}$ is meant to tend to 0, we arrive at the fixed point formulation:

$$\vec{w}(t) = \Psi\vec{w}(t), \quad \text{where} \quad \Psi\vec{v}(t) := -\int_t^{+\infty} S_L(t-s) \begin{pmatrix} 0 \\ f(v_L(s) + v(s)) \end{pmatrix} ds.$$

Let $T \in \mathbb{R}$ to be fixed later, we work in small closed balls $\overline{B}(0, \varepsilon)$ of $X([T, +\infty))$. By (7) and (9), we have for $\vec{v} \in X([T, +\infty))$,

$$\|\Psi\vec{v}\|_{X([T,+\infty))} \leqslant C\|f(v_L + v)\|_{N([T,+\infty))} \leqslant C\left(\|v_L\|^q_{W([T,+\infty))} + \|v\|^q_{W([T,+\infty))}\right).$$

Similarly,

$$\begin{aligned}\|\Psi\vec{v} - \Psi\vec{\tilde{v}}\|_{X([T,+\infty))} &\leqslant C\|f(v_L + v) - f(v_L + \tilde{v})\|_{N([T,+\infty)} \\ &\leqslant C\left(\|v_L\|^{q-1}_{W([T,+\infty))} + \|v\|^{q-1}_{W([T,+\infty))} + \|\tilde{v}\|^{q-1}_{W([T,+\infty))}\right)\|v - \tilde{v}\|_{W([T,+\infty))}.\end{aligned}$$

Let T be such that $\|v_L\|^{q-1}_{W([T,+\infty))} \leqslant \varepsilon$ be small enough, then Ψ admits a unique fixed point $\vec{w}$ in $B(0,\varepsilon)$, and $\vec{u} = \vec{v}_L + \vec{w}$ answers the question. □

Appendix A. Description of the set $P_L(R)$ of linear non radiative solutions

In this section, we gather some results of [3] where a precise description of the set $P_L(R)$ was performed for $R > 0$. This corresponds to classifying the linear solutions u that have vanishing asymptotic energy on the exterior light cone $|x| \geqslant t + R$ with $R > 0$, that is

$$E_{ext,R}(u) = 0.$$

By finite speed of propagation, initial data which are compactly supported in $|x| \leqslant R$ obviously satisfy this condition. We will call this space

$$\mathcal{K}_{R,comp} = \left\{(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d) : (u_0, u_1)|_{\{|x|>R\}} = 0\right\}.$$

where the equality is in the distributional sense.
It turns out that these are not the only examples. We will now need some further notation.
We denote $(Y_\ell)_{\ell\in\mathbb{M}}$ a countable orthonormal basis of spherical harmonics of $\mathbb{S}^{d-1}$. Y_ℓ is the restriction to $\mathbb{S}^{d-1}$ of a harmonic (homogeneous) polynomial. For short, we will denote $l = l(\ell)$ the degree of this polynomial.
The non radiative functions will be the following. Denote for $k \in \mathbb{N}$,

$$\alpha_k := -l - d + 2k + 2.$$

α_k also depends on ℓ, but here and below, we silence this dependence to keep notations light. Then let

$$g_k(x) = \mathbb{1}_{\{|x|>R\}}|x|^{\alpha_k} Y_\ell\left(\frac{x}{|x|}\right).$$

Note that $g_k \in L^2$ if and only if $\alpha_k < -d/2$. We introduce

$$\mathcal{N}^0_{R,\ell} = \mathrm{Span}\left(g_k; \text{ for } k \in \mathbb{N} \text{ such that } \alpha_k < -d/2\right)$$

Similarly, let

$$f_k(x) = \begin{cases} \left(\dfrac{|x|}{R}\right)^{\alpha_k} Y_\ell\left(\dfrac{x}{|x|}\right) & \text{for } |x| > R \\ \left(\dfrac{|x|}{R}\right)^{l} Y_\ell\left(\dfrac{x}{|x|}\right) & \text{for } |x| \leqslant R. \end{cases}$$

Note that $f_k \in \dot{H}^1$ if and only if $\alpha_k < -d/2+1$. Also, the value of f_k in $|x| \leqslant R$ is not very important; our choice allows to keep continuity and that the restriction $f_k|_{\{|x|<R\}}$ is a harmonic polynomial, so that f_k is orthogonal to (in $\dot{H}^1$) to functions with compact support in $B(0,R)$.
Let

$$\mathcal{N}^1_{R,\ell} = \mathrm{Span}\left(f_k;\ \text{for } k \in \mathbb{N} \text{ such that } \alpha_k < -d/2+1\right).$$

For any $\ell \in \mathbb{M}$, we note the space

$$P_\ell(R) = \mathcal{N}^0_{R,\ell} \times \mathcal{N}^1_{R,\ell}.$$

Remark A.1. For a fixed spherical harmonics Y_ℓ, only the value $k=0$ corresponding to $\alpha_0 = -l-d+2$ produces a solution of the stationary equation $\Delta u = 0$, and from [4] (in dimension 3), a nonlinear stationary solution defined outside a large ball: via time invariance, this yields a curve (manifold of dimension 1) of solutions stationary outside a light cone.
Theorem 1.2 constructs a non radiative solution for all elements in $P_\ell(R)$, which, except for those on the curve above, are *not* stationary outside a light cone.

One of the result of [3, Theorem 1.7] was the precise description of $P_L(R)$ in odd dimensions as follows.

$$P_L(R) = \mathcal{K}_{R,comp} \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_{\ell\in\mathbb{M}}} P_\ell(R). \tag{20}$$

(the orthogonality is related to the natural scalar product of $\dot{H}^1 \times L^2$).

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