Control and Stabilization of the Korteweg-de Vries Equation on a Periodic Domain

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Abstract

This paper aims at completing an earlier work of Russell and Zhang [38] to study internal control problems for the distributed parameter system described by the Korteweg-de Vries equation on a periodic domain $\mathbb{T}$. In [38], Russell and Zhang showed that the system is *locally* exactly controllable and *locally* exponentially stabilizable when the control acts on an arbitrary nonempty subdomain of $\mathbb{T}$. In this paper, we show that the system is in fact *globally* exactly controllable and *globally* exponentially stabilizable. The global exponential stabilizability corresponding to a natural feedback law is first established with the aid of certain properties of propagation of compactness and propagation of regularity in Bourgain spaces for solutions of the associated linear system. Then, using a different feedback law, the resulting closed-loop system is shown to be locally exponentially stable with an *arbitrarily large* decay rate. A *time-varying* feedback law is further designed to ensure a global exponential stability with an arbitrary large decay rate.

1 Introduction

The well-known Korteweg-de Vries (KdV) equation can be written as

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0, \tag{1.1}$$

where $u = u(x,t)$ denotes a real-valued function of two real variables x and t. The equation was first derived by Korteweg and de Vries [19] in 1895 (or by Boussinesq [4] in 1876 [1]) as a model for

[1]The interested readers are refereed to a nice article of de Jager [14] for the origin of the KdV equation.

propagation of some surface water waves along a channel. The KdV equation has been intensively studied from various aspects of both mathematics and physics since the 1960s when solitons were discovered through solving the KdV equation, and the inverse scattering method, a so-called nonlinear Fourier transform, was invented to seek solitons [12, 25]. It turns out that the equation is not only a good model for some water waves but also a very useful approximation model in nonlinear studies whenever one wishes to include and balance a weak nonlinearity and weak dispersive effects [25]. In particular, the equation is now commonly accepted as a mathematical model for the unidirectional propagation of small-amplitude long waves in nonlinear dispersive systems.

In this paper, we consider the KdV equation posed on the periodic domain $\mathbb{T}$:

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0, \quad x \in \mathbb{T},\ t \in \mathbb{R}. \tag{1.2}$$

The equation is known to possess an infinite set of conserved integral quantities of which the first two are

$$I_1(t) = \int_{\mathbb{T}} u(x,t)dx$$

and

$$I_2(t) = \int_{\mathbb{T}} u^2(x,t)dx.$$

From the historical origins [19, 4, 25] of the KdV equation, involving the behavior of water waves in a shallow channel, it is natural to think of I_1 and I_2 as expressing conservation of volume (or mass) and energy, respectively. The Cauchy problem for the equation (1.2) has been intensively studied for many years (see [39, 16, 3, 18] and the references therein). The best known result so far [15] is that the Cauchy problem is well-posed in the space $H^s(\mathbb{T})$ for any $s \geq -1$:

Let $s \geq -1$ and $T > 0$ be given. For any $u_0 \in H^s(\mathbb{T})$, the equation (1.2) admits a unique solution $u \in C([0,T]; H^s(\mathbb{T}))$ satisfying

$$u(x,0) = u_0(x).$$

Moreover, the corresponding solution map ($u_0 \to u$) is continuous from the space $H^s(\mathbb{T})$ to the space $C([0,T]; H^s(\mathbb{T}))$. [2]

In this paper we will study the equation (1.2) from a control point of view with a forcing term $f = f(x,t)$ added to the equation as a control input:

$$\partial_t u + u\partial_x u + \partial_x^3 u = f, \quad x \in \mathbb{T},\ t \in \mathbb{R}, \tag{1.3}$$

where f is assumed to be supported in a given open set $\omega \subset \mathbb{T}$. The following exact control problem and stabilization problem are fundamental in control theory.

Exact control problem: *Given an initial state u_0 and a terminal state u_1 in a certain space, can one find an appropriate control input f so that the equation (1.3) admits a solution u which satisfies $u(.,0) = u_0$ and $u(.,T) = u_1$?*

Stabilization problem: *Can one find a feedback control law: $f = Ku$ so that the resulting closed-loop system*

$$\partial_t u + u\partial_x u + \partial_x^3 u = Ku, \quad x \in \mathbb{T},\ t \in \mathbb{R}^+$$

is asymptotically stable as $t \to +\infty$?

[2] If $s > -\frac{1}{2}$, this solution map is, in fact, analytic.

The problems were first studied by Russell and Zhang for the KdV equation [37, 38]. In their work, in order to keep the *mass* $I_1(t)$ conserved, the control input $f(x,t)$ is chosen to be of the form

$$f(x,t) = [Gh](x,t) := g(x)\left(h(x,t) - \int_{\mathbb{T}} g(y)h(y,t)dy\right) \tag{1.4}$$

where h is considered as a new control input, and $g(x)$ is a given nonnegative smooth function such that $\{g > 0\} = \omega$ and

$$2\pi[g] = \int_{\mathbb{T}} g(x)dx = 1.$$

For the chosen g, it is easy to see that

$$\frac{d}{dt}\int_{\mathbb{T}} u(x,t)dx = \int_{\mathbb{T}} f(x,t)dx = 0 \qquad \text{for any } t \in \mathbb{R}$$

for any solution $u = u(x,t)$ of the system

$$\partial_t u + u\partial_x u + \partial_x^3 u = Gh; \tag{1.5}$$

thus the *mass* of the system is indeed conserved.

The following results are due to Russell and Zhang [38].

Theorem A: *Let $s \geq 0$ and $T > 0$ be given. There exists a $\delta > 0$ such that for any $u_0, u_1 \in H^s(\mathbb{T})$ with $[u_0] = [u_1]$ satisfying*

$$\|u_0\|_s \leq \delta, \qquad \|u_1\|_s \leq \delta,$$

one can find a control input $h \in L^2(0,T;H^s(\mathbb{T}))$ such that the system (1.5) admits a solution $u \in C([0,T];H^s(\mathbb{T}))$ satisfying

$$u(x,0) = u_0(x), \qquad u(x,T) = u_1(x).$$

In order to stabilize the system (1.5), Russell and Zhang employed a simple feedback control law

$$h(x,t) = -G^* u(x,t). \tag{1.6}$$

The resulting closed-loop system

$$\partial_t u + u\partial_x u + \partial_x^3 u = -GG^* u, \quad x \in \mathbb{T}, \ t \in \mathbb{R}. \tag{1.7}$$

is *locally* exponentially stable.

Theorem B: *Let $s = 0$ or $s \geq 1$ be given. There exist positive constants M, δ and γ such that if $u_0 \in H^s(\mathbb{T})$ satisfies*

$$\|u_0 - [u_0]\|_s \leq \delta, \tag{1.8}$$

then the corresponding solution u of (1.7) satisfies

$$\|u(\cdot,t) - [u_0]\|_s \leq Me^{-\gamma t}\|u_0 - [u_0]\|_s$$

for any $t \geq 0$.

Thus one can always find an appropriate control input h to guide the system (1.5) from a given initial state u_0 to a terminal state u_1 so long as *their amplitudes are small* and $[u_0] = [u_1]$. A question arises naturally.

Question 1: *Can one still guide the system by choosing appropriate control input h from a given initial state u_0 to a given terminal state u_1 when u_0 or u_1 have* large *amplitude?*

As for the closed-loop system (1.7), its *small amplitude* solutions decay at a uniform exponential rate to the corresponding constant state $[u_0]$ with respect to the norm in the space $H^s(\mathbb{T})$ as $t \to \infty$. One may ask naturally:

Question 2: *Do the large amplitude solutions of the closed-loop system (1.7) decay exponentially as $t \to \infty$?*

A further question is:

Question 3: *For any given number $\lambda > 0$, can we design a linear feedback control law such that the exponential decay rate of the resulting closed-loop system is λ?*

One of the main results in this paper is a positive answer to Question 1 as given below.

Theorem 1.1. *Let $s \geq 0$, $R > 0$, and $\mu \in \mathbb{R}$ be given. There exists a time $T > 0$ such that if $u_0, u_1 \in H^s(\mathbb{T})$ with $[u_0] = [u_1] = \mu$ are such that*

$$\|u_0\|_s \leq R, \qquad \|u_1\|_s \leq R,$$

then one can find a control input $h \in L^2(0, T; H^s(\mathbb{T}))$ such that the system (1.5) admits a solution $u \in C([0, T]; H^s(\mathbb{T}))$ satisfying

$$u(x, 0) = u_0(x), \qquad u(x, T) = u_1(x).$$

So the system (1.5) is *globally* exactly controllable.

As for Question 2, we have the following affirmative answer.

Theorem 1.2. *Let $s \geq 0$ and $\mu \in \mathbb{R}$ be given. There exists a constant $\kappa > 0$ such that for any $u_0 \in H^s(\mathbb{T})$ with $[u_0] = \mu$, the corresponding solution u of the system (1.7) satisfies*

$$\|u(\cdot, t) - [u_0]\|_s \leq \alpha_{s,\mu}(\|u_0 - [u_0]\|_0) e^{-\kappa t} \|u_0 - [u_0]\|_s \qquad \textit{for all } t \geq 0,$$

where $\alpha_{s,\mu} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function depending on s and μ.

Note that Theorem 1.1 follows from Theorem 1.2 and a local control result around the state $u(x) = \mu$ (similar to Theorem A) thanks to the time reversibility of the KdV equation.

The decay rate κ in Theorem 1.2 has an upper bound

$$\kappa \leq \inf\{-\operatorname{Re} \lambda : \ \lambda \in \sigma_p(A_G)\}$$

where A_G is the operator defined by

$$A_G v = -v''' - \mu v - GG^* v$$

with $\mathcal{D}(A_G) = H^3(\mathbb{T})$ as domain. In order to have the decay rate κ arbitrarily large, a different feedback control law is needed.

Theorem 1.3. *Let $\lambda > 0$, $s \geq 0$, and $\mu \in \mathbb{R}$ be given. There exists a number $\delta > 0$ and a linear bounded operator Q_λ from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$ such that if one chooses the feedback control law*

$$h = -Q_\lambda u$$

in system (1.5)-(1.4), then the solution u of the resulting closed-loop system

$$\partial_t u + u\partial_x u + \partial_x^3 u = -GQ_\lambda u, \quad u(x,0) = u_0(x), \ x \in \mathbb{T} \tag{1.9}$$

satisfies

$$\|u(\cdot,t) - [u_0]\|_s \leq Ce^{-\lambda t}\|u_0 - [u_0]\|_s \qquad \text{for all } \ t \geq 0,$$

whenever $\|u_0\|_s \leq \delta$ and $[u_0] = \mu$, $C > 0$ denoting a constant independent of u_0.

Note that this is still a *local* stabilization result. However, the feedback laws in Theorems 1.2 and 1.3 may be combined into a *time-varying* feedback law (as in [9]) ensuring a global stabilization with an arbitrary large decay rate.

Theorem 1.4. *Let $\lambda > 0$, $s \geq 0$, and $\mu \in \mathbb{R}$ be given. There exists a smooth map Q_λ from $H^s(\mathbb{T}) \times \mathbb{R}$ to $H^s(\mathbb{T})$ which is periodic with respect to the second variable, and such that the solution u of the closed-loop system*

$$\partial_t u + u\partial_x u + \partial_x^3 u = -GQ_\lambda(u,t), \qquad u(\cdot,0) = u_0$$

satisfies

$$||u(\cdot,t) - [u_0]||_s \leq \alpha_{s,\lambda,\mu}(||u_0 - [u_0]||_s)e^{-\lambda t}||u_0 - [u_0]||_s \qquad \text{for all } t \geq 0,$$

where $\alpha_{s,\lambda,\mu} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function depending on s, λ and μ.

The following remarks are in order.

Remark 1.5.

(i) In Theorem A, the control time T is independent of the initial state u_0 and the terminal state u_1 and can be, in fact, chosen arbitrarily small. By contrast, in Theorem 1.1, the control time T depends on the size of the initial state u_0 and the terminal state u_1 in the space $L^2(\mathbb{T})$. Whether the time T can be chosen independent of the size of u_0 and u_1 is an interesting open question.

(ii) While the decay rates κ in Theorem 1.2 and λ in Theorem 1.4 are independent of u_0, the constants $\alpha_{s,\mu}(\|u_0 - [u_0]\|_0)$ or $\alpha_{s,\lambda,\mu}(\|u_0 - [u_0]\|_s)$ are likely not uniformly bounded; i.e., it may happen that

$$\lim_{r\to\infty} \alpha_{s,\mu}(r) = \infty \text{ or } \lim_{r\to\infty} \alpha_{s,\lambda,\mu}(r) = \infty.$$

To prove our global controllability and stabilization results described above, we will as usual consider first the associated linear open-loop system

$$u_t + u_{xxx} = Gh \tag{1.10}$$

and the associated linear closed-loop system

$$u_t + u_{xxx} = -GQ_\lambda. \tag{1.11}$$

Without much difficulty we can show by using a standard approach in control theory of linear systems that the system (1.10) is exactly controllable in the space $H^s(\mathbb{T})$ and that the closed-loop system (1.11) is exponentially stable in the space $H^s(\mathbb{T})$ with an arbitrarily large decay rate λ. However, how to extend the linear results to the corresponding nonlinear systems is a challenging task. Indeed, after having published their linear results [37], Russell and Zhang had to wait for several years to extend their results to the nonlinear systems [38] until Bourgain [3] discovered a subtle smoothing property of solutions of the KdV equation posed on a periodic domain $\mathbb{T}$ when he showed surprisingly that the Cauchy problem of the KdV equation (1.2) is well-posed in the space $H^s(\mathbb{T})$ for $s \geq 0$. This then newly discovered smoothing property of the KdV equation has played a crucial role in the proofs of Theorem A and Theorem B in [38]. By contrast, establishing the global exact controllability and stabilizability for the nonlinear system (1.7) is even more challenging. After all, the results presented in Theorem A and Theorem B are essentially linear in nature; they are more or less small perturbation of the linear results. The global results presented in Theorem 1.1, Theorem 1.2 and Theorem 1.4 are truly nonlinear and their proofs demand new tools. The needed help turns out to be certain propagation properties of compactness and regularity for the KdV equation which are inspired by those established by Laurent in [20] for the Schrödinger equation. This strategy has already been successfully applied by Dehman, Lebeau, and Zuazua [11] for the wave equation, and by Dehman, Gérard, and Lebeau [10] and Laurent [20, 21] for the Schrödinger equation.

Note that for any solution u of the systems in consideration, its mean value $[u]$ is invariant. Thus it is convenient to introduce the number $\mu := [u] = [u_0]$, and to set

$$\tilde{u} = u - \mu.$$

Then $[\tilde{u}] = 0$ and $\tilde{u}$ solves

$$\partial_t \tilde{u} + \tilde{u}\tilde{u}_x + \partial_x^3 \tilde{u} + (\mu + \tilde{u})\partial_x \tilde{u} = Gh.$$

if u solves (1.5). Throughout the paper, μ will denote a given (real) constant, $H_0^s(\mathbb{T}) = \{u \in H^s(\mathbb{T});\ [u] = 0\}$, and $L_0^2(\mathbb{T}) = \{u \in L^2(\mathbb{T});\ [u] = 0\}$. We shall establish exponential stability results in $H_0^s(\mathbb{T})$ for the equation

$$\partial_t u + \partial_x^3 u + \mu \partial_x u + u \partial_x u = -K_\lambda u$$

that will imply all the results stated above.

The paper is outlined as follows.

- In Section 2, the exact controllability and stabilizability are presented for the associated linear systems.

- In Section 3, some preliminary results in Bourgain spaces, including the propagation of compactness and the propagation of regularity for the KdV equation, are provided.

- In Section 4, the stabilization of the KdV equation by a time invariant feedback control law is studied.

- In Section 5, the stabilization of the KdV equation by a time-varying feedback control law is investigated.

Finally we end our introduction with a few comments on the boundary controllability of the KdV equation posed on a finite interval $(0, L)$:

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad x \in (0, L),\ t > 0, \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t). \end{cases} \tag{1.12}$$

The problem was first investigated by Rosier [30] and has been intensively studied in the past decade. (See [30, 47, 31, 28, 32, 8, 26, 33, 22, 13, 5, 6, 23] and the references therein.) In contrast to control problems of other equations (parabolic equation or hyperbolic equations for instance), the boundary control system (1.12) has some interesting properties.

(i) If

$$L \in \mathcal{N} := \left\{ 2\pi\sqrt{\frac{j^2 + l^2 + jl}{3}}; j, l \in \mathbb{N}^* \right\},$$

the linear system

$$\begin{cases} u_t + u_x + u_{xxx} = 0, \quad x \in (0, L), \ t > 0, \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t). \end{cases}$$

associated to (1.12) is **not** exactly controllable if $h_1 = h_2 \equiv 0$. However, the nonlinear system (1.12) is locally exactly controllable (still with $h_1 = h_2 \equiv 0$) [30, 8, 5, 6].

(ii) The system (1.12) is exactly controllable from the right (using h_2 or h_3 as control inputs with $h_1 \equiv 0$), but **only** null controllable from the left (using h_1 as a control input with $h_2 = h_3 \equiv 0$). The system thus behaves like a parabolic system if control is acted only on the left end of the spatial domain and behaves like a hyperbolic system is control is allowed to act on the right end of the spatial domain [32, 13].

2 Linear Systems

Consideration is first given to the associate linear open loop control system

$$\partial_t v + \partial_x^3 v + \mu \partial_x v = Gh, \quad v(x, 0) = v_0(x), \quad x \in \mathbb{T}, \ t \in \mathbb{R}, \tag{2.1}$$

where the operator G is as defined in Section 1 and h is the applied control function.

Let A denote the operator

$$Aw = -w''' - \mu w'$$

with its domain $\mathcal{D}(A) = H^3(\mathbb{T})$. The operator A generates a strongly continuous group $W(t)$ on the space $L^2(\mathbb{T})$; the eigenfunctions are simply the orthonormal Fourier basis functions in $L^2(\mathbb{T})$,

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \qquad k = 0, \pm 1, \pm 2, \cdots.$$

The corresponding eigenvalue of ϕ_k is

$$\lambda_k = ik^3 - i\mu k, \ k = 0, \pm 1, \pm 2, \cdots.$$

For any $l \in \mathbb{Z}$, let

$$m(l) = \#\{k \in \mathbb{Z}; \ \lambda_k = \lambda_l\}.$$

In addition, $A^* = -A$, $G^* = G$ and $W^*(-t) = W(t)$ for any $t \in \mathbb{R}$. Using the gap condition

$$\lim_{|k| \to \infty} |\lambda_{k+1} - \lambda_k| = +\infty$$

and the fact that $m(l) \leq 3$ for any l and $m(l) = 1$ for $|l|$ large enough, we may deduce from Ingham lemma that the system (2.1) is exactly controllable in $H_0^s(\mathbb{T})$ in small time for any $s \geq 0$.

Theorem 2.1. *[38, Theorem 2.1 and Corollary 2.1] Let $s \geq 0$ and $T > 0$ be given. There exists a bounded linear operator*

$$\Phi : H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}) \mapsto L^2(0, T; H_0^s(\mathbb{T}))$$

such that for any $v_0,\ v_1 \in H_0^s(\mathbb{T})$,

$$W(T)v_0 + \int_0^T W(T-t)G(\Phi(v_0, v_1))(t)\, dt = v_1$$

and

$$\|\Phi(v_0, v_1)\|_{L^2(0,T;H^s(\mathbb{T}))} \leq C(\|v_0\|_s + \|v_1\|_s)$$

where $C > 0$ depends only on T and $\|g\|_s$.

The following estimate is a direct consequence of Theorem 2.1.

Corollary 2.2. *Let $T > 0$ be given. There exists $\delta > 0$ such that*

$$\int_0^T \|GW(t)\phi\|_0^2(t)dt \geq \delta \|\phi\|_0^2$$

for any $\phi \in L_0^2(\mathbb{T})$.

Note that the arguments presented in this paper give another proof of Corollary 2.2.

In addition, if one chooses the following simple feedback law

$$h(v) = -G^* v,$$

the resulting closed-loop system

$$\partial_t v + \partial_x^3 v + \mu \partial_x v = -GG^* v, \quad v(x, 0) = v_0(x), \quad x \in \mathbb{T} \tag{2.2}$$

is exponentially stable.

Proposition 2.3. *Let $s \geq 0$ be given. There exists a number $\kappa > 0$ independent of s such that for any $v_0 \in H_0^s(\mathbb{T})$, the corresponding solution v of (2.2) satisfies*

$$\|v(., t)\|_s \leq Ce^{-\kappa t}\|v_0\|_s$$

for any $t \geq 0$ where $C > 0$ is a constant depending only on s.

Proof. The case $s = 0$ has been proved in [37, Theorem 2]. We only provide the proof for the case $s = 3$. The case of $0 < s < 3$ follows by interpolation. The other cases of s can be proved similarly.

Pick any $v_0 \in H_0^3(\mathbb{T})$ and let $w = \partial_t v$. Then w solves

$$\partial_t w + \partial_x^3 w + \mu \partial_x w = -GG^* w, \quad w(x, 0) = w_0(x), \quad x \in \mathbb{T}$$

where $w_0(x) = -v_0'''(x) - \mu v_0' - GG^* v_0(x)$ belongs to $L_0^2(\mathbb{T})$. Thus

$$\|w(\cdot, t)\|_0 \|\partial_t v(\cdot, t)\|_0 \leq C_0 e^{-\kappa t}\|w_0\|_0$$

for any $t \geq 0$. From the equation

$$\partial_x^3 v + \mu \partial_x v = -w - GG^* v$$

it follows that

$$\|v(\cdot, t)\|_3 \leq C_3 e^{-\kappa t}\|v_0\|_3$$

for any $t \geq 0$. The proof is complete. □

Next we show that it is possible to choose an appropriate linear feedback law such that the decay rate of the resulting closed-loop system is as large as one desires.

For given $\lambda > 0$, define

$$L_\lambda \phi = \int_0^1 e^{-2\lambda\tau} W(-\tau) G G^* W^*(-\tau) \phi \, d\tau$$

for any $\phi \in H^s(\mathbb{T})$. Clearly, L_λ is a bounded linear operator from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$. Moreover, L_λ is a self-adjoint positive operator on $L_0^2(\mathbb{T})$, and so is its inverse L_λ^{-1}. L_λ is therefore an isomorphism from $L_0^2(\mathbb{T})$ onto itself. The following result claims that the same is true on $H_0^s(\mathbb{T})$.

Lemma 2.4. *L_λ is an isomorphism from $H_0^s(\mathbb{T})$ onto $H_0^s(\mathbb{T})$ for all $s \geq 0$.*

Proof. Since the result is known for $s = 0$, and L_λ maps $H_0^s(\mathbb{T})$ into itself, we only have to prove that for any $v \in L_0^2(\mathbb{T})$, $L_\lambda v \in H_0^s(\mathbb{T})$ implies $v \in H_0^s(\mathbb{T})$, i.e. $D^s v \in L^2(\mathbb{T})$. Using the continuity of L_λ^{-1} on $L_0^2(\mathbb{T})$ and a commutator estimate similar to [20, Lemma A.1], we obtain

$$\begin{aligned}
||D^s v||_0 &\leq C||L_\lambda D^s v||_0 \\
&\leq C||\int_0^1 e^{-2\lambda\tau} W(-\tau) G G^* W^*(-\tau) D^s v \, d\tau||_0 \\
&\leq C||D^s \int_0^1 e^{-2\lambda\tau} W(-\tau) G G^* W^*(-\tau) v \, d\tau||_0 \\
&\quad + C||\int_0^1 e^{-2\lambda\tau} W(-\tau) [G G^*, D^s] W^*(-\tau) v \, d\tau||_0 \\
&\leq C||L_\lambda v||_s + C_s ||v||_{s-1}.
\end{aligned}$$

The result follows at once for $s \in [0, 1]$. An induction yields the result for any $s \geq 0$. □

Choose the feedback control

$$h = -G^* L_\lambda^{-1} v.$$

The resulting closed-loop system reads:

$$\partial_t v + \partial_x^3 v + \mu \partial_x v = -K_\lambda v, \quad v(x, 0) = v_0(x), \; x \in \mathbb{T}, \tag{2.3}$$

with

$$K_\lambda := G G^* L_\lambda^{-1}.$$

If $\lambda = 0$, we define $K_0 = G G^*$.

Proposition 2.5. *Let $s \geq 0$ and $\lambda > 0$ be given. For any $v_0 \in H_0^s(\mathbb{T})$, the system (2.3) admits a unique solution $v \in C(\mathbb{R}^+; H_0^s(\mathbb{T}))$. Moreover, there exists $M = M_s$ depending on s such that*

$$\|v(., t)\|_s \leq M_s e^{-\lambda t} \|v_0\|_s$$

for any $t \geq 0$.

Proof. The case $s = 0$ follows from [41, Theorem 2.1]. The other cases of s are proved as for Proposition 2.3. □

3 Preliminaries

In this section we present some results which are essential to establish the exact controllability and stabilizability of the nonlinear systems.

3.1 The Bourgain space and its properties.

For given $b, s \in \mathbb{R}$, and a function $u : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, define the quantities

$$
\begin{aligned}
||u||_{X_{b,s}} &:= \left(\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \langle k \rangle^{2s} \left\langle \tau - k^3 + \mu k \right\rangle^{2b} \left| \widehat{\widehat{u}}(k,\tau) \right|^2 d\tau \right)^{\frac{1}{2}}, \\
||u||_{Y_{b,s}} &:= \left(\sum_{k=-\infty}^{\infty} \Big(\int_{\mathbb{R}} \langle k \rangle^{s} \left\langle \tau - k^3 + \mu k \right\rangle^{b} \left| \widehat{\widehat{u}}(k,\tau) \right| d\tau \Big)^2 \right)^{\frac{1}{2}}
\end{aligned}
$$

where $\widehat{\widehat{u}}(k,\tau)$ denotes the Fourier transform of u with respect to the space variable x and the time variable t (by contrast, $\widehat{u}(k,t)$ denotes the Fourier transform in space variable x) and $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$. Moreover, denote by D^r the operator defined on $\mathcal{D}'(\mathbb{T})$ by

$$
\begin{aligned}
\widehat{D^r u}(k) &= |k|^r \widehat{u}(k) \quad &\text{if} \quad k \neq 0, \\
&= \widehat{u}(0) \quad &\text{if} \quad k = 0.
\end{aligned} \tag{3.1}
$$

The Bourgain space $X_{b,s}$ (resp. $Y_{b,s}$) associated to the KdV equation on $\mathbb{T}$ is the completion of the space $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ under the norm $\|u\|_{X_{b,s}}$ (resp. $\|u\|_{Y_{b,s}}$). Note that for any $u \in X_{b,s}$,

$$\|u\|_{X_{b,s}} = \|W(-t)u\|_{H^b(\mathbb{R}, H^s(\mathbb{T}))}.$$

For given $b, s \in \mathbb{R}$, let

$$Z_{b,s} = X_{b,s} \cap Y_{b-\frac{1}{2},s}$$

be endowed with the norm

$$||u||_{Z_{b,s}} ||u||_{X_{b,s}} + ||u||_{Y_{b-\frac{1}{2},s}}.$$

For a given interval I, let $X_{b,s}(I)$ (resp. $Z_{b,s}(I)$) be the restriction space of $X_{b,s}$ to the interval I with the norm

$$
\begin{aligned}
\|u\|_{X_{b,s}(I)} &= \inf \left\{ \|\widetilde{u}\|_{X_{b,s}} \mid \widetilde{u} = u \text{ on } \mathbb{T} \times I \right\} \\
(\text{resp. } \|u\|_{Z_{b,s}(I)} &= \inf \left\{ \|\widetilde{u}\|_{Z_{b,s}} \mid \widetilde{u} = u \text{ on } \mathbb{T} \times I \right\}).
\end{aligned}
$$

For simplicity, we denote $X_{b,s}(I)$ (resp. $Z_{b,s}(I)$) by $X^T_{b,s}$ (resp. $Z^T_{b,s}$) if $I = (0,T)$. The following properties of the spaces $X^T_{b,s}$ are $Z^T_{b,s}$ are easily verified.

(i) $X_{b,s}(I)$ is a Hilbert space.

(ii) $D^r u \in X_{b,s-r}(I)$ for any $u \in X_{b,s}(I)$.

(iii) If $b_1 \leq b_2$ and $s_1 \leq s_2$, then X_{b_2,s_2} is continuously imbedded in the space X_{b_1,s_1}.

(iv) For a given finite interval I, if $b_1 < b_2$ and $s_1 < s_2$, then the space $X_{b_2,s_2}(I)$ is compactly imbedded in the space $X_{b_1,s_1}(I)$.

(v) $Z_{\frac{1}{2},s}(I) \subset C(\overline{I}; H^s(\mathbb{T}))$ for any $s \in \mathbb{R}$.

Lemma 3.1. *Let $b, s \in \mathbb{R}$ and $T > 0$ be given. There exists a constant $C > 0$ such that* *(i) for any $\phi \in H^s(\mathbb{T})$,*

$$\begin{aligned}\|W(t)\phi\|_{X^T_{b,s}} &\le C\|\phi\|_s;\\ \|W(t)\phi\|_{Z^T_{b,s}} &\le C\|\phi\|_s;\end{aligned}$$

(ii) for any $f \in X^T_{b-1,s}$,

$$\left\|\int_0^t W(t-\tau)f(\tau)d\tau\right\|_{X^T_{b,s}} \le C\|f\|_{X^T_{b-1,s}}$$

provided that $b > \frac{1}{2}$;

(iii) for any $f \in Z^T_{-\frac{1}{2},s}$,

$$\|\int_0^t W(t-\tau)f(\tau)\,d\tau\|_{Z^T_{\frac{1}{2},s}} \le C\|f\|_{Z^T_{-\frac{1}{2},s}}.$$

Proof. See e.g. [42] or [7]. □

Lemma 3.2. *(Strichartz estimates) The following estimates hold:*

$$||\sum_{k,l\in\mathbb{Z}} c_{k,l}e^{i(kx+lt)}||_{L^4(\mathbb{T}^2)} \le C\left(\sum_{k,l\in\mathbb{Z}}(1+|l-k^3+\mu k|)^{\frac{2}{3}}|c_{k,l}|^2\right)^{\frac{1}{2}}, \tag{3.2}$$

$$||u||_{L^4(\mathbb{T}^2)} \le C||u||_{X_{\frac{1}{3},0}}, \tag{3.3}$$

$$||u||_{L^4(\mathbb{T}\times(0,T))} \le C||u||_{X^T_{\frac{1}{3},0}}. \tag{3.4}$$

Proof. (3.2) comes from [3, Proposition 7.15]. To prove (3.3), pick any $u \in X_{\frac{1}{3},0}$ decomposed as

$$u(x,t) = \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \widehat{\widehat{u}}(k,\tau)e^{i(kx+\tau t)}d\tau.$$

Writing $\tau = l+\sigma$ with $l \in \mathbb{Z}$, $\sigma \in [0,1)$, we have that

$$u(x,t) = \int_0^1 e^{i\sigma t}\sum_{k,l\in\mathbb{Z}}\widehat{\widehat{u}}(k,l+\sigma)e^{i(kx+lt)}d\sigma.$$

Using (3.2) and Cauchy-Schwarz, we obtain

$$\begin{aligned}||u||_{L^4(\mathbb{T}^2)} &\le \int_0^1 ||\sum_{k,l\in\mathbb{Z}}\widehat{\widehat{u}}(k,l+\sigma)e^{i(kx+lt)}||_{L^4(\mathbb{T}^2)}\,d\sigma\\
&\le C\int_0^1\left(\sum_{k,l\in\mathbb{Z}}(1+|l-k^3+\mu k|)^{\frac{2}{3}}|\widehat{\widehat{u}}(k,l+\sigma)|^2\right)^{\frac{1}{2}}d\sigma\\
&\le C\left(\sum_{k\in\mathbb{Z}}\int_0^1\sum_{l\in\mathbb{Z}}(1+|l-k^3+\mu k|)^{\frac{2}{3}}|\widehat{\widehat{u}}(k,l+\sigma)|^2d\sigma\right)^{\frac{1}{2}}\\
&\le C\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}(1+|\tau-k^3+\mu k|)^{\frac{2}{3}}|\widehat{\widehat{u}}(k,\tau)|^2d\tau\right)^{\frac{1}{2}}.\end{aligned}$$

It remains to establish (3.4). Let $T>0$ and $u\in X^T_{\frac13,0}$. Pick $p\in\mathbb{N}^*$ with $T\le 2\pi p$, and an extension $\tilde u\in X_{\frac13,0}$ of u with $||\tilde u||_{X_{\frac13,0}}\le 2||u||_{X^T_{\frac13,0}}$. Then

$$||u||^4_{L^4(\mathbb{T}\times(0,T))}\le ||\tilde u||^4_{L^4(\mathbb{T}\times(0,2\pi p))}\le p(C||\tilde u||_{X_{\frac13,0}})^4\le C'||u||^4_{X^T_{\frac13,0}}.$$

Note that C' depends only on T. $\square$

Lemma 3.3 (Bilinear estimates)**.** *Let $s\ge 0$, $T\in(0,1)$, and $u,v\in X^T_{\frac12,s}\cap L^2(0,T;L^2_0(\mathbb{T}))$. Then there exist some constants $\theta>0$ and $C>0$ independent of T and u,v such that*

$$\|(uv)_x\|_{Z^T_{-\frac12,s}}\le CT^\theta\|u\|_{X^T_{\frac12,s}}\|v\|_{X^T_{\frac12,s}} \tag{3.5}$$

The proof of Lemma 3.3 can be found in [3] with $\theta=1/12$ (see also [7]).

To end this section, we prove a multiplication property of the Bourgain space $X^T_{b,s}$. If $\psi=\psi(t)$ is any C^∞ function, then $\psi u\in X^T_{b,s}$ for any $u\in X^T_{b,s}$. However, if $\phi=\phi(x)\in C^\infty(\mathbb{T})$, then ϕu may not belong to the space $X^T_{b,s}$ for $u\in X^T_{b,s}$. Some regularity in the index b is lost due to the fact that the multiplication by a (smooth) function of x does not keep the structure in time of the harmonics. This loss is, in fact, unavoidable. For instance, for $k\ge1$, let $u_k=\psi(t)e^{ikx}e^{i(k^3-\mu k)t}$, where $\psi\in C_0^\infty(\mathbb{R})$ takes the value 1 on $[-1,1]$. The sequence $\{u_k\}$ is uniformly bounded in the space $X_{b,0}$ for every $b\ge0$. However, multiplying u_k by $\phi(x)=e^{ix}$, we observe that $\left\|e^{ix}u_k\right\|_{X_{b,0}}\approx k^{2b}$.

The next lemma shows that this is the worst case.

Lemma 3.4. *Let $-1\le b\le 1$, $s\in\mathbb{R}$ and $\varphi\in C^\infty(\mathbb{T})$. Then, for any $u\in X_{b,s}$, $\varphi(x)u\in X_{b,s-2|b|}$. Similarly, the multiplication by φ maps $X^T_{b,s}$ into $X^T_{b,s-2|b|}$.*

Proof. We first consider the case of $b=0$ and $b=1$. The other cases of b will be derived later by interpolation and duality.

For $b=0$, $X_{0,s}=L^2(\mathbb{R},H^s(\mathbb{T}))$ and the result is obvious. For $b=1$, note that $u\in X_{1,s}$ if and only if

$$u\in L^2(\mathbb{R},H^s(\mathbb{T}))\text{ and }\partial_t u+\partial_x^3u+\mu\partial_x u\in L^2(\mathbb{R},H^s(\mathbb{T})),$$

and that

$$\|u\|^2_{X_{1,s}}=\|u\|^2_{L^2(\mathbb{R},H^s(\mathbb{T}))}+\left\|\partial_t u+\partial_x^3u+\mu\partial_x u\right\|^2_{L^2(\mathbb{R},H^s(\mathbb{T}))}.$$

Thus,

$$\begin{aligned}
\|\varphi(x)u\|^2_{X_{1,s-2}} &= \|\varphi u\|^2_{L^2(\mathbb{R},H^{s-2}(\mathbb{T}))}+\left\|\partial_t(\varphi u)+\partial_x^3(\varphi u)+\mu\partial_x(\varphi u)\right\|^2_{L^2(\mathbb{R},H^{s-2}(\mathbb{T}))}\\
&\le C\Big(\|u\|^2_{L^2(\mathbb{R},H^{s-2}(\mathbb{T}))}+\left\|\varphi\left(\partial_t u+\partial_x^3u+\mu\partial_x u\right)\right\|^2_{L^2(\mathbb{R},H^{s-2}(\mathbb{T}))}\\
&\quad+\left\|\left[\varphi,\partial_x^3+\mu\partial_x\right]u\right\|^2_{L^2(\mathbb{R},H^{s-2}(\mathbb{T}))}\Big)\\
&\le C\Big(\|u\|^2_{L^2(\mathbb{R},H^{s-2}(\mathbb{T}))}+\left\|\partial_t u+\partial_x^3u+\mu\partial_x u\right\|^2_{L^2(\mathbb{R},H^{s-2}(\mathbb{T}))}+\|u\|^2_{L^2(\mathbb{R},H^{s}(\mathbb{T}))}\Big)\\
&\le C\,\|u\|^2_{X_{1,s}}.
\end{aligned}$$

Here, we have used the fact that

$$\left[\varphi,\partial_x^3+\mu\partial_x\right]=-3(\partial_x\varphi)\partial_x^2-3(\partial_x^2\varphi)\partial_x-\partial_x^3\varphi-\mu\partial_x\varphi$$

is a differential operator of order 2. To conclude, we prove that the $X_{b,s}$ spaces are in interpolation. First, using Fourier transform, $X_{b,s}$ may be viewed as the weighted L^2 space $L^2(\mathbb{R}_\tau \times \mathbb{Z}_k, \langle k\rangle^{2s}\langle \tau - k^3 + \mu k\rangle^{2b}\lambda\otimes\delta)$, where λ is the Lebesgue measure on $\mathbb{R}$ and δ is the discrete measure on $\mathbb{Z}$. Then, we use the complex interpolation theorem of Stein-Weiss for weighted L^p spaces (see [1, p. 114]): for $0<\theta<1$

$$\left(X_{0,s}, X_{1,s'}\right)_{[\theta]} \approx L^2\left(\mathbb{R}\times\mathbb{Z}, \langle k\rangle^{2s(1-\theta)+2s'\theta}\left\langle \tau - k^3+\mu k\right\rangle^{2\theta}\mu\otimes\delta\right) \approx X_{\theta, s(1-\theta)+s'\theta}.$$

Since the multiplication by φ maps $X_{0,s}$ into $X_{0,s}$ and $X_{1,s}$ into $X_{1,s-2}$, we conclude that for $0\le b\le 1$, it maps $X_{b,s} = (X_{0,s}, X_{1,s})_{[b]}$ into $(X_{0,s}, X_{1,s-2})_{[b]} = X_{b,s-2b}$, which yields the $2b$ loss of regularity as announced.
Then, by duality, this also implies that for $0\le b\le 1$, the multiplication by $\varphi(x)$ maps $X_{-b,-s+2b}$ into $X_{-b,-s}$. As the number s may take arbitrary values in $\mathbb{R}$, we also have the result for $-1\le b\le 0$ with a loss of $-2b = 2|b|$.
To get the same result for the restriction spaces $X^T_{b,s}$, we write the estimate for an extension $\tilde{u}$ of u, which yields

$$\|\varphi u\|_{X^T_{b,s-2|b|}} \le \|\varphi\tilde{u}\|_{X_{b,s-2|b|}} \le C\,\|\tilde{u}\|_{X_{b,s}}.$$

Taking the infimum on all the $\tilde{u}$, we get the claimed result. □

3.2 Propagation of compactness and regularity

In this subsection, we present some properties of propagation of compactness and regularity for the linear differential operator $L\partial_t + \partial_x^3 + \mu\partial_x$ associated with the KdV equation. Those propagation properties will play a key role when studying the global stabilizability of the KdV equation.

Proposition 3.5. *Let $T>0$ and $0\le b'\le b\le 1$ be given (with $b>0$) and suppose that $u_n \in X^T_{b,0}$ and $f_n\in X^T_{-b,-2+2b}$ satisfy*

$$\partial_t u_n + \partial_x^3 u_n + \mu\partial_x u_n = f_n$$

for $n=1,2,...$ Assume that there exists a constant $C>0$ such that

$$\|u_n\|_{X^T_{b,0}} \le C \qquad \text{for all } n\ge 1, \tag{3.6}$$

and that

$$\|u_n\|_{X^T_{-b,-2+2b}} + \|f_n\|_{X^T_{-b,-2+2b}} + \|u_n\|_{X^T_{-b',-1+2b'}} \to 0 \text{ as } n\to\infty. \tag{3.7}$$

In addition, assume that for some nonempty open set $\omega\subset\mathbb{T}$ it holds

$$u_n \to 0 \text{ strongly in } L^2(0,T;L^2(\omega)).$$

Then

$$u_n\to 0 \text{ strongly in } L^2_{loc}((0,T);L^2(\mathbb{T})).$$

Proof. Pick $\varphi\in C^\infty(\mathbb{T})$ and $\psi\in C_0^\infty((0,T))$ real valued and set

$$B = \varphi(x)D^{-2} \text{ and } A = \psi(t)B.$$

Then

$$A^* = \psi(t)D^{-2}\varphi(x).$$

For $\varepsilon > 0$, let $A_\varepsilon = Ae^{\varepsilon\partial_x^2} = \psi(t)B_\varepsilon$ be a regularization of A. Then

$$\begin{aligned}\alpha_{n,\varepsilon} &:= ([A_\varepsilon, L]u_n, u_n)_{L^2(\mathbb{T}\times(0,T))} \\ &= ([A_\varepsilon, \partial_x^3 + \mu\partial_x]u_n, u_n) - (\psi'(t)B_\varepsilon u_n, u_n).\end{aligned}$$

On the other hand,

$$\alpha_{n,\varepsilon} = (f_n, A_\varepsilon^* u_n)_{L^2(\mathbb{T}\times(0,T))} + (A_\varepsilon u_n, f_n)_{L^2(\mathbb{T}\times(0,T))}$$

since $Lu_n = f_n$ and $L^* = -L$. By Lemma 3.4,

$$\begin{aligned}\left|(f_n, A_\varepsilon^* u_n)_{L^2(\mathbb{T}\times(0,T))}\right| &\leq \|f_n\|_{X^T_{-b,-2+2b}} \|A_\varepsilon^* u_n\|_{X^T_{b,2-2b}} \\ &\leq \|f_n\|_{X^T_{-b,-2+2b}} \|u_n\|_{X^T_{b,0}} \qquad (3.8)\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \sup_{0<\varepsilon\leq 1} \left|(f_n, A_\varepsilon^* u_n)_{L^2(\mathbb{T}\times(0,T))}\right| = 0.$$

Similarly, we have

$$\lim_{n\to\infty} \sup_{0<\varepsilon\leq 1} \left|(A_\varepsilon u_n, f_n)_{L^2(\mathbb{T}\times(0,T))}\right| = 0$$

and

$$\lim_{n\to\infty} \sup_{0<\varepsilon\leq 1} \left|(\psi'(t)B_\varepsilon u_n, u_n)\right| = 0.$$

Thus

$$\lim_{n\to\infty} \sup_{0<\varepsilon\leq 1} |\alpha_{n,\varepsilon}| = 0$$

and therefore

$$\lim_{n\to\infty} \sup_{0<\varepsilon\leq 1} \left|([A_\varepsilon, \partial_x^3 + \mu\partial_x]u_n, u_n)\right| = 0.$$

In particular,

$$\lim_{n\to\infty} ([A, \partial_x^3 + \mu\partial_x]u_n, u_n)_{L^2(\mathbb{T}\times(0,T))} = 0.$$

As D^{-2} commutes with ∂_x, we have

$$[A, \partial_x^3 + \mu\partial_x] = -3\psi(t)(\partial_x\varphi)\partial_x^2 D^{-2} - 3\psi(t)(\partial_x^2\varphi)\partial_x D^{-2} - \psi(t)(\partial_x^3\varphi + \mu\partial_x\varphi)D^{-2}.$$

Using the same argument as in (3.8), we get

$$(\psi(t)(\partial_x^3\varphi + \mu\partial_x\varphi)D^{-2}u_n, u_n)_{L^2(\mathbb{T}\times(0,T))} \to 0.$$

However, for the second term, the loss of regularity is too large if we use the estimates with the same b. Using the index b' instead, we have

$$\begin{aligned}(\psi(t)(\partial_x^2\varphi)\partial_x D^{-2}u_n, u_n) &\leq \|\psi(t)(\partial_x^2\varphi)\partial_x D^{-2}u_n\|_{X^T_{b',1-2b'}} \|u_n\|_{X^T_{-b',-1+2b'}} \\ &\leq \|u_n\|_{X^T_{b',0}} \|u_n\|_{X^T_{-b',-1+2b'}}\end{aligned}$$

which tends to 0 as $n \to \infty$, by (3.6)-(3.7). Note that $-\partial_x^2 D^{-2}$ is the orthogonal projection on the subspace of functions with $\widehat{u}(0) = 0$. Using Rellich Theorem combined to the fact that $b > 0$, we easily see that $\widehat{u_n}(0,t)$ tends to 0 in $L^2(0,T)$ (strongly), and hence

$$(\psi(t)(\partial_x \varphi)\widehat{u_n}(0,t), u_n)_{L^2(\mathbb{T}\times(0,T))} \to 0.$$

We have thus proved that for any $\varphi \in C^\infty(\mathbb{T})$ and any $\psi \in C_0^\infty((0,T))$

$$(\psi(t)(\partial_x \varphi)u_n, u_n)_{L^2(\mathbb{T}\times(0,T))} \to 0.$$

Note that a function $\phi \in C^\infty(\mathbb{T})$ can be written in the form $\partial_x \varphi$ for some function $\varphi \in C^\infty(\mathbb{T})$ if and only if $\int_{\mathbb{T}} \phi(x)\,dx = 0$. Thus, for any $\chi \in C_0^\infty(\omega)$ and any $x_0 \in \mathbb{T}$, $\phi(x)\chi(x) - \chi(x - x_0)$ can be written as $\phi = \partial_x \varphi$ for some $\varphi \in C^\infty(\mathbb{T})$.

Since u_n is strongly convergent to 0 in $L^2(0,T;L^2(\omega))$,

$$\lim_{n\to\infty} (\psi(t)\chi u_n, u_n)_{L^2(\mathbb{T}\times(0,T))} = 0.$$

Therefore, for any $x_0 \in \mathbb{T}$,

$$\lim_{n\to\infty} (\psi(t)\chi(\cdot - x_0)u_n, u_n)_{L^2(\mathbb{T}\times(0,T))} = 0.$$

The proof is then completed by constructing a partition of unity of $\mathbb{T}$ involving functions of the form $\chi_i(\cdot - x_0^i)$ with $\chi_i \in C_0^\infty(\omega)$ and $x_0^i \in \mathbb{T}$. □

Next we investigate the propagation of regularity for the operator $L = \partial_t + \partial_x^3 + \mu\partial_x$.

Proposition 3.6. *Let $T > 0$, $0 \le b < 1$, $r \in \mathbb{R}$ and $f \in X_{-b,r}^T$ be given. Let $u \in X_{b,r}^T$ be a solution of*

$$\partial_t u + \partial_x^3 u + \mu\partial_x u = f.$$

If there exists a nonempty open set ω of $\mathbb{T}$ such that $u \in L^2_{loc}((0,T), H^{r+\rho}(\omega))$ for some ρ with

$$0 < \rho \le \min\{1 - b, \frac{1}{2}\},$$

then $u \in L^2_{loc}((0,T), H^{r+\rho}(\mathbb{T}))$.

Proof. Set $s = r + \rho$ and for $n = 1, 2, ...$

$$u_n = e^{\frac{1}{n}\partial_x^2} u =: \Xi_n u,\ f_n = \Xi_n f = L u_n.$$

There exists a constant $C > 0$ such that

$$\|u_n\|_{X_{b,r}^T} \le C, \quad \|f_n\|_{X_{-b,r}^T} \le C \qquad \forall\, n \ge 1.$$

Pick $\varphi \in C^\infty(\mathbb{T})$ and $\psi \in C_0^\infty((0,T))$ as in the proof of Proposition 3.5, and set

$$B = D^{2s-2}\varphi(x) \text{ and } A = \psi(t)B.$$

We have

$$\begin{aligned} &(Lu_n, A^* u_n)_{L^2(\mathbb{T}\times(0,T))} + (Au_n, Lu_n)_{L^2(\mathbb{T}\times(0,T))} \\ = \;& ([A, \partial_x^3 + \mu\partial_x]u_n, u_n)_{L^2(\mathbb{T}\times(0,T))} - (\psi'(t)Bu_n, u_n) \end{aligned}$$

$$
\begin{aligned}
|(Au_n, f_n)_{L^2(\mathbb{T}\times(0,T))}| &\leq \|Au_n\|_{X^T_{b,-r}}\|f_n\|_{X^T_{-b,r}} \\
&\leq \|u_n\|_{X^T_{b,r+2\rho-2+2b}}\|f_n\|_{X^T_{-b,r}} \\
&\leq C\|u_n\|_{X^T_{b,r}}\|f_n\|_{X^T_{-b,r}} \\
&\leq C
\end{aligned}
$$

since $r+2\rho-2+2b \leq r$. The same estimates for the other terms imply that

$$
\left|([A,\partial_x^3+\mu\partial_x]u_n, u_n)_{L^2(\mathbb{T}\times(0,T))}\right| \leq C.
$$

Note that

$$
[A,\partial_x^3+\mu\partial_x] = -3\psi(t)D^{2s-2}(\partial_x\varphi)\partial_x^2 - 3\psi(t)D^{2s-2}(\partial_x^2\varphi)\partial_x - \psi(t)D^{2s-2}(\partial_x^3\varphi+\mu\partial_x\varphi)
$$

and $2s-2+1 = 2r+2\rho-1 \leq 2r$. We have

$$
\begin{aligned}
&|(\psi(t)D^{2s-2}(\partial_x^2\varphi)\partial_x u_n, u_n)_{L^2(\mathbb{T}\times(0,T))}| \\
&\quad \leq C\|\psi(t)D^{2s-2}(\partial_x^2\varphi)\partial_x u_n\|_{L^2(0,T;H^{-r}(\mathbb{T}))}\|u_n\|_{L^2(0,T;H^r(\mathbb{T}))} \\
&\quad \leq C\|u_n\|^2_{L^2(0,T;H^r(\mathbb{T}))} \\
&\quad \leq C
\end{aligned}
$$

and

$$
\begin{aligned}
&|(\psi(t)D^{2s-2}(\partial_x^3\varphi+\mu\partial_x\varphi)u_n, u_n)_{L^2(\mathbb{T}\times(0,T))}| \\
&\quad \leq C\|\psi(t)D^{2s-2}(\partial_x^3\varphi+\mu\partial_x\varphi)u_n\|_{L^2(0,T;H^{-r}(\mathbb{T}))}\|u_n\|_{L^2(0,T;H^r(\mathbb{T}))} \\
&\quad \leq C\|u_n\|^2_{L^2(0,T;H^r(\mathbb{T}))} \\
&\quad \leq C
\end{aligned}
$$

for any $n \geq 1$. Thus

$$
|(\psi(t)D^{2s-2}(\partial_x\varphi)\partial_x^2 u_n, u_n)| \leq C. \tag{3.9}
$$

For any $\chi \in C_0^\infty(\omega)$,

$$
\begin{aligned}
&(\psi(t)D^{2s-2}\chi^2\partial_x^2 u_n, u_n) \\
&\quad = (\psi(t)D^{s-2}\chi\partial_x^2 u_n, \chi D^s u_n) + (\psi(t)[D^{s-2},\chi]\chi\partial_x^2 u_n, D^s u_n) \\
&\quad = (\psi(t)D^{s-2}\chi\partial_x^2 u_n, D^s\chi u_n) + (\psi(t)D^{s-2}\chi\partial_x^2 u_n, [\chi, D^s]u_n) \\
&\qquad +(\psi(t)[D^{s-2},\chi]\chi\partial_x^2 u_n, D^s u_n) =: I_1 + I_2 + I_3.
\end{aligned}
$$

We infer from the assumptions that $\chi u \in L^2_{loc}((0,T), H^s(\mathbb{T}))$ and that $\chi\partial_x^2 u \in L^2_{loc}((0,T), H^{s-2}(\mathbb{T}))$. Thus

$$
\chi u_n = \Xi_n \chi u + [\chi, \Xi_n]u
$$

is uniformly bounded in $L^2_{loc}((0,T), H^s(\mathbb{T}))$ by [20, Lemma A.3] and the fact that $s \leq r+1$. Applying the same argument to $\chi\partial_x^2 u_n$, we obtain

$$
|I_1| \leq C.
$$

It follows from [20, Lemma A.1] and the fact that $u \in L^2(0,T;H^r(\mathbb{T}))$ that

$$
\begin{aligned}
|I_2| &\leq C\left\|D^{r-2}\chi\partial_x^2 u_n\right\|_{L^2(0,T;L^2(\mathbb{T}))}\left\|D^\rho[\chi, D^s]u_n\right\|_{L^2(0,T;L^2(\mathbb{T}))} \\
&\leq C\left\|u_n\right\|_{L^2(0,T;H^r(\mathbb{T}))}\left\|u_n\right\|_{L^2(0,T;H^{s-1+\rho}(\mathbb{T}))} \leq C.
\end{aligned}
$$

A similar bound may be obtained for $|I_3|$. Consequently,

$$\left|(\psi(t)D^{2s-2}\chi^2\partial_x^2 u_n, u_n)\right| \le C$$

for any $n \ge 1$. Then, using (3.9) with $\partial_x\varphi = \chi^2(x) - \chi^2(x - x_0)$ yields

$$\left|(\psi(t)D^{2s-2}\chi^2(\cdot - x_0)\partial_x^2 u_n, u_n)\right| \le C$$

for any $n \ge 1$. Using a partition of unity as in the proof of Proposition 3.5, we obtain

$$|(\psi(t)D^{2s-2}\partial_x^2 u, u)| \le C,$$

that is

$$\int_0^T \psi(t)\big(\sum_{k\neq 0} |k|^{2s}|\widehat{u}(k,t)|^2\big)\, dt \le C.$$

The proof is thus complete. □

Corollary 3.7. *Let $u \in X^T_{\frac{1}{2},0}$ be a solution of*

$$\partial_t u + \partial_x^3 u + \mu\partial_x u + u\partial_x u = 0 \text{ on } \mathbb{T}\times(0,T). \tag{3.10}$$

Assume that $u \in C^\infty(\omega\times(0,T))$, where ω is a nonempty open set in $\mathbb{T}$. Then $u \in C^\infty(\mathbb{T}\times(0,T))$.

Proof. Recall that the mean value $[u]$ is conserved. Changing μ into $\mu + [u]$ if needed, we may assume that $[u] = 0$. We have $u\partial_x u \in X^T_{-\frac{1}{2},0}$ by Lemma 3.3. It follows from Proposition 3.6 that $u \in L^2_{loc}((0,T), H^{\frac{1}{2}}(\mathbb{T}))$. Choose t_0 such that $u(t_0) \in H^{\frac{1}{2}}(\mathbb{T})$. We can then solve (3.10) in $X^T_{\frac{1}{2},\frac{1}{2}}$ with the initial data $u(t_0)$. By uniqueness of the solution in $X^T_{\frac{1}{2},0}$, we conclude that $u \in X^T_{\frac{1}{2},\frac{1}{2}}$. An iterated application of Proposition 3.6 yields that $u \in L^2(0,T; H^r(\mathbb{T}))$ for every $r \in \mathbb{R}$, and hence $u \in C^\infty(\mathbb{T}\times(0,T))$. □

Corollary 3.8. *Let ω be a nonempty open set in $\mathbb{T}$ and let $u \in X^T_{\frac{1}{2},0}$ be a solution of*

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu\partial_x u + u\partial_x u &= 0 \quad \text{on } \mathbb{T}\times(0,T) \\ u &= c \quad \text{on } \omega\times(0,T) \end{cases}$$

where $c \in \mathbb{R}$ denotes some constant. Then $u(x,t) = c$ on $\mathbb{T}\times(0,T)$

Proof. Using Corollary 3.7, we infer that $u \in C^\infty(\mathbb{T}\times(0,T))$. It follows that $u \equiv c$ on $\mathbb{T}\times(0,T)$ by the unique continuation property for the KdV equation (see [40, 32]). □

4 Nonlinear systems

In this section, we are concerned with the stability properties of the closed loop system

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu\partial_x u + u\partial_x u = -K_\lambda u, \qquad x \in \mathbb{T},\ 0 < t < T, \\ \\ u(x,0) = u_0(x), \qquad x \in \mathbb{T}, \end{cases} \tag{4.1}$$

where $\lambda \ge 0$ is a given number and $u_0 \in L^2_0(\mathbb{T})$.

We first check that the system is globally well-posed in the space $H^s_0(\mathbb{T})$ for any $s \ge 0$.

Theorem 4.1. *Let $\lambda \geq 0$ and $s \geq 0$ be given. Then for any $T > 0$ and any $u_0 \in H_0^s(\mathbb{T})$, there exists a unique solution $u \in Z^T_{\frac{1}{2},s} \cap C([0,T]; L^2_0(\mathbb{T}))$ of (4.1). Furthermore, the following estimate holds*

$$\|u\|_{Z^T_{\frac{1}{2},s}} \leq \alpha_{T,s}(\|u_0\|_0)\|u_0\|_s \tag{4.2}$$

where $\alpha_{T,s} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function depending only on T and s.

Proof. We shall first establish the existence and uniqueness of a solution $u \in Z^T_{\frac{1}{2},s} \cap L^2(0,T; L^2_0(\mathbb{T}))$ of (4.1) for $T > 0$ small enough. Then we shall show that T can be taken as large as one wishes.

Let $u_0 \in H_0^s(\mathbb{T})$. Rewrite system (4.1) in its integral form

$$u(t) = W(t)u_0 - \int_0^t W(t-\tau)(u\partial_x u)(\tau)d\tau - \int_0^t W(t-\tau)[K_\lambda u](\tau)d\tau \tag{4.3}$$

where $W(t) = e^{-t(\partial_x^3 + \mu\partial_x)}$. For given u_0, define the map

$$\Gamma(v) = W(t)u_0 - \int_0^t W(t-\tau)(v\partial_x v)(\tau)d\tau - \int_0^t W(t-\tau)[K_\lambda v](\tau)d\tau.$$

The following estimate is needed.

Lemma 4.2. *For any $\varepsilon > 0$ there exists a positive constant $C(\varepsilon)$ such that*

$$|| \int_0^t W(t-\tau)[K_\lambda v](\tau)d\tau||_{Z^T_{\frac{1}{2},s}} \leq C(\varepsilon)T^{1-\varepsilon}||v||_{Z^T_{\frac{1}{2},s}}. \tag{4.4}$$

Proof of Lemma 4.2. Let $v \in Z^T_{\frac{1}{2},s}$. Pick an extension of v to $\mathbb{T} \times \mathbb{R}$, still denoted by v, and such that

$$||v||_{Z_{\frac{1}{2},s}} \leq 2||v||_{Z^T_{\frac{1}{2},s}}.$$

Pick any $\eta \in C^\infty(\mathbb{R})$ with $\eta(t) = 1$ for $|t| \leq 1$ and $\eta(t) = 0$ for $|t| \geq 2$. By Lemma 3.1, it is clearly sufficient to prove that

$$||\eta^2(t/T)K_\lambda v||_{Z_{-\frac{1}{2},s}} \leq CT^{1-\varepsilon}||v||_{Z_{\frac{1}{2},s}}. \tag{4.5}$$

Let us first estimate $||\eta^2(t/T)K_\lambda v||_{X_{-\frac{1}{2},s}}$. We have that

$$||\eta^2(t/T)K_\lambda v||_{X_{-\frac{1}{2},s}} \leq ||\eta^2(t/T)K_\lambda v||_{X_{\frac{-1+\varepsilon}{2},s}} \leq CT^{\frac{1-\varepsilon}{2}}||\eta(t/T)K_\lambda v||_{X_{0,s}} \leq CT^{1-\varepsilon}||v||_{X_{\frac{1}{2},s}}$$

where we used [42, Lemma 2.11] twice and Lemma 2.4. This yields also

$$||\eta^2(t/T)K_\lambda v||_{Y_{-1,s}} \leq ||\eta^2(t/T)K_\lambda v||_{X_{\frac{-1+\varepsilon}{2},s}} \leq CT^{1-\varepsilon}||v||_{X_{\frac{1}{2},s}}.$$

and (4.5) follows. The proof of Lemma 4.2 is complete. □

It follows then from Lemmas 3.1, 3.3 and 4.2 that there exist some positive constants θ, C_1, C_2 and C_3 such that

$$||\Gamma(v)||_{Z^T_{\frac{1}{2},s}} \leq C_1\|u_0\|_s + C_2T^\theta||v||^2_{Z^T_{\frac{1}{2},s}} + C_3T^{1-\varepsilon}||v||_{Z^T_{\frac{1}{2},s}} \tag{4.6}$$

$$||\Gamma(v_1) - \Gamma(v_2)||_{Z^T_{\frac{1}{2},s}} \leq C_2T^\theta||v_1+v_2||_{Z^T_{\frac{1}{2},s}}||v_1-v_2||_{Z^T_{\frac{1}{2},s}} + C_3T^{1-\varepsilon}||v_1-v_2||_{Z^T_{\frac{1}{2},s}} \tag{4.7}$$

for any $v, v_1, v_2 \in Z^T_{\frac12,s} \cap L^2(0,T;L^2_0(\mathbb{T}))$. Pick $d = 2C_1\|u_0\|_s$ and $T>0$ such that

$$2C_2 dT^\theta + C_3 T^{1-\varepsilon} \le \frac12. \tag{4.8}$$

Then

$$||\Gamma(v)||_{Z^T_{\frac12,s}} \le d$$

and

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{Z^T_{\frac12,s}} \le \frac12 \|v_1 - v_2\|_{Z^T_{\frac12,s}}$$

whenever $||v||_{Z^T_{\frac12,s}} \le d$, $||v_1||_{Z^T_{\frac12,s}} \le d$, and $||v_2||_{Z^T_{\frac12,s}} \le d$. Thus the map Γ is a contraction in the closed ball $B_d(0)$ of $Z^T_{\frac12,s} \cap L^2(0,T;L^2_0(\mathbb{T}))$ for the $||\cdot||_{Z^T_{\frac12,s}}$ norm. Its fixed point u is the desired solution of (4.1) in the space $Z^T_{\frac12,s} \cap L^2(0,T;L^2_0(\mathbb{T}))$. It follows from the property (v) of the Bourgain space $Z^T_{b,s}$ recalled in the previous section that $u \in C([0,T];H^s_0(\mathbb{T}))$ with

$$||u||_{L^\infty(0,T;H^s(\mathbb{T}))} \le C_4 ||u||_{Z^T_{\frac12,s}} \le 2C_1C_4||u_0||_s.$$

Let us now pass to the global existence of the solution. Assume first that $s=0$. The solution of (4.1) satisfies

$$||u(.,t)||_0^2 = ||u||_0^2 - \int_0^t (GL_\lambda^{-1}u, Gu)_0(\tau)d\tau \quad \forall t \ge 0$$

which yields with Gronwall lemma

$$||u(.,t)||_0^2 \le ||u_0||_0^2\, e^{Ct} \tag{4.9}$$

with $C = ||G||^2||L_\lambda^{-1}||$. A standard continuation argument shows that (4.1) is globally well-posed in $L^2_0(\mathbb{T})$. (Note that $||u(.,t)||_0 \le ||u_0||_0$ when $\lambda = 0$ and $t \ge 0$.) Next, we show that (4.1) is globally well-posed in the space $H^3_0(\mathbb{T})$. For a smooth solution u of (4.1), let $v = u_t$. Then

$$\begin{cases} \partial_t v + \partial_x^3 v + \mu\partial_x v + \partial_x(uv) = -K_\lambda v, & x \in \mathbb{T},\ 0<t<T, \\ v(x,0) = v_0(x), & x \in \mathbb{T}, \end{cases} \tag{4.10}$$

where

$$v_0 = -K_\lambda u_0 - u_0''' - \mu u_0' - u_0 u_0'.$$

For T fulfilling (4.8), we have

$$||u||_{Z^T_{\frac12,0}} \le d = 2C_1||u_0||_0.$$

The same computations as those leading to (4.6) yield

$$||v||_{Z^T_{\frac12,0}} \le C_1||v_0||_0 + (4C_1C_2T^\theta||u_0||_0 + C_3T^{1-\varepsilon})||v||_{Z^T_{\frac12,0}}$$

and hence

$$||v||_{Z^T_{\frac12,0}} \le 2C_1||v_0||_0$$

for $0<T<T_1(||u_0||_0)$, where $T_1(\cdot)$ is a continuous nonincreasing function. Therefore,

$$||v||_{L^\infty(0,T;L^2(\mathbb{T}))} \le C_4||v||_{Z^T_{\frac12,0}} \le C_1'||v_0||_0$$

for $0 < T < T_1$ and $C_1' = 2C_1C_4$. ¿From the equation

$$\partial_x^3 u = -K_\lambda u - v - \mu\partial_x u - u\partial_x u,$$

we infer that for $0 < t < T < T_1$

$$\begin{aligned} ||\partial_x^3 u||_0 &\le C_7||u||_0 + ||v||_0 + (C_8 + ||u||_0)||\partial_x u||_{L^\infty_x} \\ &\le C_7||u||_0 + ||v||_0 + C_9(1+||u||_0)||u||_0^{\frac{1}{2}}||\partial_x^3 u||_0^{\frac{1}{2}} \\ &\le \frac{1}{2}||\partial_x^3 u||_0 + ||v||_0 + C_{10}(||u||_0 + ||u||_0^3). \end{aligned}$$

Consequently,

$$||u||_{L^\infty(0,T;H^3(\mathbb{T}))} \le \alpha_{T,3}(||u_0||_0)||u_0||_3$$

for $T < T_1(||u_0||_0)$. Combined to (4.9), this shows that $u \in C(\mathbb{R}^+; H_0^3(\mathbb{T}))$ and that (4.2) holds true for $s = 3$. A similar result can be obtained for any $s \in 3\mathbb{N}^*$. For other values of s, the global well-posedness follows by nonlinear interpolation [43, 2]. The proof is complete. □

Next we prove a local exponential stability result when applying the feedback law $h = -K_\lambda u$.

Theorem 4.3. *Let $0 < \lambda' < \lambda$ and $s \ge 0$ be given. There exists $\delta > 0$ such that for any $u_0 \in H_0^s(\mathbb{T})$ with $||u_0||_s \le \delta$, the corresponding solution u of (4.1) satisfies*

$$||u(.,t)||_s \le Ce^{-\lambda' t}||u_0||_s \qquad \text{for all } t \ge 0$$

where $C > 0$ is a constant independent of u_0.

Proof. We proceed as in [34, 36]. System (4.1) can be rewritten in an equivalent integral form

$$u(t) = W_\lambda(t)u_0 - \int_0^t W_\lambda(t-\tau)(uu_x)(\tau)\,d\tau \tag{4.11}$$

where $W_\lambda(t) = e^{-t(\partial_x^3 + \mu\partial_x + K_\lambda)}$. At this point we need to extend some estimates in Lemmas 3.1-3.3 for the C^0-group $W_\lambda(t)$.

Lemma 4.4. *Let $s \ge 0$, $\lambda \ge 0$ and $T > 0$ be given. Then there exists a constant $C > 0$ such that (i) for any $\phi \in H_0^s(\mathbb{T})$*

$$||W_\lambda(t)\phi||_{Z^T_{\frac{1}{2},s}} \le C||\phi||_s.$$

(ii) For any $u, v \in Z^T_{\frac{1}{2},s}$

$$||\int_0^t W_\lambda(t-\tau)(uv)_x(\tau)d\tau||_{Z^T_{\frac{1}{2},s}} \le C||u||_{Z^T_{\frac{1}{2},s}}||v||_{Z^T_{\frac{1}{2},s}}.$$

Proof of Lemma 4.4: An application of Duhamel formula gives

$$W_\lambda(t)\phi = W(t)\phi - \int_0^t W(t-\tau)[K_\lambda W_\lambda(\tau)\phi]d\tau. \tag{4.12}$$

Using Lemma 4.2, this yields

$$\begin{aligned} ||W_\lambda(t)\phi||_{Z^T_{\frac{1}{2},s}} &\le ||W(t)\phi||_{Z^T_{\frac{1}{2},s}} + ||\int_0^t W(t-\tau)[K_\lambda W_\lambda(\tau)\phi]d\tau||_{Z^T_{\frac{1}{2},s}} \\ &\le C||\phi||_s + CT^{1-\varepsilon}||W_\lambda(t)\phi||_{Z^T_{\frac{1}{2},s}} \end{aligned}$$

(i) follows at once if T is small enough, say $T < T_0$. For $T \geq T_0$, the result follows from an easy induction. To prove (ii), we use the identity

$$\int_0^t W_\lambda(t-\tau)f(\tau)\,d\tau = \int_0^t W(t-\tau)f(\tau)\,d\tau - \int_0^t W(t-\tau)K_\lambda\left(\int_0^\tau W_\lambda(\tau-\sigma)f(\sigma)\,d\sigma\right)d\tau$$

which gives with $f = (uv)_x$

$$\begin{aligned}
&||\int_0^t W_\lambda(t-\tau)(uv)_x(\tau)\,d\tau||_{Z^T_{\frac{1}{2},s}} \\
&\quad \leq ||\int_0^t W(t-\tau)(uv)_x(\tau)d\tau||_{Z^T_{\frac{1}{2},s}} + ||\int_0^t W(t-\tau)K_\lambda\left(\int_0^\tau W_\lambda(\tau-\sigma)(uv)_x(\sigma)\,d\sigma\right)d\tau||_{Z^T_{\frac{1}{2},s}} \\
&\quad \leq C||u||_{Z^T_{\frac{1}{2},s}}||v||_{Z^T_{\frac{1}{2},s}} + CT^{1-\varepsilon}||\int_0^t W_\lambda(t-\tau)(uv)_x(\tau)\,d\tau||_{Z^T_{\frac{1}{2},s}}
\end{aligned}$$

(ii) follows again if T is small enough, say $T < T_0$. For $T \geq T_0$, the result follows from (i) and an easy induction. $\Box$

For given $s \geq 0$, there exists by Proposition 2.5 some constant $C > 0$ such that

$$||W_\lambda(t)u_0||_s \leq Ce^{-\lambda t}||u_0||_s \qquad \forall t \geq 0.$$

Pick $T > 0$ such that

$$2Ce^{-\lambda T} \leq e^{-\lambda' T}.$$

We seek a solution u to the integral equation (4.11) as a fixed point of the map

$$\Gamma(v) = W_\lambda(t)u_0 - \int_0^t W_\lambda(t-\tau)(vv_x)(\tau)\,d\tau$$

in some closed ball $B_M(0)$ in the space $Z^T_{\frac{1}{2},s} \cap L^2(0,T;L^2_0(\mathbb{T}))$ for the $||v||_{Z^T_{\frac{1}{2},s}}$ norm. This will be done provided that $||u_0||_s \leq \delta$ where δ is a small number to be determined. Furthermore, to ensure the exponential stability with the claimed decay rate, the numbers δ and M will be chosen in such a way that

$$||u(T)||_s \leq e^{-\lambda' T}||u_0||_s.$$

By Lemma 4.4, there exist some positive constants C_1, C_2 (independent of δ and M) such that

$$||\Gamma(v)||_{Z^T_{\frac{1}{2},s}} \leq C_1||u_0||_s + C_2||v||^2_{Z^T_{\frac{1}{2},s}}$$

and

$$||\Gamma(v_1) - \Gamma(v_2)||_{Z^T_{\frac{1}{2},s}} \leq C_2||v_1 + v_2||_{Z^T_{\frac{1}{2},s}}||v_1 - v_2||_{Z^T_{\frac{1}{2},s}}.$$

On the other hand, since $Z^T_{\frac{1}{2},s} \subset C([0,T];H^s(\mathbb{T}))$, we have for some constant $C' > 0$ and all $v \in B_M(0)$

$$\begin{aligned}
||\Gamma(v)(T)||_s &\leq ||W_\lambda(T)u_0||_s + ||\int_0^T W_\lambda(T-t)(vv_x)(\tau)d\tau||_s \\
&\leq Ce^{-\lambda T}\delta + C'M^2.
\end{aligned}$$

Pick $\delta = C_4 M^2$, where C_4 and M are chosen so that

$$\frac{C'}{C_4} \le Ce^{-\lambda T}, \quad (C_1C_4 + C_2)M^2 \le M, \quad \text{and} \quad 2C_2 M \le \frac{1}{2}.$$

Then we have

$$\begin{aligned} ||\Gamma(v)||_{Z^T_{\frac{1}{2},s}} &\le M \qquad \forall v \in B_M(0), \\ ||\Gamma(v_1) - \Gamma(v_2)||_{Z^T_{\frac{1}{2},s}} &\le \frac{1}{2}||v_1 - v_2||_{Z^T_{\frac{1}{2},s}} \qquad \forall v_1, v_2 \in B_M(0). \end{aligned}$$

Therefore, Γ is a contraction in $B_M(0)$. Furthermore, its unique fixed point $u \in B_M(0)$ fulfills

$$||u(T)||_s = ||\Gamma(u)(T)||_s \le e^{-\lambda' T}\delta.$$

Assume now that $0 < ||u_0||_s < \delta$. Changing δ into $\delta' := ||u_0||_s$ and M into $M' = (\delta'/\delta)^{\frac{1}{2}} M$, we infer that $||u(T)||_s \le e^{-\lambda' T}||u_0||_s$, and an obvious induction yields $||u(nT)||_s \le e^{-\lambda' nT}||u_0||_s$ for any $n \ge 0$. As $Z^T_{\frac{1}{2},s} \cap L^2(0,T;L^2_0(\mathbb{T})) \subset C([0,T];H^s_0(\mathbb{T}))$, we infer by the semigroup property that there exists some constant $C' > 0$ such that

$$||u(t)||_s \le C' e^{-\lambda' t}||u_0||_s$$

provided that $||u_0||_s \le \delta$. The proof is complete. □

The stability result presented in Theorem 4.3 was local. We extend it to a global stability result in the following theorem.

Theorem 4.5. *Assume $\lambda = 0$ in (4.1).*[3] *There exists a $\kappa > 0$ such that for any $R_0 > 0$, there exists a constant $C > 0$ such that for any $u_0 \in L^2_0(\mathbb{T})$ with*

$$\|u_0\|_0 \le R_0,$$

the corresponding solution u of (4.1) (with $\lambda = 0$) satisfies

$$\|u(\cdot,t)\|_0 \le Ce^{-\kappa t}\|u_0\|_0 \qquad \textit{for all} \;\; t \ge 0. \tag{4.13}$$

Theorem 4.5 is a direct consequence of the following observability inequality.

Proposition 4.6. *Let $T > 0$ and $R_0 > 0$ be given. There exists a constant $\beta > 1$ such that for any $u_0 \in L^2_0(\mathbb{T})$ satisfying*

$$\|u_0\|_0 \le R_0,$$

the corresponding solution u of (4.1) satisfies

$$\|u_0\|_0^2 \le \beta \int_0^T ||Gu||_0^2(t)dt. \tag{4.14}$$

Indeed, if (4.14) holds, then it follows from the energy estimate

$$\|u(\cdot,t)\|_0^2 = \|u_0\|_0^2 - \int_0^t ||Gu||_0^2(\tau)d\tau \quad \forall t \ge 0 \tag{4.15}$$

[3] Recall that $K_0 = GG^*$.

that

$$\|u(\cdot,T)\|_0^2 \le (1-\beta^{-1})\|u_0\|_0^2.$$

Thus

$$\|u(\cdot,mT)\|_0^2 \le (1-\beta^{-1})^m\|u_0\|_0^2$$

which gives (4.13) by the semigroup property. We obtain a constant κ independent of R_0 by noticing that for $t > c(\|u_0\|_0)$, the L^2 norm of $u(.,t)$ is smaller than 1, so that we can take the κ corresponding to $R_0 = 1$. □

Now we present a proof of Proposition 4.6.

Proof of Proposition 4.6: We prove the estimate (4.14) by contradiction. If (4.14) is not true, then for any $n \ge 1$, (4.1) admits a solution $u_n \in Z^T_{\frac{1}{2},0} \cap C([0,T];L^2_0(\mathbb{T}))$ satisfying

$$\|u_n(0)\|_0 \le R_0$$

and

$$\int_0^T ||Gu_n||_0^2 dt < \frac{1}{n}\|u_{0,n}\|_0^2 \tag{4.16}$$

where $u_{0,n} = u_n(0)$. Since $\alpha_n := \|u_{0,n}\|_0 \le R_0$, one can choose a subsequence of $\{\alpha_n\}$, still denoted by $\{\alpha_n\}$, such that

$$\lim_{n\to\infty} \alpha_n = \alpha.$$

There are two possible cases: (i) $\alpha > 0$ and (ii) $\alpha = 0$.

<u>(i) $\alpha > 0$</u>

Note that the sequence $\{u_n\}$ is bounded in both spaces $L^\infty(0,T;L^2(\mathbb{T}))$ and $X^T_{\frac{1}{2},0}$. By Lemma 3.3, the sequence $\{\partial_x(u_n^2)\}$ is bounded in the space $X^T_{-\frac{1}{2},0}$. On the other hand, the space $X^T_{\frac{1}{2},0}$ is compactly imbedded in the space $X^T_{0,-1}$. Therefore, we can extract a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$\begin{aligned} u_n &\to u \quad \text{weakly in } X^T_{\frac{1}{2},0}, \text{ and strongly in } X^T_{0,-1}, \\ -\frac{1}{2}\partial_x(u_n^2) &\to f \quad \text{weakly in } X^T_{-\frac{1}{2},0}, \end{aligned}$$

where $u \in X^T_{\frac{1}{2},0}$ and $f \in X^T_{-\frac{1}{2},0}$. Furthermore, since $X^T_{\frac{1}{2},0}$ is continuously imbedded in $L^4(\mathbb{T}\times(0,T))$ by (3.4), u_n^2 is bounded in $L^2(\mathbb{T}\times(0,T))$. It follows that $\partial_x(u_n^2)$ is bounded in

$$L^2(0,T;H^{-1}(\mathbb{T})) = X^T_{0,-1}.$$

Conducting interpolation between $X^T_{-\frac{1}{2},0}$ and $X^T_{0,-1}$, we obtain that $\partial_x(u_n^2)$ is bounded in $X^T_{-\frac{1-\theta}{2},-\theta} = X^T_{-\frac{1}{2}+\frac{\theta}{2},-\theta}$ for $\theta \in [0,1]$. As $X^T_{-\frac{1}{2}+\frac{\theta}{2},-\theta}$ is compactly imbedded in $X^T_{-\frac{1}{2},-1}$ for $0<\theta<1$, we can extract a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $-\frac{1}{2}\partial_x(u_n^2)$ converges to f strongly in $X^T_{-\frac{1}{2},-1}$. It follows from (4.16) that

$$\int_0^T ||Gu_n||_0^2\,dt \longrightarrow \int_0^T ||Gu||_0^2\,dt = 0,$$

which implies that $u(x,t) = c(t)$ on $\omega \times (0,T)$ for some function $c(t)$. Thus, passing to the limit in (4.1), we obtain

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu \partial_x u &= f \qquad \text{on } \mathbb{T} \times (0,T), \\ u &= c(t) \qquad \text{on } \omega \times (0,T). \end{cases} \tag{4.17}$$

Let $w_n = u_n - u$ and $f_n = -\frac{1}{2}\partial_x(u_n^2) - f - K_0 u_n$. Note first that

$$\int_0^T ||Gw_n||_0^2 dt \int_0^T ||Gu_n||_0^2 dt + \int_0^T ||Gu||_0^2 dt - 2\int_0^T (Gu_n, Gu)_0 dt \to 0 \tag{4.18}$$

Since $w_n \to 0$ weakly in $X^T_{\frac{1}{2},0}$, we infer from Rellich theorem that $\int_{\mathbb{T}} g(y) w_n(y,t)dy \to 0$ strongly in $L^2(0,T)$. Combined to (4.18), this yields

$$\int_0^T \int_{\mathbb{T}} g(x)^2 w_n(x,t)^2 dxdt \to 0.$$

Thus

$$\partial_t w_n + \partial_x^3 w_n + \mu \partial_x w_n = f_n$$

and

$$f_n \underset{X^T_{-\frac{1}{2},-1}}{\longrightarrow} 0, \quad w_n \underset{L^2(0,T;L^2(\tilde{\omega}))}{\longrightarrow} 0,$$

where $\tilde{\omega} := \{g > ||g||_{L^\infty(\mathbb{T})}/2\}$.

Applying Proposition 3.5 with $b = \frac{1}{2}$ and $b' = 0$ yields that

$$w_n \underset{L^2_{loc}((0,T);L^2(\mathbb{T}))}{\longrightarrow} 0.$$

Consequently, u_n^2 tends to u^2 in $L^1_{loc}((0,T);L^1(\mathbb{T}))$ and $\partial_x(u_n^2)$ tends to $\partial_x(u^2)$ in the distributional sense. Therefore $f = -\frac{1}{2}\partial_x(u^2)$ and $u \in X^T_{\frac{1}{2},0}$ satisfies

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu \partial_x u + \frac{1}{2}\partial_x(u^2) &= 0 \qquad \text{on } \mathbb{T} \times (0,T), \\ u &= c(t) \qquad \text{on } \omega \times (0,T). \end{cases}$$

The first equation gives $c'(t) = 0$ which, combined to Corollary 3.8, yields that $u(x,t) \equiv c$ for some constant $c \in \mathbb{R}$. Since $[u] = 0$, $c = 0$, and u_n converges strongly to 0 in $L^2_{loc}((0,T), L^2(\mathbb{T}))$. We can pick some time $t_0 \in [0,T]$ such that $u_n(t_0)$ tends to 0 strongly in $L^2(\mathbb{T})$. Since

$$\|u_n(0)\|_0^2 = \|u_n(t_0)\|_0^2 + \int_0^{t_0} ||Gu_n||_0^2 dt,$$

it is inferred that $\alpha_n = \|u_n(0)\|_0 \to 0$ which is a contradiction to the assumption $\alpha > 0$.

(ii) $\alpha = 0$.

Note first that $\alpha_n > 0$ for all n. Set $v_n = u_n/\alpha_n$ for all $n \geq 1$. Then

$$\partial_t v_n + \partial_x^3 v_n + \mu \partial_x v_n + K_0 v_n + \frac{\alpha_n}{2}\partial_x(v_n^2) = 0$$

and

$$\int_0^T ||Gv_n||_0^2 dt < \frac{1}{n}. \tag{4.19}$$

Because of

$$\|v_n(0)\|_0 = 1, \tag{4.20}$$

the sequence $\{v_n\}$ is bounded in both spaces $L^\infty(0,T;L^2(\mathbb{T}))$ and $X^T_{\frac{1}{2},0}$. Indeed, $||v_n(t)||_0$ is a nonincreasing function of t, and the boundedness of $||v_n||_{X^T_{\frac{1}{2},0}}$ for small values of T follows from an estimate similar to (4.6) (since α_n is bounded). We can extract a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, such that $v_n \to v$ weakly in the space $X^T_{\frac{1}{2},0}$ and strongly in the spaces $X^T_{-\frac{1}{2},-1}$ and $X^T_{0,-1}$. Moreover, the sequence $\{\partial_x(v_n^2)\}$ is bounded in the space $X^T_{-\frac{1}{2},0}$, and therefore $\alpha_n\partial_x(v_n^2)$ tends to 0 in the space $X^T_{-\frac{1}{2},0}$. Finally, $\int_0^T ||Gv||_0^2 dt = 0$. Thus, v solves

$$\begin{cases} \partial_t v + \partial_x^3 v + \mu\partial_x v &= 0 \qquad \text{on } \mathbb{T}\times(0,T) \\ v &= c(t) \qquad \text{on } \omega\times(0,T). \end{cases} \tag{4.21}$$

We infer that $v(x,t) = c(t) = c$ thanks to Holmgren Theorem, and that $c = 0$ because of $[v] = 0$.

According to (4.19)

$$\int_0^T ||Gv_n||_0^2 dt \longrightarrow 0$$

and so $K_0 v_n$ converges strongly to 0 in $X^T_{-\frac{1}{2},-1}$. Then, an application of Proposition 3.5 as in (i) shows that v_n converges to 0 in $L^2_{loc}((0,T), L^2(\mathbb{T}))$. Thus we can pick a time $t_0 \in (0,T)$ such that $v_n(t_0)$ converges to 0 strongly in $L^2(\mathbb{T})$. Since

$$\|v_n(0)\|_0^2 = \|v_n(t_0)\|_0^2 + \int_0^{t_0} ||Gv_n||_0^2 dt,$$

we infer from (4.19) that $\|v_n(0)\|_0 \to 0$ which is a contradiction to (4.20). The proof is complete. $\square$

Next we show that the solution u of (4.1) (with $\lambda = 0$) decays exponentially in any space $H^s(\mathbb{T})$.

Theorem 4.7. *Assume that $\lambda = 0$ in (4.1), and let $\kappa > 0$ be the infimum of the numbers κ given respectively in Proposition 2.3 and in Theorem 4.5. Let $s \geq 0$ and let $\kappa' \in (0,\kappa)$ be given. Then there exists a nondecreasing continuous function $\alpha_{s,\kappa'} : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $u_0 \in H_0^s(\mathbb{T})$, the corresponding solution u of (4.1) satisfies*

$$\|u(\cdot,t)\|_s \leq \alpha_{s,\kappa'}(\|u_0\|_0)e^{-\kappa' t}\|u_0\|_s$$

for all $t \geq 0$.

Proof. The result for $s = 0$ has already been established in Theorem 4.5 with $\kappa' = \kappa$. Let us consider now the case $s = 3$. Pick any number $R_0 > 0$ and any $u_0 \in H_0^3(\mathbb{T})$ with $||u_0||_0 \leq R_0$. Let u denote the solution of (4.1) emanating from u_0 at $t = 0$, and let $v = u_t$. Then v solves

$$\partial_t v + \partial_x^3 v + \mu\partial_x v + \partial_x(uv) = -K_0 v,\ v(x,0) = v_0(x),\ x\in\mathbb{T},\ t>0, \tag{4.22}$$

where $v_0 = -K_0 u_0 - \mu u_0' - u_0 u_0' - u_0'''$. According to (4.2) and (4.13), for any $T > 0$ there exists a number $C > 0$ depending only on R_0 and T such that

$$\|u(\cdot,t)\|_{Z^{[t,t+T]}_{\frac{1}{2},0}} \leq Ce^{-\kappa t}||u_0||_0 \qquad \text{for all } t\ \geq 0.$$

Thus, for any $\epsilon > 0$, there exists a $t^* > 0$ such that if $t \geq t^*$, one has

$$\|u(\cdot,t)\|_{Z^{[t,t+T]}_{\frac{1}{2},0}} \leq \epsilon.$$

At this point we need an exponential stability result for the linearized system

$$\partial_t w + \partial_x^3 w + \mu \partial_x w + \partial_x (aw) = -K_0 w, \ w(x,0) = w_0(x), \ x \in \mathbb{T}, \ t > 0 \tag{4.23}$$

where $a \in Z^T_{\frac{1}{2},s} \cap L^2(0,T; L^2_0(\mathbb{T}))$ is a given function.

Lemma 4.8. *Let $s \geq 0$ and $a \in Z^T_{\frac{1}{2},s} \cap L^2(0,T; L^2_0(\mathbb{T}))$ for all $T > 0$. Then for any $\kappa' \in (0,\kappa)$ there exist $T > 0$, $\beta > 0$ such that if*

$$\sup_{n \geq 1} \|a\|_{Z^{[nT,(n+1)T]}_{\frac{1}{2},s}} \leq \beta,$$

then

$$\|w(\cdot,t)\|_s \leq C e^{-\kappa' t} \|w_0\|_s \qquad \textit{for all} \ \ t \geq 0,$$

where $C > 0$ is a constant independent of w_0.

Proof of Lemma 4.8: First, a proof similar to those of Theorem 4.1 shows that for any $T > 0$ and any $s \geq 0$, if $a \in Z^T_{\frac{1}{2},s} \cap L^2(0,T; L^2_0(\mathbb{T}))$, then (4.23) admits a unique solution $w \in Z^T_{\frac{1}{2},s} \cap L^2(0,T; L^2_0(\mathbb{T}))$ and

$$\|w\|_{Z^T_{\frac{1}{2},s}} \leq \mu(\|a\|_{Z^T_{\frac{1}{2},s}}) \|w_0\|_s \tag{4.24}$$

where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function. Rewrite (4.23) in its integral form

$$w(t) = W_0(t) w_0 - \int_0^t W_0(t-\tau) \partial_x (aw)(\tau) d\tau$$

where $W_0(t) = e^{-t(\partial_x^3 + \mu \partial_x + K_0)}$. Thus, for any $T > 0$, by Proposition 2.3, Lemma 4.4 and (4.24),

$$\begin{aligned} \|w(\cdot,T)\|_s &\leq C_1 e^{-\kappa T} \|w_0\|_s + C_2 \|a\|_{Z^T_{\frac{1}{2},s}} \|w\|_{Z^T_{\frac{1}{2},s}} \\ &\leq C_1 e^{-\kappa T} \|w_0\|_s + C_2 \|a\|_{Z^T_{\frac{1}{2},s}} \mu(\|a\|_{Z^T_{\frac{1}{2},s}}) \|w_0\|_s \end{aligned}$$

where $C_1 > 0$ is independent of T while C_2 may depend on T. Let

$$y_n = w(\cdot, nT) \quad \text{for} \quad n = 1, 2, ...$$

Then, using the semigroup property of the system (4.23),

$$\|y_{n+1}\|_s \leq C_1 e^{-\kappa T} \|y_n\|_s + C_2 \|a\|_{Z^{[nT,(n+1)T]}_{\frac{1}{2},s}} \mu(\|a\|_{Z^{[nT,(n+1)T]}_{\frac{1}{2},s}}) \|y_n\|_s$$

for $n \geq 1$. Choose $T > 0$ large enough and $\beta > 0$ small enough so that

$$C_1 e^{-\kappa T} + C_2 \beta \mu(\beta) = e^{-\kappa' T}$$

Then

$$\|y_{n+1}\|_s \leq e^{-\kappa' T} \|y_n\|_s$$

for any $n \geq 1$ as long as

$$\sup_{n\geq 1} \|a\|_{Z^{[nT,(n+1)T]}_{\frac{1}{2},s}} \leq \beta.$$

Thus

$$\|y_n\|_s \leq e^{-n\kappa' T}\|y_0\|_s$$

for any $n \geq 1$, which implies that

$$\|w(\cdot,t)\|_s \leq Ce^{-\kappa' t}\|w_0\|_s$$

for all $t \geq 0$. The proof is complete. □

Choose $\epsilon < \beta$, and then apply Lemma 4.8 to (4.22) to obtain

$$\|v(\cdot,t)\|_0 \leq Ce^{-\kappa'(t-t^*)}\|v(\cdot,t^*)\|_0$$

for any $t \geq t^*$, or

$$\|v(\cdot,t)\|_0 \leq C_1 e^{-\kappa' t}\|v_0\|_0$$

for any $t \geq 0$, where $C_1 > 0$ depends only on R_0. It then follows from the equation

$$\partial_x^3 u = -K_0 u - u\partial_x u - \mu\partial_x u - v$$

and Theorem 4.5 that

$$\|u(\cdot,t)\|_3 \leq Ce^{-\kappa' t}\|u_0\|_3$$

for any $t \geq 0$, where $C > 0$ depends only on R_0.

Thus the theorem has been proved for $s = 0$ and $s = 3$. Using the same argument for $u_1 - u_2$ and $a = u_1 + u_2$ for two different solutions u_1 and u_2, we obtain the Lipchitz stability estimate needed for interpolation:

$$\|(u_1-u_2)(\cdot,t)\|_0 \leq Ce^{-\kappa' t}\|(u_1-u_2)(\cdot,0)\|_0.$$

The case of $0 < s < 3$ follows by interpolation. The other cases can be proved similarly. □

5 Time-varying feedback law

In this section we prove that it is possible to design a smooth time-varying feedback law ensuring a semiglobal stabilization with an arbitrary large decay rate.

Let $\lambda > 0$ and $s \geq 0$ be given. According to Theorem 4.7, there exists a number $\kappa > 0$ and a nondecreasing function α_s such that any solution u of

$$\partial_t u + \partial_x^3 u + \mu\partial_x u + u\partial_x u = -GG^* u \tag{5.1}$$

emanating from $u_0 \in H_0^s(\mathbb{T})$ at $t = 0$ fulfills

$$||u(t)||_s \leq \alpha_s(||u_0||_0)e^{-\kappa t}||u_0||_s. \tag{5.2}$$

On the other hand, it follows from Theorem 4.3 that for any fixed $\lambda' \in (0,\lambda)$, any solution u of

$$\partial_t u + \partial_x^3 u + \mu\partial_x u + u\partial_x u = -K_\lambda u \tag{5.3}$$

emanating from $u_0 \in H_0^s(\mathbb{T})$ at $t = 0$ fulfills

$$||u(t)||_s \leq C_s e^{-\lambda' t}||u_0||_s \tag{5.4}$$

provided that $||u_0||_s \le r_0$, for some constant C_s and some number $r_0 \in (0,1)$. Pick any function $\theta \in C^\infty(\mathbb{R};[0,1])$ fulfilling the following properties:

$$\theta(t) = 1 \qquad \text{for } \delta \le t \le 1-\delta \tag{5.5}$$

$$\theta(t) = 0 \qquad \text{for } 1 \le t \le 2 \tag{5.6}$$

$$\theta(t+2) = \theta(t) \qquad \text{for all } t \in \mathbb{R} \tag{5.7}$$

where $\delta \in (0,1/10)$ is a number whose value will be specified later. Pick a function $\rho \in C^\infty(\mathbb{R}^+;[0,1])$ such that

$$\rho(r) = 1 \quad \text{for } r \le r_0, \qquad \rho(r) = 0 \quad \text{for } r \ge 1. \tag{5.8}$$

Let $T > 0$ be given. We consider the following time-varying feedback law

$$\begin{aligned} K(u,t) &= \rho(||u||_s^2)\,[\theta(\frac{t}{T})K_\lambda u + \theta(\frac{t-T}{T})GG^*u] + (1-\rho(||u||_s^2))GG^*u \\ &= GG^*\,\{\rho(||u||_s^2)\,[\theta(\frac{t}{T})L_\lambda^{-1}u + \theta(\frac{t-T}{T})u] + (1-\rho(||u||_s^2))u\}. \end{aligned} \tag{5.9}$$

The following result indicates that a semiglobal stabilization with an arbitrary decay rate can be obtained.

Theorem 5.1. *Let $\lambda > 0$ and let $K = K(u,t)$ be as given in* (5.9). *Pick any $\lambda' \in (0,\lambda)$ and any $\lambda'' \in (\lambda'/2, (\lambda'+\kappa)/2)$. Then there exists a time $T_0 > 0$ such that for $T > T_0$, $t_0 \in \mathbb{R}$ and $u_0 \in H_0^s(\mathbb{T})$, the unique solution of the closed-loop system*

$$\partial_t u + \partial_x^3 u + \mu\partial_x u + u\partial_x u = -K(u,t), \qquad u(t_0) = u_0 \tag{5.10}$$

satisfies

$$||u(.,t)||_s \le \gamma_s(||u_0||_s)e^{-\lambda''(t-t_0)}||u_0||_s \qquad \text{for all } t \ge t_0 \tag{5.11}$$

where γ_s is a nondecreasing continuous function.

Proof. First, proceeding as for Theorem 4.1, we check that the system (5.10) is globally well-posed in $H_0^s(\mathbb{T})$. Next, rough estimates for $||u(.,t)||_s$ are established for the times t when both K_λ and GG^* are active.

Lemma 5.2. *Pick any pair $(t_0,u_0) \in \mathbb{R}\times H_0^s(\mathbb{T})$. Then the system* (5.10) *admits a unique solution $u : \mathbb{T}\times[t_0,+\infty) \to \mathbb{R}$ fulfilling*

$$u \in Z_{\frac{1}{2},s}^{[t_0,t_0+T]} \cap L^2(t_0,t_0+T;L_0^2(\mathbb{T})) \qquad \text{for all } T > 0.$$

The following a priori estimates hold true

$$\text{If} \quad ||u_0||_s \le 1, \qquad ||u(.,t)||_s \le \alpha_s(1) \qquad \text{for all } t \ge t_0; \tag{5.12}$$

$$\text{If} \quad ||u_0||_s > 1, \qquad ||u(.,t)||_s \le \alpha_s(||u_0||_0)||u_0||_s \qquad \text{for all } t \ge t_0; \tag{5.13}$$

$$\text{If} \quad ||u_0||_s \le R, \qquad ||u(.,t)||_s \le K_s\, e^{d_s(t-t_0)}||u_0||_s \qquad \text{for all } t \ge t_0, \tag{5.14}$$

where K_s and d_s denote some positive constants depending only on s and R.

Proof of Lemma 5.2: Let us begin with the local existence of a solution. Pick any pair $(t_0, u_0) \in \mathbb{R} \times H_0^s(\mathbb{T})$. It may be seen that

$$||K(v_1,t) - K(v_2,t)||_s \le c||v_1 - v_2||_s \qquad \text{for all } \ v_1, v_2 \in H_0^s(\mathbb{T}),\ t \in \mathbb{R}$$

where c denotes a positive constant independent of v_1, v_2 and t. Defining the map

$$\Gamma(v)(t) = W(t-t_0)u_0 - \int_{t_0}^t W(t-\tau)(v\partial_x v)(\tau)\,d\tau - \int_{t_0}^t W(t-\tau)K(v(\tau),\tau)\,d\tau,$$

we infer as in the proof of Theorem 4.1 that (4.6) and (4.7) hold for all $v, v_1, v_2 \in Z_{\frac{1}{2},s}^{[t_0,t_0+\tilde{T}]} \cap L^2(t_0, t_0+\tilde{T}; L_0^2(\mathbb{T}))$. Moreover, the involved constants only depend on θ for its L^∞ norm and not on δ. Let $d = 2C_1||u_0||_s$ and $\tilde{T} > 0$ be such that

$$2C_2 d\tilde{T}^\theta + C_3\tilde{T}^{1-\varepsilon} \le \frac{1}{2}\cdot$$

Then the map Γ is a contraction in the closed ball $B_d(0)$ of $Z_{\frac{1}{2},s}^{[t_0,t_0+\tilde{T}]} \cap L^2(t_0, t_0+\tilde{T}; L_0^2(\mathbb{T}))$ for the $||v||_{Z_{\frac{1}{2},s}^{[t_0,t_0+\tilde{T}]}}$ norm. Its fixed point is the desired solution of (5.10). Note that for some constant $C_4 > 0$ we have that

$$||u||_{L^\infty(t_0,t_0+\tilde{T};H^s(\mathbb{T}))} \le C_4||u||_{Z_{\frac{1}{2},s}^{[t_0,t_0+\tilde{T}]}} \le 2C_1C_4||u_0||_s.$$

Noticing that $K(u,t) = GG^*u$ for $||u||_s > 1$ and using (5.2), we infer that the solution u of (5.10) is defined for all $t \ge t_0$. Moreover, (5.2) yields (5.12) and (5.13). Let

$$d' 2C_1 \max(\alpha_s(1), \alpha_s(||u_0||_0)||u_0||_s).$$

Note that d' depends only on R and s. Replacing $\tilde{T}$ by T' satisfying

$$2C_2 d' {T'}^\theta + C_3 {T'}^{1-\varepsilon} \le \frac{1}{2}$$

in the application of the contraction mapping principle, we infer that the (unique) solution u of (5.10) fulfills

$$||u||_{Z_{\frac{1}{2},s}^{[t_0+kT',t_0+(k+1)T']}} \le 2C_1||u(.,t_0+kT')||_s.$$

This gives

$$||u||_{L^\infty(t_0+kT',t_0+(k+1)T',H^s(\mathbb{T}))} \le 2C_1C_4||u(.,t_0+kT')||_s$$

and

$$||u(.,t)||_s \le K_s e^{d_s(t-t_0)}||u_0||_s$$

for some constants $K_s > 0$, $d_s > 0$ depending only on s and R. □

Given λ'' as in the statement of the theorem, we pick $\delta > 0$ such that

$$\lambda'' < -2\delta d_s + (1-2\delta)\frac{\kappa+\lambda'}{2} \qquad \text{and} \qquad \delta d_s - (1-2\delta)\kappa < 0\cdot \tag{5.15}$$

Next, choose $r_1 \in (0, r_0)$ such that

$$\alpha_s(\alpha_s(1))C_s K_s^4 e^{4\delta T d_s} r_1 < r_0, \tag{5.16}$$

and $T_0 > 0$ such that

$$\alpha_s(1)\alpha_s(\alpha_s(1))K_s e^{[\delta d_s-(1-2\delta)\kappa]T} \leq r_1, \tag{5.17}$$

$$\alpha_s(1)C_sK_s^4 e^{[4\delta d_s-(1-2\delta)(\kappa+\lambda')]T} \leq e^{-2\lambda''T} \tag{5.18}$$

for all $T \geq T_0$. Note that T_0 exists by (5.15). Pick any $u_0 \in H_0^s(\mathbb{T})$ and any time $t_0 \in \mathbb{R}$. The proof rests on a series of claims.

Claim 1. There exists a time $t_1 \in [t_0, t_0 + \kappa^{-1}\ln(\alpha_s(||u_0||_0)||u_0||_s)]$ such that

$$||u(t_1)||_s \leq 1. \tag{5.19}$$

Without loss of generality we may assume that $||u(t_0)||_s \geq 1$. Then the dynamics of u is governed by (5.1) as long as $||u(t)||_s \geq 1$. By (5.2), we have (5.19) for some time t_1 with

$$\alpha_s(||u_0||_0)e^{-\kappa(t_1-t_0)}||u_0||_s \leq 1.$$

Therefore, Claim 1 holds. □

Claim 2. There exists a time $t_2 \in 2\mathbb{Z}T \cap [t_1, t_1 + 3T]$ such that

$$||u(t_2)||_s \leq r_1. \tag{5.20}$$

From the fact that $||u(t_1)||_0 \leq 1$ and (5.2) we have that

$$||u(t)||_s \leq \alpha_s(1) \qquad \text{for all } t \geq t_1.$$

Pick $R = \alpha_s(1)$ and let K_s and d_s be as given in Lemma 5.2 for that choice of R. Let $t_1' \geq t_1$ denote the first time of the form $t_1' = (2k+1)T + \delta$ with $k \in \mathbb{Z}$, and let $t_2 = (2k+2)T$. Then it follows from (5.2), (5.14) and (5.17) that

$$||u(t_2)||_s \leq K_s e^{\delta T d_s}\alpha_s(\alpha_s(1))e^{-\kappa(1-2\delta)T}||u(t_1')||_s \leq r_1.$$

Claim 3. $||u(t)||_s \leq r_0$ for all $t \geq t_2$ and $||u(t_2+2kT)||_s \leq e^{-2k\lambda''T}||u(t_2)||_s$ for all $k \in \mathbb{N}$.

First, we notice that the dynamics of u is governed by (5.3) (resp. by (5.1)) when $t \in (t_2 + \delta T, t_2 + (1-\delta)T)$ (resp. when $t \in (t_2 + (1+\delta)T, t_2 + (2-\delta)T)$), as long as $||u(t)||_s \leq r_0$. Therefore, using (5.2), (5.4), (5.14), and (5.16) we obtain that

$$||u(t)||_s \leq (\alpha_s(\alpha_s(1))K_s^2 e^{2\delta T d_s})(C_sK_s^2 e^{2\delta T d_s})||u(t_2)||_s < r_0 \qquad \text{for all } t \in [t_2, t_2 + 2T].$$

On the other hand, by (5.18),

$$\begin{aligned} ||u(t_2+2T)||_s &\leq (\alpha_s(1)e^{-\kappa(1-2\delta)T}K_s^2 e^{2\delta T d_s})(C_s e^{-\lambda'(1-2\delta)T}K_s^2 e^{2\delta T d_s})||u(t_2)||_s \\ &\leq e^{-2\lambda''T}||u(t_2)||_s \\ &\leq r_1. \end{aligned}$$

The claim follows by an obvious induction. □

It follows from Claim 3 that for $t \geq t_2$

$$||u(t)||_s \leq c\, e^{-\lambda''(t-t_2)}||u(t_2)||_s$$

for some constant c independent of t and u_0. Since

$$t_2 - t_0 \leq 3T + \kappa^{-1}\ln(\alpha_s(||u_0||_0)||u_0||_s),$$

the theorem follows. □

Remark 5.3.

- *A natural idea to combine both feedback controls would be to consider a discontinuous feedback control which agrees with $K_0 u$ when $||u||_s$ is large, and with $K_\lambda u$ when $||u||_s$ is small. The main difficulty is then to define properly what we mean by a solution of the closed-loop system. In finite dimension, the Filippov solutions are widely used by the control community to deal with discontinuous systems. (See [9] for the definition of a Filippov solution.) The main advantage of the time-varying feedback law considered here is its regularity, which guarantees the existence and uniqueness of "classical" solutions for the closed-loop system.*

- *It would be interesting to see whether a smooth time-invariant feedback law ensuring a semi-global exponential stabilization with an arbitrary decay rate can be designed.*

- *A simpler, but less efficient, time-varying feedback law is*

$$K(u,t) := \theta(\frac{t}{T})\rho(||u||_s^2)\, K_\lambda u + \theta(\frac{t-T}{T})GG^* u.$$

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