INVISIBLE SOLUTIONS FOR SCHRÖDINGER AND WAVE EQUATIONS

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Abstract. On the Euclidean disk, we construct a nontrivial solution of the Schrödinger (resp. wave) equation with Dirichlet boundary conditions which vanishes identically on a smaller concentric disk for all times. The equation has an additional potential term which is a smooth function having Gevrey $2+\varepsilon$ (resp. $1+\varepsilon$) regularity in time. This shows optimality of the local and global results of the authors [FLL25] (resp. of Tataru-Robbiano-Zuily-Hörmander [Tat95, RZ98, Hör97, Tat99]), stating that Gevrey 2 (resp. Gevrey 1) regularity of the potential yields unique continuation. In turn, this proves that Gevrey 2 (resp. Gevrey 1) regularity in time is critical for the approximate controllability of the Schrödinger (resp. wave) equation. Our construction builds on those of Kumano-go [Kg63] and Takase [Tak21].

Contents

1. Introduction 1
1.1. Results 1
1.2. Comments and related literature 4
1.3. Sketch of proofs and plan of the paper 6
2. Construction of the solution u and the potential V 8
2.1. Definition of the oscillation scales, modified Bessel functions, and cutoff functions 8
2.2. Definition of the solution u 10
2.3. Definition of the potential V 12
3. Regularity of the solution u 12
4. Regularity of the potential V 14
5. End of proof of Theorem 1.1 19
6. The wave equation 22
Appendix A. Bessel functions and the functions F_m 24
A.1. Bessel functions J_ν 24
A.2. The functions F_m 24
Appendix B. Gevrey type functional spaces 27
B.1. Abstract Banach valued Gevrey spaces 27
B.2. Gevrey spaces with values in C^ℓ 30
Appendix C. The maximization lemma 32
Appendix D. Perturbation of observability 32
References 33

1. Introduction

1.1. **Results.** This article is devoted to the construction of nice solutions to the Schrödinger and the wave equations, which vanish identically on a large part of the spatial domain for all times, violating unique continuation. The equation has a potential with critical (Gevrey) regularity. All examples are constructed on the (closed) Euclidean unit disk

$$\mathbb{D} := \{x = (x_1, x_2) \in \mathbb{R}^2, |x|^2 := x_1^2 + x_2^2 \leq 1\} \subset \mathbb{R}^2.$$

Our first result concerns the Schrödinger equation.

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Theorem 1.1 (Invisible solution for the Schrödinger equation)**.** *For any $\rho_0 \in (0,1)$, $\varepsilon > 0$, $\eta > 0$, $\ell_0 \in \mathbb{N}$, there exist ρ_1, ρ_2 with $\rho_0 < \rho_1 < \rho_2 < 1$, and functions*

$$\mathsf{q} \in \bigcap_{\ell \in \mathbb{N}} \mathcal{G}^{2+\varepsilon}(\mathbb{R}_t; C^\ell(\mathbb{D};\mathbb{C})), \quad \mathsf{u} \in \bigcap_{\ell \in \mathbb{N}} \mathcal{G}^{2+\varepsilon}(\mathbb{R}_t; C^\ell(\mathbb{D};\mathbb{C}))$$

such that

$$\begin{cases} (-i\partial_t - \Delta + \mathsf{q})\mathsf{u} = 0, & \text{in } \mathbb{R} \times \mathrm{Int}(\mathbb{D}), \\ \mathsf{u} = 0, & \text{on } \mathbb{R} \times \partial\mathbb{D}, \end{cases} \tag{1.1}$$

satisfying in addition

$$\begin{aligned} &\mathrm{supp}(\mathsf{u}) = \{x \in \mathbb{D}, \rho_1 \leq |x| \leq 1\}, \quad \mathrm{supp}(\mathsf{q}) \subset \{x \in \mathbb{D}, \rho_1 \leq |x| \leq \rho_2\}, \\ &\|\mathsf{q}\|_{C^{\ell_0}(\mathbb{R}_t \times \mathrm{Int}(\mathbb{D});\mathbb{C})} \leq \eta, \\ &\|\mathsf{u}(t,\cdot)\|_{L^2(\mathbb{D})} = 1, \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

We refer to Figure 1 for a schematic representation of the support of the functions q and u. In this statement, $C^\ell(\mathbb{D};\mathbb{C})$ denotes the space of restrictions to $\mathbb{D}$ of functions of class C^ℓ in a neighborhood of $\mathbb{D}$ in $\mathbb{R}^2$. The notation $\|\cdot\|_{C^{\ell_0}(\mathbb{R}\times\mathrm{Int}(\mathbb{D});\mathbb{C})}$ stands for a norm (on the space $C_b^{\ell_0}(\mathbb{R} \times \mathrm{Int}(\mathbb{D});\mathbb{C})$ of functions in C^{ℓ_0} having all derivatives of order $\leq \ell_0$ bounded on the open set $\mathbb{R} \times \mathrm{Int}(\mathbb{D})$) equivalent to

$$\|\mathsf{q}\|_{C^{\ell_0}(\mathbb{R}\times\mathrm{Int}(\mathbb{D});\mathbb{C})} = \max_{|\alpha| \leq \ell_0} \sup_{(t,x) \in \mathbb{R}\times\mathrm{Int}(\mathbb{D})} |\partial^\alpha \mathsf{q}(t,x)|.$$

We have also used the following definition of Gevrey functions (with "global bounds").

Definition 1.2. Given $\mathcal{J} \subset \mathbb{R}$ an open interval, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ a Banach space and $s > 0$, we say that f is a globally s-Gevrey function valued in $\mathcal{B}$, denoted $f \in \mathcal{G}^s(\mathcal{J};\mathcal{B})$, if $f \in C_b^\infty(\mathcal{J};\mathcal{B})$ is such that there are constants $C, L > 0$ such that for all $\alpha \in \mathbb{N}$

$$\sup_{t \in \mathcal{J}} \left\| f^{(\alpha)}(t) \right\|_{\mathcal{B}} \leq C L^\alpha \alpha!^s.$$

We refer e.g. to [Hör90] or [Rod93] for more properties of Gevrey functions Notice that $s_1 \leq s_2 \implies \mathcal{G}^{s_1}(\mathcal{J};\mathcal{B}) \subset \mathcal{G}^{s_2}(\mathcal{J};\mathcal{B})$ and for $s = 1$, $\mathcal{G}^1(\mathcal{J};\mathcal{B})$ is a space of real-analytic $\mathcal{B}$-valued functions.

Our second main result is an analogue of Theorem 1.1 for the wave equation, in which the critical Gevrey regularity $2+\varepsilon$ is replaced by $1+\varepsilon$.

Theorem 1.3 (Invisible solution for the wave equation)**.** *For any $\rho_0 \in (0,1)$, $\varepsilon > 0$, $\eta > 0$, $\ell_0 \in \mathbb{N}$, there exist ρ_1, ρ_2 with $\rho_0 < \rho_1 < \rho_2 < 1$, and functions*

$$\mathsf{q}^W \in \bigcap_{\ell \in \mathbb{N}} \mathcal{G}^{1+\varepsilon}(\mathbb{R}; C^\ell(\mathbb{D};\mathbb{C})), \quad \mathsf{u}^W \in \bigcap_{\ell \in \mathbb{N}} \mathcal{G}^{1+\varepsilon}(\mathbb{R}; C^\ell(\mathbb{D};\mathbb{C}))$$

such that

$$\begin{cases} (\partial_t^2 - \Delta + \mathsf{q}^W)\mathsf{u}^W = 0, & \text{in } \mathbb{R} \times \mathrm{Int}(\mathbb{D}), \\ \mathsf{u}^W = 0, & \text{on } \mathbb{R} \times \partial\mathbb{D}, \end{cases} \tag{1.2}$$

satisfying in addition

$$\begin{aligned} &\mathrm{supp}(\mathsf{u}^W) = \{x \in \mathbb{D}, \rho_1 \leq |x| \leq 1\}, \quad \mathrm{supp}(\mathsf{q}^W) \subset \{x \in \mathbb{D}, \rho_1 \leq |x| \leq \rho_2\}, \\ &\|\mathsf{q}^W\|_{C^{\ell_0}(\mathbb{R}\times\mathbb{D};\mathbb{C})} \leq \eta, \\ &\|\partial_t \mathsf{u}^W(t,\cdot)\|^2_{L^2(\mathbb{D})} + \|\nabla \mathsf{u}^W(t,\cdot)\|^2_{L^2(\mathbb{D})} = 1, \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

The interest of Theorems 1.1 and 1.3, when compared to the existing literature, is twofold:

(1) Firstly, they provide with counterexamples to local unique continuation with critical Gevrey regularity.

Theorem 1.1 is a counterpart to the following result of the authors in [FLL25, Theorem 1.5] (see also [FLL24, Theorem 2.2]), proving that $\mathsf{q} \in \mathcal{G}^2(L^\infty)$ implies local/global unique continuation for the Schrödinger equation in a general setting.

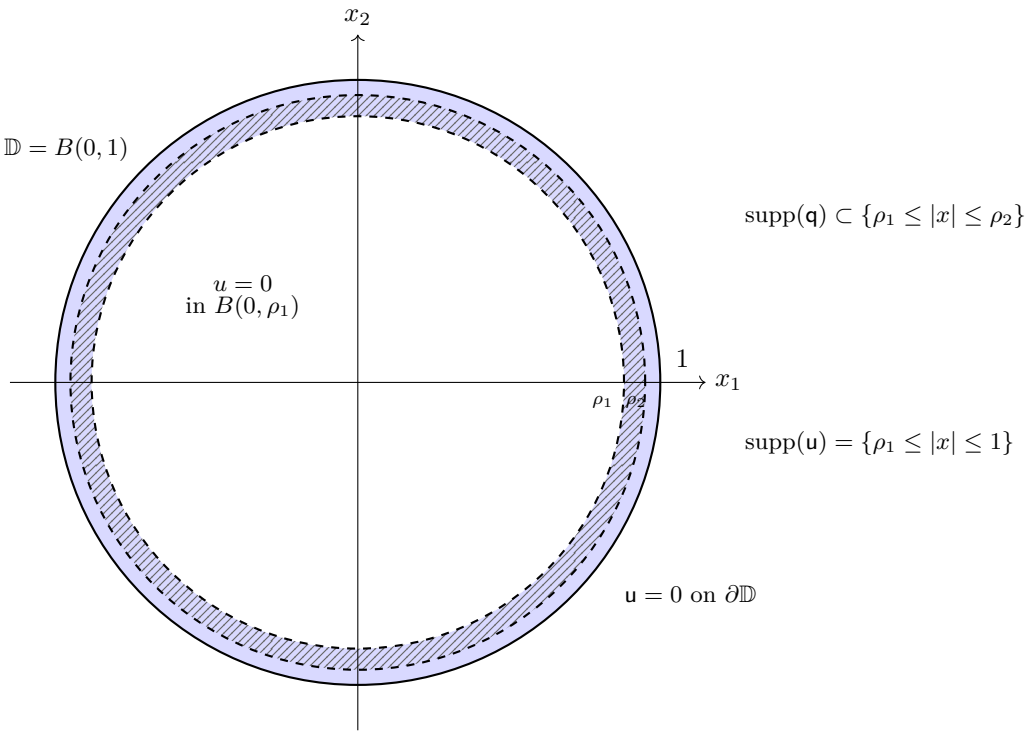


FIGURE 1. Schematic representation of the support properties of u and q in Theorem 1.1 (resp. of u^W and q^W in Theorem 1.3). The purple region corresponds to the support of the solution u. The support of the potential q is contained in the purple dashed region.

Theorem 1.4. *Let $T > 0$ and $\mathcal{M} = \mathrm{Int}(\mathcal{M}) \sqcup \partial\mathcal{M}$ be a connected smooth manifold with or without boundary $\partial\mathcal{M}$. Suppose that $g \in W^{1,\infty}_{\mathrm{loc}}(\mathrm{Int}(\mathcal{M}))$ is a Riemannian metric on $\mathrm{Int}(\mathcal{M})$, that $\mathsf{q} \in \mathcal{G}^2_{\mathrm{loc}}((0,T); L^\infty_{\mathrm{loc}}(\mathrm{Int}(\mathcal{M}); \mathbb{C}))$, Then, denoting by Δ_g the Laplace-Beltrami operator on $\mathrm{Int}(\mathcal{M})$, given ω any nonempty open set of $\mathcal{M}$, we have*

$$\begin{cases} (-i\partial_t - \Delta_g + \mathsf{q})u = 0 \text{ in } (0,T) \times \mathrm{Int}(\mathcal{M}) \\ u \in L^2_{\mathrm{loc}}((0,T) \times \mathrm{Int}(\mathcal{M})) \\ u = 0 \text{ in } (0,T) \times \omega \end{cases} \implies u = 0 \text{ in } (0,T) \times \mathrm{Int}(\mathcal{M}).$$

In this statement $\mathcal{G}^2_{\mathrm{loc}}((0,T); L^\infty_{\mathrm{loc}}(\mathrm{Int}(\mathcal{M}); \mathbb{C}))$ stands for the space of functions that belong to $\mathcal{G}^2_{\mathrm{loc}}(\mathcal{J}; L^\infty(W; \mathbb{C}))$ for all open sets $\mathcal{J}$ with $\overline{\mathcal{J}} \subset (0,T)$ and open sets W with $\overline{W} \subset \mathrm{Int}(\mathcal{M})$.

As for the wave equation, Theorem 1.3 is a counterpart to the Tataru-Robbiano-Zuily-Hörmander theorem [Tat95, RZ98, Hör97, Tat99] (see e.g. [LL19] for global quantitative statement) stating that $\mathsf{q} \in \mathcal{G}^1(L^\infty)$ implies local unique continuation for the wave equation across any noncharacteristic hypersurface and global unique continuation from any open set in sufficiently long time. That $\mathcal{G}^1$ is the critical regularity for *local* unique continuation was already proved by Hörmander [Hör00]. This results however only disproves *local* unique continuation; the solution constructed is however *not* solution to a well-posed Cauchy problem, hence not disproving global unique continuation.

(2) Secondly, the functions u, u^W in Theorems 1.1 and 1.3 are solution of a well-posed Cauchy problem. Hence Theorem 1.1 (for instance, a similar discussion holds for Theorem 1.3) implies existence of q (as in Theorem 1.1) and $\mathsf{u}_0 \in L^2(\mathbb{D})$ such that the unique solution u

$$\begin{cases} (-i\partial_t - \Delta + \mathsf{q})\mathsf{u} = 0, & \text{in } \mathbb{R} \times \mathrm{Int}(\mathbb{D}), \\ \mathsf{u} = 0, & \text{on } \mathbb{R} \times \partial\mathbb{D}, \\ \mathsf{u}(0,\cdot) = \mathsf{u}_0, & \text{in } \mathbb{D}, \end{cases}$$

(that is to say, to (1.1) with $\mathsf{u}(0,\cdot) = \mathsf{u}_0$) satisfies $\mathrm{supp}(\mathsf{u}) = \{x \in \mathbb{D}, \rho_1 \leq |x| \leq 1\}$ (one indeed only have to take $\mathsf{u}(0,\cdot) := \mathsf{u}_0$). The construction of such global counterexamples seems to be new and suggests that these solutions are "physical". Moreover, it also contradicts some approximate controllability results as described in Section 1.2.1 below.

Remark 1.5 (Neumann boundary conditions)**.** Theorem 1.1 (resp. Theorem 1.3) remains valid if we replace the Dirichlet boundary condition $\mathsf{u}|_{\mathbb{R}\times\partial\mathbb{D}} = 0$ in (1.1) by the Neumann boundary condition $\partial_\nu \mathsf{u}|_{\mathbb{R}\times\partial\mathbb{D}} = 0$, where $\nu = \frac{x}{|x|}$ is the outward pointing unit normal vector field to $\partial\mathbb{D}$. The proof simply uses Lemma 5.4 instead of Lemma 5.3 below.

Remark 1.6 (Conserved quantities)**.** The solutions in Theorems 1.1 and 1.3 actually have many additional preserved quantities. In fact, they share many common properties with the free Schrödinger and wave equations. They are indeed constructed as superposition of truncated free solutions, as described below. Quite surprisingly, the cutoff functions do not kill the orthogonality between the different free solutions, see Lemma 5.1.

1.2. **Comments and related literature.** In all the article, we write

$$\mathbb{D}_R := \{x \in \mathbb{R}^2, |x| \leq R\} = \overline{B}(0,R), \quad R > 0,$$

for the closed disk of radius R. With this notation, $\mathbb{D} = \mathbb{D}_1$.

1.2.1. *About approximate controllability.* Theorem 1.1 has the following corollary concerning (lack of) approximate controllability. We recall that the controlled Schrödinger equation

$$\begin{cases} i\partial_t v + \Delta v + \mathfrak{q} v = \mathbb{1}_\omega f, & \text{in } (0,T)\times \mathrm{Int}(\mathbb{D}), \\ v = 0, & \text{on } (0,T)\times\partial\mathbb{D}, \\ v(0,\cdot) = 0, & \text{in } \mathrm{Int}(\mathbb{D}), \end{cases} \tag{1.3}$$

is approximately controllable in time T if for any target $v_T \in L^2(\Omega)$, there exists a control force $f \in L^2((0,T)\times\omega)$ such that the associated solution v of (1.3) satisfies $v(T,\cdot) = v_T$.

Corollary 1.7. *For any $\rho_0 \in (0,1)$, $\varepsilon > 0$, $\eta > 0$, $\ell_0 \in \mathbb{N}$, there is $\mathfrak{q} \in \bigcap_{\ell\in\mathbb{N}} \mathcal{G}^{2+\varepsilon}(\mathbb{R}; C^\ell(\mathbb{D};\mathbb{C}))$ with $\|\mathfrak{q}\|_{C^{\ell_0}(\mathbb{R}\times\mathbb{D};\mathbb{C})} \leq \eta$ such that for any open set $\omega \subset \mathbb{D}_{\rho_0}$ and any time $T > 0$, the Schrödinger equation (1.3) is* not *approximately controllable from ω in time T.*

Corollary 1.7 is a direct consequence of Theorem 1.1 with $\mathfrak{q} = \overline{\mathfrak{q}}$, the fact that the solution u in this result is actually the unique solution to the well-posed Cauchy problem (1.1) with fixed $u(0,\cdot) \in L^2(\mathbb{D})$, and the link between unique continuation and approximate controllability, see e.g. [FLL24, Lemma 1.1]. This proves that the regularity $\mathcal{G}^2(L^\infty)$ is critical for approximate controllability in general (that is to say, without any further geometric assumption).

Similarly, we recall that the controlled wave equation

$$\begin{cases} \partial_t^2 v - \Delta v + \mathfrak{q} v = \mathbb{1}_\omega f, & \text{in } (0,T)\times \mathrm{Int}(\mathbb{D}), \\ v = 0, & \text{on } (0,T)\times\partial\mathbb{D}, \\ (v,\partial_t v)(0,\cdot) = 0, & \text{in } \mathrm{Int}(\mathbb{D}), \end{cases} \tag{1.4}$$

is approximately controllable in time T if for any target $(v_T^0, v_T^1) \in H_0^1(\mathbb{D}) \times L^2(\mathbb{D})$, there exists a control force $f \in L^2((0,T)\times\omega)$ such that the associated solution v of (1.4) satisfies $v(T,\cdot) = v_T$.

Corollary 1.8. *For any $\rho_0 \in (0,1)$, $\varepsilon > 0$, $\eta > 0$, $\ell_0 \in \mathbb{N}$, there is $\mathfrak{q} \in \bigcap_{\ell\in\mathbb{N}} \mathcal{G}^{1+\varepsilon}(\mathbb{R}; C^\ell(\mathbb{D};\mathbb{C}))$ with $\|\mathfrak{q}\|_{C^{\ell_0}(\mathbb{R}\times\mathbb{D};\mathbb{C})} \leq \eta$ such that for any open set $\omega \subset \mathbb{D}_{\rho_0}$ and any time $T > 0$, the wave equation (1.4) is* not *approximately controllable from ω in time T.*

Corollary 1.8 is a direct consequence of Theorem 1.3 with $\mathfrak{q} = \overline{\mathfrak{q}}$, the fact that the solution u in this result is actually the unique solution to the well-posed Cauchy problem (1.2) with initial data, and the link between unique continuation and approximate controllability, see e.g. [LL23, Section 1.1.2].

This proves that the regularity $\mathcal{G}^1(L^\infty)$ is critical for approximate controllability in general (that is to say, without any further geometric assumption).

1.2.2. *About unique continuation for the Schrödinger equation on the disk.* Theorem 1.1 completes the characterization of subsets of the disk which enjoy "stable unique continuation". Namely, we recall the following result, consequence of [ALM16, Theorem 1.2] (combined with a perturbation argument).

Theorem 1.9. *Let $T > 0$, $\omega \subset \mathbb{D}$ be an open set of $\mathbb{D}$ such that $\omega \cap \partial\mathbb{D} \neq \emptyset$. Then there is $\varepsilon_0 > 0$ such that the following unique continuation property*

$$\begin{cases} (-i\partial_t - \Delta + \mathfrak{q})u = 0 \text{ in } (0,T)\times \mathrm{Int}(\mathbb{D}) \\ u \in C^0([0,T]; L^2(\mathbb{D})) \\ u = 0 \text{ (weakly) on } (0,T)\times\partial\mathbb{D}, \\ u = 0 \text{ in } (0,T)\times\omega \end{cases} \implies u = 0 \text{ in } (0,T)\times\mathbb{D}.$$

holds for any $\mathfrak{q} \in L^1(0,T;L^\infty(\mathbb{D}))$ such that $\|\mathfrak{q}\|_{L^1(0,T;L^\infty(\mathbb{D}))} < \varepsilon_0$.

Theorem 1.9 is a direct consequence of [ALM16, Theorem 1.2] (with zero potential) together with a perturbation argument proved in Proposition D.1. Actually, these two results imply an observability inequality, which is a linear quantification of the unique continuation statement (1.9).

Theorem 1.1 provides with a counterpart of this result: in case $\overline{\omega} \cap \partial\mathbb{D} = \emptyset$, stable uniqueness in the sense of Theorem 1.9 does not hold.

1.2.3. *About pseudoconvexity.* We recall here for comparison the general results of stable local unique continuation for the wave and Schrödinger operators, across a hypersurface. We begin with the wave equation where the statements are easier to present.

For a general second-order operator, the classical Hörmander Theorem [Hör94, Chapter 28] assumes that the oriented hypersurface $S = \{\Psi = 0\}$ is strongly pseudconvex for P at $\mathbf{x}_0 = (t_0, x_0) \in I \times V$ (see [LL23, Section 2]):

$$p_2(\mathbf{x}_0,\xi) = \{p_2,\Psi\}(\mathbf{x}_0,\xi) = 0 \Longrightarrow \{p_2,\{p_2,\Psi\}\}(\mathbf{x}_0,\xi) > 0, \text{ for all } \xi \in T^*_{\mathbf{x}_0}(I\times V)\backslash\{0\}. \quad (1.5)$$

Here, $\{\cdot,\cdot\}$ denotes the Poisson bracket, and the geometric content of this condition is explained e.g. in [LL23, Section 2]. In the particular case of the wave operator $P = \partial_t^2 - \Delta$, then $p_2(t,x,\xi_t,\xi_x) = -\xi_t^2 + |\xi_x|^2$, and, if the oriented surface is $\mathbb{R}\times\partial\mathbb{D}_{\rho_1} = \{\Psi_\pm = 0\}$ with $\Psi_\pm(t,x) = \pm(|x| - \rho_1)$, then (1.5) can be equivalently rephrased at the point (t_0,x_0) as

$$(|\xi_t| = |\xi_x|, \quad \xi_x \cdot x_0 = 0) \Longrightarrow \pm|\xi_x|^2 > 0, \qquad \text{for all } (\xi_t,\xi_x) \in T^*_{\mathbf{x}_0}(I\times V)\backslash\{0\}. \quad (1.6)$$

The latter is satisfied for Ψ_+, whence stable unique continuation holds from $|x| > \rho_1$ to $|x| < \rho_1$, but not for Ψ_-. The Hörmander Theorem [Hör94, Chapter 28] does not apply to prove unique continuation from $|x| < \rho_1$ to $|x| > \rho_1$ and Theorem 1.3 actually proves that stable unique continuation does not hold.

The classical Hörmander Theorem [Hör94, Chapter 28] does not apply to the Schrödinger operator $-i\partial_t - \Delta$. Taking advantage of the anisotropic (or quasi-homogeneous) nature of the Schrödinger operator, Lascar and Zuily proved in [LZ82] that the results of Hörmander [Hör94, Chapter 28] can be generalized to the anisotropic case with an appropriate modification of the symbol classes and Poisson bracket, see also [Deh84], [Isa93] and [Tat97] for later results in this direction. In the context of the Schrödinger operator $-i\partial_t - \Delta$, at a point $\mathbf{x}_0 = (t_0,x_0) \in I\times V$, the related sufficient condition for unique continuation across the oriented time-invariant hypersurface $S = \{\Psi = 0\}$ for $\Psi(t,x) = \Psi(x)$ is $d\Psi(x_0) \neq 0$ and

$$\xi_x \cdot \partial_x\Psi(x_0) = 0 \Longrightarrow \mathrm{Hess}\,\Psi(x_0)(\xi_x,\xi_x) > 0, \qquad \text{for all } \xi_x \in T_{x_0}V\backslash\{0\}.$$

As in the case of the wave equation with $\Psi_\pm(t,x) = \pm(|x| - \rho_1)$, this condition reads $\xi_x \cdot x_0 = 0 \Longrightarrow \pm|\xi_x|^2 > 0$ for all $\xi_x \in T_{x_0}V\backslash\{0\}$. Thus it holds for Ψ_+, whence stable unique continuation holds from $|x| > \rho_1$ to $|x| < \rho_1$, but not for Ψ_-. The result of [LZ82] does not apply to prove unique continuation from $|x| < \rho_1$ to $|x| > \rho_1$ and Theorem 1.1 actually proves that stable unique continuation does not hold.

In the case of partial analyticity of the coefficients with respect to the time for the wave equation, or partial Gevrey 2 regularity for the Schrödinger equation, the pseudoconvexity needs to be checked only in $\{\xi_t = 0\}$. This leads to much less restrictive geometric conditions as in Theorem 1.4 for the Schrödinger equation or the Tataru-Robbiano-Zuily-Hörmander theorem [Tat95, RZ98, Hör97, Tat99] for the wave equation.

1.2.4. *On previous counterexamples to unique continuation.* There are mainly two families of counterexamples in the literature, which are reviewed [LL23, Section 2.4] in the case of the wave operator.

The first family concerns *local* constructions of counterexamples near a point. A very general theorem of Alinhac [Ali83, Théorème 2] holds for all second order operator P with nondegenerate real symbol p_2. It states that if $\Psi \in C^\infty(\Omega;\mathbb{R})$ is such that $\Psi(x_0)=0$ and $p_2(x_0,d\Psi(x_0)) \neq 0$, and if there exists $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ such that

$$p_2(x_0,\xi_0) = \{p_2,\Psi\}(x_0,\xi_0) = 0 \quad \text{and} \quad \{p_2,\{p_2,\Psi\}\}(x_0,\xi_0) < 0, \tag{1.7}$$

then, there exist $U \subset \Omega$ a neighborhood of x_0 and $q,u \in C^\infty(U;\mathbb{C})$ such that

$$Pu + qu = 0 \text{ in } U, \quad \text{and} \quad \operatorname{supp}(u) \cap U = \{\Psi \geq 0\} \cap U.$$

In particular, under Assumption (1.7) on the oriented hypersurface $\{\Psi = 0\}$, unique continuation from $\{\Psi > 0\}$ to a neighborhood of x_0 does not hold for the operator $P+q$. This applies to the wave operator. Condition (1.7) is a strong negation of strong pseudoconvexity (1.5), which is satisfied in the setting of the Theorem 1.3, see (1.6) in case Ψ_-.

In the particular case of the wave operator $P = \partial_t^2 - \Delta$ in $\mathbb{R}^{1+2}$, Alinhac-Baouendi [AB79, Théorème 2] and [AB95, Theorem] actually pushed the local counterexamples to the limit case where $\{p_2,\{p_2,\Psi\}\}(x_0,\xi_0) = 0$ (improving on [Hör63, Theorem 8.9.4] where an additional first-order term appears). They prove that there exist an open set $U \subset \mathbb{R}^{1+d}$ with $0 \in U$ and $q,u \in C^\infty(U;\mathbb{C})$ such that

$$Pu + qu = 0 \text{ in } U, \quad \text{and} \quad \operatorname{supp}(u) \cap U = \{x_1 \geq 0\} \cap U.$$

This implies that unique continuation fails for the operator $P+q$ across the hypersurface $\{x_1 = 0\}$, even though P has constant coefficients and the perturbation q is of lower-order and smooth. Note that the principal symbol of $P+q$ is $p_2(t,x,\xi_t,\xi_x) = -\xi_t^2 + |\xi_x|^2$ and the hypersurface $\{x_1 = 0\}$ barely fails to be strongly pseudoconvex (1.5). The regularity in this result has been pushed by Hörmander [Hör00], showing that the potential q and the solution u in the Alinhac-Baouendi result can be taken in the Gevrey class $\mathcal{G}^{1+\varepsilon}$ for any $\varepsilon > 1$.

Lascar-Zuily proved in [LZ82] an analogue of the result of [Ali83] in the setting of the Schrödinger equation, showing that stable local unique continuation does not hold if (1.6) is "strongly violated".

Finally, Guisset and Shao [GS24] generalized the constructions of [AB79, AB95] to geometric wave equations having a singular potential.

The second family of counterexamples (to which the present work belongs) is concerned with *semiglobal* counterexamples. They have been constructed by Kumano-go [Kg63] (inspired by a construction by John [Joh60] and Pliś [Pli61]) and Takase [Tak21]. Kumano-go [Kg63] proved the existence of $u,g,q \in C^\infty(\mathbb{R}^{1+2};\mathbb{R})$ *real-valued* solving (for the Euclidean metric) the damped wave equation $(\partial_t^2 - \Delta + g\partial_t + q)u = 0$ in $\mathbb{R}^{1+2}$ with $\operatorname{supp}(u) = \mathbb{R} \times \big(\mathbb{R}^2 \setminus B(0,1)\big)$. Takase [Tak21], proved existence of $u,q \in C^\infty(\mathbb{R}^{1+2};\mathbb{C})$ solving (for the Euclidean metric) $(\partial_t^2 - \Delta + q)u = 0$ (resp. $(-i\partial_t - \Delta + q)u = 0$) in $\mathbb{R}^{1+2}$ and $\operatorname{supp}(u) = \mathbb{R} \times \big(\mathbb{R}^2 \setminus B(0,1)\big)$. As already mentioned, the construction in the proof of Theorems 1.1 and 1.3 builds on those of Kumano-go [Kg63] and Takase [Tak21].

1.3. **Sketch of proofs and plan of the paper.**

1.3.1. *On the construction and Gevrey regularity.* We start by explaining informally the idea behind the constructions appearing in [Kg63] and [Tak21], which are closely related to the ideas of [Joh60], and upon which the present paper builds. We focus on the wave operator

$$\Box := \partial_t^2 - \Delta = \partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2.$$

Using a scaling argument $(t,x) \mapsto \left(\frac{t}{\rho_1},\frac{x}{\rho_1}\right)$ leaving unchanged the operator $\Box$, we are left with constructing a solution w to $\Box w + Vw = 0$, with V to be determined, having $\operatorname{supp}(w) = \mathbb{R}_t \times \big(\mathbb{R}^2 \setminus B(0,1)\big)$. In particular, the boundary of the support is the spacetime cylinder $\mathbb{R} \times \{|x| = 1\}$. In this situation, it is natural to work in usual polar coordinates

$$\mathbb{R}^2 \setminus \{0\} \ni x = (r\cos\theta, r\sin\theta), \quad \text{with } r \in \mathbb{R}_+^*, \theta \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

In this setting, the building blocks of the the construction of the solution w are the functions $(t,r,\theta)\mapsto J_n(nr)e^{in(t+\theta)}$ where J_n is the Bessel function of order $n\in\mathbb{N}$. From the Bessel equation satisfied by J_n, the latter are *exact solutions* of the wave equation

$$\Box\left(J_n(nr)e^{in(t+\theta)}\right)=0,\quad (t,r,\theta)\in\mathbb{R}\times\mathbb{R}_+^*\times\mathbb{S}^1.$$

Now, $J_n(nr)e^{in(t+\theta)}$ solves the wave equation but has full support in $\mathbb{R}\times\mathbb{R}_+^*\times\mathbb{S}^1$. We therefore truncate them by using a family of radial cut-offs $r\mapsto\chi_n(r)$, with $\mathrm{supp}(\chi_0)\subset(\frac{3}{2},+\infty)$ and $\mathrm{supp}(\chi_n)\subset(1+\frac{1}{n+2},1+\frac{1}{n})$, $n\in\mathbb{N}^*$. This forces the support of $\chi_n(r)J_n(nr)e^{in(t+\theta)}$ outside the cylinder $r<1$, at the price of not solving anymore the free wave equation since

$$\Box\left(\chi_n(r)J_n(nr)e^{in(t+\theta)}\right)=[\Box,\chi_n]\left(J_n(nr)e^{in(t+\theta)}\right)\neq 0.$$

In order to be able to divide $\Box w$ by the function w in the construction, as well as to impose that w has full support in $r\geq 1$, one defines

$$w(t,r,\theta):=\sum_n\chi_n(r)J_n(nr)e^{in(t+\theta)}.$$

This is an infinite but locally finite sum, and the supports of the radial cut-off functions χ_n cover the whole set $r\geq 1$, while accumulating to $r\to 1^+$ as $n\to\infty$. They are also trivially zero for $r<1$. The function w constructed in this way is no longer an exact solution of the wave equation, and this is where the potential comes into play. We *define* the potential V by $V=\frac{-\Box w}{w}$ so that $\Box w+Vw=0$. The delicate part is then to verify that w has indeed the desired support properties and that such a potential V is well defined and regular.

A crucial point of this construction is that the family $r\mapsto J_n(nr)$, $n\geq 1$, changes behavior radically at $r=1$: for $r<1$ the functions are positive and decay exponentially as $n\to\infty$, while for $r>1$ they strongly oscillate and decay like a negative power of n. This motivates the introduction of two oscillation parameters λ_m and k_m in the above construction, and work with the functions

$$w(t,r,\theta):=\sum_{m\geq M}a_m\chi_m(r)J_{\lambda_m}(k_mr)e^{i(\lambda_m\theta+k_mt)},$$

where $a_m\in\mathbb{R}^+$ are correction parameters. Here, λ_m is the main oscillation scale, satisfying $\lambda_m\in\mathbb{N}$, $\lambda_m\to+\infty$ and $k_m\sim\lambda_m$ is of same order. Each building block $J_{\lambda_m}(k_mr)e^{i(\lambda_m\theta+k_mt)}$ remains an exact solution to $\Box\left(J_{\lambda_m}(k_mr)e^{i(\lambda_m\theta+k_mt)}\right)=0$. The concentration scale in the variable r remains fixed by $\mathrm{supp}(\chi_m)\subset(1+\frac{1}{m+2},1+\frac{1}{m})$. Imposing $k_m/\lambda_m=1-1/m$ ensures that on the support of χ_m the function

$$J_{\lambda_m}(k_mr)=J_{\lambda_m}\left(\lambda_m\cdot\big(1-\frac{1}{m}\big)r\right)$$

decays exponentially as $m\to\infty$ since $\big(1-\frac{1}{m}\big)r\leq\big(1-\frac{1}{m}\big)\big(1+\frac{1}{m}\big)=1-\frac{1}{m^2}<1$ for $r\in\mathrm{supp}\,\chi_m$. This is what makes w smooth at $r=1$. The parameters a_m are there to ensure that V grows in a controlled way, and guarantee that it is sufficiently smooth.

To establish our main theorems, we need to examine each of the objects arising in the constructions above in detail. Obtaining Gevrey regularity considerably complicates this task, as it necessitates quantitative control of derivative growth with respect to the various parameters involved. The treatment of the potential is particularly delicate and was the primary motivation for employing an abstract Gevrey framework.

In addition, carefully choosing the initial cut-off χ_M in the definition of w and exploiting the behavior of the first zero of the function $J_{\lambda_M}(\cdot)$ allows to force $w=0$ (resp. $\partial_r w=0$) at some $r=\mathsf{R}_M>1$ (as close to 1 as wished by choosing M large). This provides with a solution satisfying a *well posed Cauchy problem* with Dirichlet (or Neumann) boundary conditions. The statements in a disk of radius 1 finally follow from the scaling argument described above.

1.3.2. *Link with complex WKB solutions.* It is interesting to relate the construction used in this paper (together with [Kg63, Tak21]) with the counterexample machinery developed in [Ali83, AB79, AB95]. The latter articles produce a general mechanism resulting in non uniqueness examples and they are based on the construction of Complex WKB (or Complex Geometric Optics) solutions, which are approximate solutions of $\Box v=0$ of the form $v=\alpha_m e^{i\lambda\tilde{\phi}}$ with a *complex valued* phase

$\tilde{\phi}$ and a large parameter $\lambda \to +\infty$. Here α_m is the amplitude of the approximate solution and one can imagine that $\alpha_m \sim 1$. Hence, in the case of a *real valued phase* one obtains solutions whose magnitude is roughly speaking equal to 1. Allowing the phase to take complex values with positive imaginary part gives some decay on the modulus of the solution and it is important for the construction of those examples. We refer to [Ler19, Chapter 6.1.2] for a detailed presentation of this approach.

In these WKB constructions, one considers a real-valued function ϕ satisfying the eikonal equation $p(x, d\phi) = 0$, where p denotes the principal symbol of the operator under consideration. In case the operator is $\Box$ in polar coordinates, then $p(t, r, \theta, \xi_t, \xi_r, \xi_\theta) = -\xi_t^2 + \xi_r^2 + \frac{\xi_\theta^2}{r^2}$. The function ϕ then plays the role of the real part of the complex phase $\tilde{\phi}$. In the case of the wave operator in dimension 1+2 the eikonal equation reads in polar coordinates

$$(\partial_t \phi)^2 = (\partial_r \phi)^2 + \frac{(\partial_\theta \phi)^2}{r^2},$$

and we see that for $r = 1$, this is satisfied by $\phi = t + \theta$. This makes a connection with the construction used in this article and allows to interpret it as WKB solution as well. Indeed, looking at $J_{\lambda_m}(k_m r)e^{i\lambda_m(t+\theta)}$ we see that $\phi(t, r, \theta) = t+\theta$ corresponds to the real part of the phase close to $r = 1$. The imaginary part is already encoded $J_{\lambda_m}(k_m r)$ in the sense that $J_{\lambda_m}(k_m r)$ has an exponential decay, similarly to the general complex WKB constructions.

1.3.3. *Plan of the paper.* In Section 2 we define the main parameters and construct the solution and the associated potential in the case of the Schrödinger equation. In Sections 3 and 4 we prove the Gevrey regularity properties for the solution and the potential respectively. In Section 5, we prove that the solution just constructed has many conserved quantities, and correct it so that it satisfies a Dirichlet/Neumann problem. We then put all the pieces together and finish the proof of Theorem 1.1 on the Schrödinger equation. We explain the modifications needed for the proof of Theorem 1.3 (wave equation) in Section 6. Appendix A contains properties of the (modified) Bessel functions and Appendix B collects various technical statements on $\mathcal{B}$-valued Gevrey spaces, which are then specialized to the case $\mathcal{B} = C^\ell$. In Appendix C we provide with a technical maximization lemma transforming parameter depending derivative estimates into Gevrey classes statement, and we conclude with Appendix D, which proves that observability is stable by perturbation.

2. Construction of the solution u and the potential V

We start by collecting basic informations about the objects involved in the construction. We shall use polar coordinates $(r, \theta) \in \mathbb{R}_+^* \times \mathbb{S}^1$, $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, for the space variable in $\mathbb{R}^2 \setminus \{0\}$.

2.1. Definition of the oscillation scales, modified Bessel functions, and cutoff functions.

As already mentioned, there are several oscillation scales, which we introduce now. The main oscillation scale is the oscillation scale λ_m in the angular variable θ. The second main oscillation scale is the oscillation scale k_m (or rather k_m^2 for the Schrödinger equation) in the time variable t.

Assumption 2.1 (Oscillation scale λ_m)**.** *We assume that there is $p \in (0, \frac{1}{5})$ and $A \geq 1$ such that for all $m \in \mathbb{N}^*$ the sequence $(\lambda_m)_{m\in\mathbb{N}}$ satisfies the following properties*

- $\lambda_m \in \mathbb{N}$,
- $\lambda_m < \lambda_{m+1}$ *for all* $m \in \mathbb{N}^*$,
- $m^{\frac{2}{p}} \leq \lambda_m \leq A m^{\frac{2}{p}}$ *for all* $m \in \mathbb{N}^*$,
- $\lim_{m\to\infty} \dfrac{\lambda_{m+1}}{\lambda_m} = 1$.

One can take for instance $\lambda_m = \lfloor m^{\frac{2}{p}} \rfloor + 1$. The parameter p will be chosen at the end close enough to zero only depending on ε, see (5.5). So, we will consider it fixed from now on and we will drop the dependency of the constants on p and also on A which is arbitrary. We consider as well the sequence of real numbers $(k_m)_{m\in\mathbb{N}}$, defined by

$$k_m := \lambda_m \left(1 - \frac{1}{m}\right) \qquad \text{for } m \in \mathbb{N}. \tag{2.1}$$

Note that we will only use in the construction the set of indices $m \geq M$ for $M \in \mathbb{N}$ to be fixed later on (sufficiently large). The condition $\lambda_m < \lambda_{m+1}$ will only be used in proving that the L^2-norm

of the solution $u(t,\cdot)$ is preserved (see Lemma 5.1 below). Recalling the definition of the Bessel functions J_ν in (A.1), we set

$$F_m(r) = J_{\lambda_m}(k_m r),$$

a real-valued real-analytic function, and observe from the Bessel equation (A.2) that

$$F_m''(r) + \frac{1}{r}F_m' + \left(k_m^2 - \frac{\lambda_m^2}{r^2}\right)F_m(r) = 0. \tag{2.2}$$

Now for $(t,r,\theta) \in \mathbb{R}\times\mathbb{R}_+^*\times\mathbb{S}^1$, we define

$$u_m(t,r,\theta) := F_m(r)e^{i(\lambda_m\theta - k_m^2 t)}. \tag{2.3}$$

Notice that the condition $\lambda_m \in \mathbb{N}$ implies that $\theta \to e^{i\lambda_m\theta}$ is well defined on $\mathbb{S}^1$ and hence that $u_m \in C^\infty(\mathbb{R}\times\mathbb{R}_+^*\times\mathbb{S}^1)$. We recall that the Laplace operator in polar coordinates reads

$$\Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2,$$

and we set

$$P := -i\partial_t - \Delta.$$

Using (2.2) we see that

$$Pu_m(t,r,\theta) = (-i\partial_t - \Delta)u_m(t,r,\theta) = 0, \quad \text{for} \quad (t,r,\theta)\in\mathbb{R}\times\mathbb{R}_+^*\times\mathbb{S}^1, m\in\mathbb{N}. \tag{2.4}$$

For $m \in \mathbb{N}^*$ we consider the intervals

$$I_m := \left[1 + \frac{1}{m+1}, 1 + \frac{1}{m}\right].$$

Notice that

$$|I_m| = 1 + \frac{1}{m} - \left(1 + \frac{1}{m+1}\right) = \frac{1}{m(m+1)}.$$

We divide each interval I_m into 4 sub-intervals $I_{m,j} \subset I_m$, of equal length (see Figure 2),

$$I_{m,j} := \left[1 + \frac{1}{m} - \frac{j}{4m(m+1)}, 1 + \frac{1}{m} - \frac{j-1}{4m(m+1)}\right], \quad j \in \{1,2,3,4\}.$$

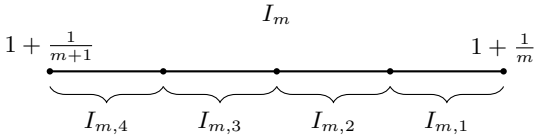


FIGURE 2. The interval I_m is divided in 4 sub-intervals $I_{m,j}$.

For $s \in [0,1]$ we define

$$r_m(s) := 1 + \frac{1}{m} - \frac{s}{m(m+1)}, \tag{2.5}$$

so that $r_m(s)$ parametrizes points in the interval I_m from the right endpoint of I_m at $s = 0$ to the left endpoint at $s = 1$, namely

$$r_m(0) = 1 + \frac{1}{m}, \quad r_m(1) = 1 + \frac{1}{m+1}, \quad r_m'(s) = -\frac{1}{m(m+1)}.$$

In particular, we have for $j \in \{1,2,3,4\}$

$$r_m(s) \in I_{m,j} \quad \Longleftrightarrow \quad s \in \left[\frac{j-1}{4}, \frac{j}{4}\right].$$

We define the sequence $(\gamma_m)_{m\in\mathbb{N}^*}$ by

$$\gamma_{m+1} = \frac{F_m(r_{m+1}(2^{-1}))}{F_{m+1}(r_{m+1}(2^{-1}))}. \tag{2.6}$$

Notice that $r_{m+1}(2^{-1})$ is the middle point of the interval I_{m+1}, and that there is $M_0 > 0$ such that $F_{m+1}(r_{m+1}(2^{-1})) > 0$ for all $m \geq M_0$, according to Lemma A.3, whence γ_{m+1} is well-defined

for $m > M_0$. Relative asymptotic estimates of the quantities $\gamma_{m+1}, F_{m+1}(r_{m+1}(s)), F_m(r_{m+1}(s))$ as $m \to +\infty$ are given in Lemma A.4.

We now take $M \in \mathbb{N}$ (M will be taken large enough in the proofs) and define a smooth function $\check{A}_M$ satisfying

$$\check{A}_M(r) = \begin{cases} 1, & r \geq 1 + \frac{1}{M+2} + \frac{1}{4(M+1)(M+2)}, \\ 0, & 0 \leq r \leq 1 + \frac{1}{M+2}, \end{cases} \qquad \check{A}_M \in C^\infty(\mathbb{R}). \tag{2.7}$$

We also define, for $m \in \mathbb{N}$ smooth functions A_m by

$$A_m(r) = \begin{cases} 1, & r \in (I_{m+1} \backslash I_{m+1,4}) \cup (I_m \backslash I_{m,1}), \\ 0, & r \in [0, 1 + \frac{1}{m+2}] \cup (1 + \frac{1}{m}, \infty), \end{cases} \qquad A_m \in C_c^\infty(\mathbb{R}). \tag{2.8}$$

The shape of the function A_m is illustrated on Figure 3.

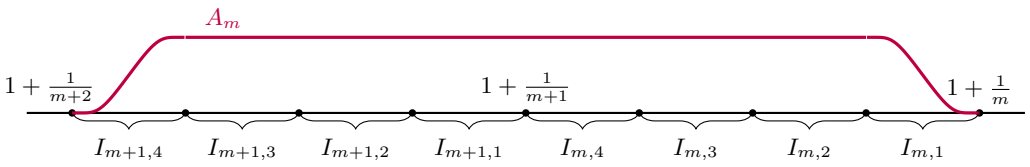


FIGURE 3. Shape of the cut-off function A_m.

In particular, $\mathrm{supp}(A_m) \subset I_m \cup I_{m+1}$ and $\mathrm{supp}(A_m') \subset I_{m,1} \cup I_{m+1,4}$. By rescaling a fixed function, we can choose A_m so that for any $\ell \in \mathbb{N}$, it also satisfies for some $C_\ell > 0$,

$$\left\| A_m^{(\ell)} \right\|_{L^\infty(\mathbb{R})} \leq C_\ell m^{2\ell}, \quad \text{for all } m \in \mathbb{N}. \tag{2.9}$$

2.2. **Definition of the solution u.** Equipped with the oscilation scales λ_m, k_m, the modified Bessel functions F_m and cutoff functions A_m, we are now prepared to define the solution u. Note that this function will be chosen to have support in $r \in [1, +\infty)$, and our final counterexamples u in Theorem 1.1 will have the announced support thanks to a scaling argument. We define

$$u(t, r, \theta) := \check{A}_M(r) u_M(t, r, \theta) + \sum_{m=M+1}^{\infty} \left(\prod_{n=M+1}^{m} \gamma_n \right) A_m(r) u_m(t, r, \theta). \tag{2.10}$$

Note that u depends implicitly on the parameter M, which we already assume $M \geq M_0$. The parameter M will be taken large at several places in the proof. Note also that only the functions A_m for $m \geq M + 1$ are used in the definition.

Lemma 2.2. *For all $(t, r, \theta) \in \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{S}^1$ and $M \geq 1$, the series defining u in (2.10) converges. There is $M_0 > 0$ such that for all $M \geq M_0$, the function u in (2.10) satisfies* $\mathrm{supp}(u) = \mathbb{R} \times [1, +\infty) \times \mathbb{S}^1$.

We will actually provide a proof in Section 3 (see Proposition 3.1) of the fact that the function u in (2.10) is smooth.

Proof. As already mentioned right after (2.6), γ_{M+1} is defined and $\gamma_{M+1} > 0$ from Lemma A.3 for all $m \geq M_0$, whence all terms in the sum (2.10) defining u are well-defined. Next, we let $M \geq M_0$ be fixed and notice from the definition of $\check{A}_M$ and A_m that

$$\{\check{A}_M \neq 0\} \subset \left(1 + \frac{1}{M+2}, +\infty\right), \quad \{A_m \neq 0\} \subset \left(1 + \frac{1}{m+2}, 1 + \frac{1}{m}\right). \tag{2.11}$$

As a consequence of (2.11), given $r_0 > 1$ we have $A_m(r_0) \neq 0$ for at most two indices m, so that the infinite sum in (2.10) is locally finite and hence the series defining $u(t, r, \theta)$ converges for $r > 1$. On the other hand, all terms vanish if $r \in (0, 1]$ so the series converges as well there.

The rest of the proof is devoted to proving $\mathrm{supp}(u) = \mathbb{R} \times [1, +\infty) \times \mathbb{S}^1$. From (2.11) we first notice that $u(t, r, \theta) = 0$ if $r \leq 1$. We now study vanishing of $u(t, r, \theta)$ on different intervals of r, noticing that $\mathbb{R}_+ \subset [0, 1] \cup \bigcup_{m \geq M+1} I_m \cup \left[1 + \frac{1}{M+1}, +\infty\right)$.

(1) On the interval $r \in \left[1+\frac{1}{M+1}, +\infty\right)$, we have $A_m(r) = 0$ for all $m \geq M+1$ from (2.11), hence, recalling (2.3), $u(t,r,\theta) = 0$ if and only if

$$0 = \check{A}_M(r) u_M(t,r,\theta) = \check{A}_M(r) F_M(r) e^{i(\lambda_M \theta - k_M^2 t)}.$$

Recalling from (2.7) that $\check{A}_M(r) = 1$ for $r \geq 1+\frac{1}{M+1} > 1+\frac{1}{M+2}+\frac{1}{4(M+1)(M+2)}$, we deduce that this is equivalent to $0 = F_M(r) = J_{\lambda_M}(k_M r)$. The set of r satisfying this equation is discrete (see e.g. Lemma A.2), whence the statement for $r \geq 1+\frac{1}{M+1}$.

(2) For $r \in I_{m_0+1} = \left[1+\frac{1}{m_0+2}, 1+\frac{1}{m_0+1}\right]$ for a fixed $m_0 \geq M+1$, we notice first that $\check{A}_M = 0$ and that $A_m = 0$ for all $m \notin \{m_0, m_0+1\}$. In this interval, we thus have from (2.10)

$$u(t,r,\theta) = \left(\prod_{n=M+1}^{m_0} \gamma_n\right) A_{m_0}(r) u_{m_0}(t,r,\theta) + \left(\prod_{n=M+1}^{m_0+1} \gamma_n\right) A_{m_0+1}(r) u_{m_0+1}(t,r,\theta),$$

and, given that $\gamma_n > 0$, $u(t,r,\theta) = 0$ is equivalent to

$$0 = A_{m_0}(r) u_{m_0}(t,r,\theta) + \gamma_{m_0+1} A_{m_0+1}(r) u_{m_0+1}(t,r,\theta),$$

that is to say

$$0 = A_{m_0}(r) F_{m_0}(r) e^{i(\lambda_{m_0}\theta - k_{m_0}^2 t)} + \gamma_{m_0+1} A_{m_0+1}(r) F_{m_0+1}(r) e^{i(\lambda_{m_0+1}\theta - k_{m_0+1}^2 t)}. \tag{2.12}$$

We now make the distinction between three different subsets of the interval I_{m_0+1}.

(a) On the central part $I_{m_0+1,2} \cup I_{m_0+1,3}$, we have both $A_{m_0} = A_{m_0+1} = 1$, so that this reformulates as

$$0 = F_{m_0}(r) e^{i(\lambda_{m_0}\theta - k_{m_0}^2 t)} + \gamma_{m_0+1} F_{m_0+1}(r) e^{i(\lambda_{m_0+1}\theta - k_{m_0+1}^2 t)},$$

but the function on the right-hand side is real-analytic (since J_ν is, see e.g. (A.1)) and non identically vanishing, so has full support in this interval.

(b) On $I_{m_0+1,1}$, we have $A_{m_0} = 1$. Moreover, since $m_0 \geq M+1 \geq M_0+1$, Inequality (A.4) in Lemma A.4 yields $0 < \gamma_{m_0+1} F_{m_0+1}(r) \leq C e^{-\mu \lambda_{m_0} m_0^{-3}} F_{m_0}(r)$, for all $r \in I_{m_0+1,1}$. In particular, we may divide by F_{m_0} on $I_{m_0+1,1}$ and take modulus in (2.12) to obtain

$$1 = \gamma_{m_0+1} A_{m_0+1}(r) \frac{F_{m_0+1}(r)}{F_{m_0}(r)} \leq C e^{-\mu \lambda_{m_0} m_0^{-3}}.$$

Thanks to Assumptions 2.1 and $0 < p < 2/3$, this is a contradiction if $M \geq M_1 = M_1(C, \mu, M_0)$, so u does not vanish on $I_{m_0+1,1}$.

(c) on $I_{m_0+1,4}$, we have $A_{m_0+1} = 1$. Moreover, since $m_0+1 \geq m_0 \geq M+1 \geq M_0+1$, Inequality (A.5) in Lemma A.4 yields $0 < F_m(r) \leq C e^{-\mu \lambda_m m^{-3}} \gamma_{m+1} F_{m+1}(r)$, for $r \in I_{m_0+1,4}$. In particular, we may divide by $\gamma_{m_0+1} F_{m_0+1}$ on $I_{m_0+1,4}$ and take modulus in (2.12) to obtain

$$1 = A_{m_0}(r) \frac{F_{m_0}(r)}{\gamma_{m_0+1} F_{m_0+1}(r)} \leq C e^{-\mu \lambda_m m^{-3}}.$$

This is again a contradiction if $M \geq M_1$, so u does not vanish on $I_{m_0+1,4}$.

(3) Finally, for $r \in I_{M+1} = \left[1+\frac{1}{M+2}, 1+\frac{1}{M+1}\right]$, we obtain from the definition of u in (2.10) together with (2.11) that

$$u(t,r,\theta) := \check{A}_M(r) u_M(t,r,\theta) + \gamma_{M+1} A_{M+1}(r) u_{M+1}(t,r,\theta),$$

see Figure 4. The situation is then exactly the same as in the previous case $r \in I_{m_0+1}$: the function u is real analytic for $r \in I_{M+1,2} \cup I_{M+1,3}$, and it does not vanish for $r \in I_{M+1,1}$ and $r \in I_{M+1,4}$ thanks to the asymptotics of Lemma A.4.

This concludes the proof of the lemma. □

2.3. **Definition of the potential V.** Since $Pu_m = Pu_M = 0$ from (2.4), we have that

$$Pu = [P, \check{A}_M]u_M + \sum_{m \geq M+1} \gamma_{M+1} \cdot \ldots \cdot \gamma_m [P, A_m] u_m.$$

Notice that for $m \geq M+1$

$$[P, A_m] = [-i\partial_t - \Delta, A_m] = -\left[\partial_r^2 + r^{-1}\partial_r, A_m\right] = -\left(2A_m'\partial_r + \frac{A_m'}{r} + A_m''\right), \tag{2.13}$$

is a first order differential operator on $\mathbb{R} \times \mathbb{R}_+^* \times \mathbb{S}^1$, supported on $r \in \mathrm{supp}(A_m')$. The same computation and remark holds for $[P, \check{A}_M]$. That is to say, $[P, \check{A}_M]$ is supported on

$$r \in \left[1 + \frac{1}{M+2}, 1 + \frac{1}{M+2} + \frac{1}{4(M+1)(M+2)}\right] = I_{M+1,4}$$

and for all $m \geq M+1$, $[P, A_m]$ is supported in $r \in I_{m+1,4} \cup I_{m,1}$. This implies

$$\mathrm{supp}(Pu) \subset \bigcup_{m \geq M+1} I_{m,1} \cup I_{m,4}. \tag{2.14}$$

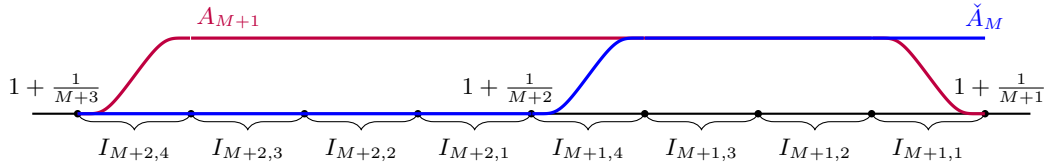


FIGURE 4. In blue the graph of $\check{A}_M$ and in purple the graph of A_m for $m = M+1$.

We now define the potential V. Consider the set

$$K = [0,1] \cup \left[1 + \frac{1}{M+1}, +\infty\right) \cup \bigcup_{m \geq M+1} (I_{m,2} \cup I_{m,3}), \tag{2.15}$$

and define finally the potential V by

$$V(t,r,\theta) = \begin{cases} 0, & r \in K, \\ -\frac{Pu}{u}, & r \in [0,\infty)\backslash K. \end{cases} \tag{2.16}$$

Remark that for $r \in K$ we have that $Pu = 0$ thanks to (2.14) and the definition of K. Together with (2.16), this yields

$$(Pu)(t,r,\theta) + V(t,r,\theta)u(t,r,\theta) = 0, \qquad \text{for } (t,r,\theta) \in \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{S}^1. \tag{2.17}$$

For the proof of Theorem 1.1 to be complete, we need to check the regularity of the solution u and the potential V, especially the Gevrey regularity. This will be done in the next two sections.

3. Regularity of the solution u

The present section is devoted to proving the following regularity property of the solution u defined in (2.10). Checking the regularity of the potential V is more intricate and postponed to Section 4 below. We recall that the space of Gevrey functions $\mathcal{G}^s(\mathbb{R}; \mathcal{B})$ is defined in Definition 1.2.

Proposition 3.1. *There exists $M_1 \in \mathbb{N}$ such that if $M \geq M_1$ in the definition (2.10) of the function u, we have $u \in \mathcal{G}^{\frac{4}{2-3p}}\left(\mathbb{R}; C^\ell([0,R] \times \mathbb{S}^1; \mathbb{C})\right)$ for all $\ell \in \mathbb{N}$ and for all $R > 0$.*

We will see in the proof that the parameters γ_m are not really used: the latter are useful for the regularity of the potential V in Section 4 below.

Proof. We fix ℓ for the rest of the proof. We define $\check{I}_M = [1 + \frac{1}{M+1}, R]$ so that $[0,R] = [0,1] \cup \check{I}_M \cup \bigcup_{m \geq M+1} I_m$.

Let $(t,r,\theta) \in \mathbb{R} \times I_{m_0} \times \mathbb{S}^1$ for some $m_0 \geq M+2$.

Since $\mathrm{supp}(A_m) \subset I_m \cup I_{m+1}$ for any $m \geq M+1$, we have $A_m(r) = 0$ except for $m = m_0$ or $m = m_0 - 1$. Moreover, $\check{A}_M = 0$ on I_{m_0} since $m_0 \geq M+2$. In particular, formula (2.10) becomes

$$u(t,r,\theta) = \sum_{m\in\{m_0,m_0-1\}} \gamma_{M+1}\cdot\ldots\cdot\gamma_m A_m(r) u_m(t,r,\theta).$$

Recall that by definition (2.3), $\partial_t^j u_m = \left(-ik_m^2\right)^j F_m(r) e^{i(\lambda_m\theta - k_m^2 t)}$. Using the notations of Lemma B.7 and the algebra property we find,

$$\left|\partial_t^j u(t)\right|_{C^\ell((r,\theta))} \leq \sum_{m\in\{m_0,m_0-1\}} \gamma_{M+1}\cdot\ldots\cdot\gamma_m k_m^{2j} \left|A_m\right|_{C^\ell(r)} \left|F_m(r)\right|_{C^\ell(r)} \left|e^{i\lambda_m\theta}\right|_{C^\ell(\theta)}. \quad (3.1)$$

Inequality (2.9) allows to estimate $|A_m|_{C^\ell(r)} \leq C_\ell m^{2\ell}$. By Assumption 2.1, we get $\left|e^{i\lambda_m\theta}\right|_{C^\ell(\theta)} \leq C_\ell \lambda_m^\ell \leq C_\ell A^\ell m^{\frac{4\ell}{p}}$. Combined with (2.1), it also provides $k_m \leq \lambda_m \leq A m^{\frac{2}{p}}$. Moreover, Estimate (A.13) of Lemma A.7 gives $|F_m|_{C^\ell(r)} \leq C_\ell \lambda_m^\ell |F_m(r)| \leq C_\ell A^\ell m^{\frac{2\ell}{p}} |F_m(r)|$. We finally get, for $\beta(\ell,p) = 2\ell + \frac{6\ell}{p}$ and with some different constants C_ℓ,

$$\left|\partial_t^j u(t)\right|_{C^\ell((r,\theta))} \leq C_\ell \sum_{m\in\{m_0,m_0-1\}} \gamma_{M+1}\cdot\ldots\cdot\gamma_m |F_m(r)| m^{\beta(\ell)} A^{2j} m^{\frac{4j}{p}}$$

where we have used (2.1).

Now, we notice that that since $s \in [0,1]$ is equivalent to $r_m(s) \in I_m$, Lemma A.3 implies that there exist $C, c > 0$ depending on $p \in (0, 1/5)$ only such that for $m > M$

$$|F_m(r)| \leq C e^{-c\lambda_m m^{-3}}, \quad \forall r \in I_m.$$

If $M \geq M_0$, (A.3) implies $\gamma_m \leq 1$ and more precisely $\gamma_{M+1}\cdot\ldots\cdot\gamma_m \leq \gamma_m \leq e^{-c\lambda_m m^{-3}}$. We finally obtained, since $\lambda_m \geq m^{\frac{2}{p}}$,

$$\left|\partial_t^j u(t)\right|_{C^\ell((r,\theta))} \leq C A^{2j} m_0^{\frac{4j}{p}+\beta(\ell,p)} e^{-c m_0^{\frac{2}{p}-3}},$$

and therefore, since the estimate is uniform for $(r,\theta) \in I_{m_0,j}$, we get for any $m \geq M+2$,

$$\left\|\partial_t^j u(t)\right\|_{C^\ell(I_m\times\mathbb{S}^1;\mathbb{C})} \leq C A^{2j} m^{\frac{4j}{p}+\beta(\ell,p)} e^{-c m^{\frac{2}{p}-3}}. \quad (3.2)$$

In particular, since $\frac{2}{p} - 3 > 0$ by choice of p, it implies the fact, already proved in Takase [Tak21] that the constructed function $\partial_t^j u(t)$, extended by zero for $r \leq 1$ is actually in $C^\ell([0,R]\times\mathbb{S}^1;\mathbb{C})$. Therefore, the decomposition $[0,R] = [0,1] \cup \check{I}_M \cup \bigcup_{m\geq M+1} I_m$ allows to write

$$\begin{aligned}\left\|\partial_t^j u(t)\right\|_{C^\ell([0,R]\times\mathbb{S}^1;\mathbb{C})} &= \max\left\{\left\|\partial_t^j u(t)\right\|_{C^\ell([0,1]\times\mathbb{S}^1;\mathbb{C})}, \left\|\partial_t^j u(t)\right\|_{C^\ell(\check{I}_M\times\mathbb{S}^1;\mathbb{C})}, \right.\\ &\qquad\qquad \left. \sup_{m\geq M+1}\left\|\partial_t^j u(t)\right\|_{C^\ell(I_m\times\mathbb{S}^1;\mathbb{C})}\right\}\\ &= \max\left\{\left\|\partial_t^j u(t)\right\|_{C^\ell(\check{I}_M\times\mathbb{S}^1;\mathbb{C})}, \sup_{m\geq M+1}\left\|\partial_t^j u(t)\right\|_{C^\ell(I_m\times\mathbb{S}^1;\mathbb{C})}\right\},\end{aligned}$$

since $u(t) = 0$ on $[0,1]\times\mathbb{S}^1$. An estimate like (3.2) is also true uniformly in $j \in \mathbb{N}$ for $m = M$ and $m = M+1$, up to changing the constants. So, we conclude

$$\left\|\partial_t^j u(t)\right\|_{C^\ell([0,R]\times\mathbb{S}^1;\mathbb{C})} \leq C A^{2j} \sup_{m\geq M+1} m^{\frac{4j}{p}+\beta(\ell,p)} e^{-c m^{\frac{2}{p}-3}}, \quad \forall j \in \mathbb{N}.$$

Using the optimization performed in Lemma C.2, and using the notation Γ_λ from Appendix B.1, we obtain the bound

$$\left\|\partial_t^j u(t)\right\|_{C^\ell([0,R]\times\mathbb{S}^1;\mathbb{C})} \leq C A^{2j} L^j \Gamma_\lambda(j), \qquad \text{for all } j \in \mathbb{N},$$

with $\lambda = \frac{4}{2-3p} > 2$ and some appropriate constants C, $L > 0$. This yields the stated Gevrey regularity and concludes the proof of the proposition. □

4. Regularity of the potential V

The present section is devoted to proving the following regularity property of the potential V defined in (2.16).

Theorem 4.1. *There exists $M_1 \in \mathbb{N}$ such that if $M \geq M_1$ in the definition (2.10) of the function u, the potential V defined in (2.16) satisfies the following statement. For all $\ell \in \mathbb{N}$, $R > 1$,*

$$V \in \mathcal{G}^{\frac{4}{2-3p}}\big(\mathbb{R}; C^\ell([0,R]\times\mathbb{S}^1;\mathbb{C})\big).$$

Moreover, given $\eta > 0$ and $\ell_0 \in \mathbb{N}$ one can find $M_2 > 0$ and $L > 0$ such that if $M \geq M_2$ in the definition (2.10) of u, the potential satisfies in addition the bounds

$$\|V\|_{\mathcal{G}_L^{\frac{4}{2-3p}}\left(\mathbb{R};C^{\ell_0}([0,R]\times\mathbb{S}^1)\right)} \leq \eta, \quad \|V\|_{C^{\ell_0}(\mathbb{R}\times(0,R)\times\mathbb{S}^1)} \leq \eta.$$

Proof. Recalling the support property (2.14) and the definition of V in (2.16), we study the term $-V = \frac{Pu}{u}$ on the intervals $I_{m+1,1}$ and $I_{m+1,4}$ for $m \geq M$ (where Pu is non zero).

Step 1: Computation of V in $I_{m+1,1}$.

Let us consider $r \in I_{m+1,1}$, $m \geq M+1$. Then, the non zero terms in the sum defining u are A_{m+1} and A_m. But A_m is constant equal to 1 in $I_{m+1,1}$. As a consequence, for $r \in I_{m+1,1}$ we deduce that

$$\begin{aligned} Pu &= \gamma_{M+1}\cdot\ldots\cdot\gamma_m[P,A_m]u_m + \gamma_{M+1}\cdot\ldots\cdot\gamma_{m+1}[P,A_{m+1}]u_{m+1} \\ &= \gamma_{M+1}\cdot\ldots\cdot\gamma_{m+1}[P,A_{m+1}]u_{m+1} \\ &= \gamma_{M+1}\cdot\ldots\cdot\gamma_{m+1}[P,A_{m+1}]F_{m+1}(r)e^{i(\lambda_{m+1}\theta - k_{m+1}^2 t)}, \end{aligned} \tag{4.1}$$

has a single term (here we used $Pu_m = 0$). Next, on the interval $I_{m+1,1}$,

$$\begin{aligned} u &= \gamma_{M+1}\cdot\ldots\cdot\gamma_{m+1}A_{m+1}u_{m+1} + \gamma_{M+1}\cdot\ldots\cdot\gamma_m A_m u_m \\ &= \gamma_{M+1}\cdot\ldots\cdot\gamma_{m+1}A_{m+1}u_{m+1} + \gamma_{M+1}\cdot\ldots\cdot\gamma_m u_m, \end{aligned}$$

since $A_m = 1$ on $I_{m+1,1}$. As a consequence, recalling that $M+1 \leq m \leq m+1$, we obtain on $I_{m+1,1}$,

$$\frac{Pu}{u} = \frac{\gamma_{M+1}\cdot\ldots\cdot\gamma_{m+1}[P,A_{m+1}]u_{m+1}}{\gamma_{M+1}\cdot\ldots\cdot\gamma_{m+1}A_{m+1}u_{m+1} + \gamma_{M+1}\cdot\ldots\cdot\gamma_m u_m} = \frac{\gamma_{m+1}[P,A_{m+1}]u_{m+1}}{\gamma_{m+1}A_{m+1}u_{m+1} + u_m},$$

after simplification by $\gamma_{M+1}\cdot\ldots\cdot\gamma_m$. We notice as well that this formula also holds for $m = M$.

We next recall that $[P, A_{m+1}]$ is a differential operator of order one in the variable r, and $u_m = F_m(r)e^{i\lambda_m\theta - ik_m^2 t}$. This implies

$$\frac{Pu}{u} = \frac{\gamma_{m+1}([P,A_{m+1}]F_{m+1})e^{i\lambda_{m+1}\theta - ik_{m+1}^2 t}}{\gamma_{m+1}A_{m+1}F_{m+1}e^{i\lambda_{m+1}\theta - ik_{m+1}^2 t} + F_m e^{i\lambda_m\theta - ik_m^2 t}}, \qquad \text{for all } m \geq M.$$

From (A.4), for $m \geq M$ we may divide by $F_m e^{i\lambda_m\theta - ik_m^2 t}$, and we finally obtain for $r \in I_{m+1,1}$,

$$\frac{Pu}{u}(t,r,\theta) = \frac{\frac{\gamma_{m+1}}{F_m(r)}([P,A_{m+1}]F_{m+1})(r)E_{m+1}(t,\theta)}{1 + \frac{\gamma_{m+1}F_{m+1}(r)}{F_m(r)}A_{m+1}(r)E_{m+1}(t,\theta)},$$

with

$$E_{m+1}(t,\theta) = e^{i(\lambda_{m+1}-\lambda_m)\theta - i(k_{m+1}^2 - k_m^2)t}. \tag{4.2}$$

This can be rewritten as

$$\frac{Pu}{u} = \frac{\mathcal{A}_{m+1,1}}{\mathcal{C}_{m+1,1}}, \qquad \text{on} \qquad \mathbb{R}\times I_{m+1,1}\times\mathbb{S}^1, \quad m \geq M,$$

with

$$\mathcal{A}_{m+1,1}(t,r,\theta) := \frac{\gamma_{m+1}}{F_m(r)}([P,A_{m+1}]F_{m+1})(r)E_{m+1}(t,\theta), \quad \text{defined for } (t,r,\theta) \in \mathbb{R}\times\mathbb{R}_+^*\times\mathbb{S}^1,$$

$$\mathcal{C}_{m+1,1}(t,r,\theta) := 1 + \frac{\gamma_{m+1}F_{m+1}(r)}{F_m(r)}A_{m+1}(r)E_{m+1}(t,\theta), \quad \text{defined for } (t,r,\theta) \in \mathbb{R}\times\mathbb{R}_+^*\times\mathbb{S}^1.$$

Step 2: Computation of V in $I_{m+1,4}$.
We now consider the case $r \in I_{m+1,4}$, $m \geq M$. The non zero terms in the sum defining u are still A_{m+1} and A_m but now A_{m+1} is constant equal to 1 in $I_{m+1,4}$. Similarly to (4.1) we find that

$$Pu = \gamma_{M+1} \cdot \ldots \cdot \gamma_m [P, A_m] F_m(r) e^{i(\lambda_m \theta - k_m^2 t)}.$$

On the interval $I_{m+1,4}$,

$$\begin{aligned} u &= \gamma_{M+1} \cdot \ldots \cdot \gamma_{m+1} A_{m+1} u_{m+1} + \gamma_{M+1} \cdot \ldots \cdot \gamma_m A_m u_m \\ &= \gamma_{M+1} \cdot \ldots \cdot \gamma_{m+1} u_{m+1} + \gamma_{M+1} \cdot \ldots \cdot A_m \gamma_m u_m, \end{aligned}$$

since $A_{m+1} = 1$ on $I_{m+1,4}$. We get, for $m \geq M$,

$$\frac{Pu}{u} = \frac{\gamma_{M+1} \cdot \ldots \cdot \gamma_m [P, A_m] u_m}{\gamma_{M+1} \cdot \ldots \cdot \gamma_{m+1} u_{m+1} + \gamma_{M+1} \cdot \ldots \cdot A_m \gamma_m u_m} = \frac{[P, A_m] u_m}{\gamma_{m+1} u_{m+1} + A_m u_m},$$

after simplification by $\gamma_{M+1} \cdot \ldots \cdot \gamma_m$.

From (A.5), for $m \geq M$ we may divide by $\gamma_{m+1} u_{m+1} = \gamma_{m+1} F_{m+1} e^{i\lambda_{m+1}\theta - i k_{m+1}^2 t}$, and, using that $[P, A_m]$ is a differential operator of order one in the variable r, we finally obtain for $r \in I_{m+1,4}$ and $m \geq M$

$$\frac{Pu}{u}(t, r, \theta) = \frac{\frac{1}{\gamma_{m+1} F_{m+1}(r)} ([P, A_m] F_m)(r) \tilde{E}_{m+1}(t, \theta)}{1 + \frac{F_m(r)}{\gamma_{m+1} F_{m+1}(r)} A_m(r) \tilde{E}_{m+1}(t, \theta)},$$

with

$$\tilde{E}_{m+1}(t, \theta) = e^{-i(\lambda_{m+1} - \lambda_m)\theta + i(k_{m+1}^2 - k_m^2)t}.$$

This can be rewritten as

$$\frac{Pu}{u} = \frac{\mathcal{A}_{m+1,4}}{\mathcal{C}_{m+1,4}}, \quad \text{on} \quad \mathbb{R} \times I_{m+1,4} \times \mathbb{S}^1, \quad m \geq M,$$

with

$$\mathcal{A}_{m+1,4}(t, r, \theta) := \frac{1}{\gamma_{m+1} F_{m+1}(r)} ([P, A_m] F_m)(r) \tilde{E}_{m+1}(t, \theta), \quad \text{defined for } (t, r, \theta) \in \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{S}^1,$$

$$\mathcal{C}_{m+1,4}(t, r, \theta) := 1 + \frac{F_m(r)}{\gamma_{m+1} F_{m+1}(r)} A_m(r) \tilde{E}_{m+1}(t, \theta), \quad \text{defined for } (t, r, \theta) \in \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{S}^1.$$

Step 3: Reformulation of the problem.
As a consequence of the above discussion, we deduce that $\operatorname{supp}\left(\mathcal{A}_{m+1,4}, \mathcal{A}_{m,1}\right) \subset \operatorname{supp}[P, A_m] \subset I_{m,1} \cup I_{m+1,4}$ for $m > M$ and $\operatorname{supp} \mathcal{A}_{M+1,4} \subset \operatorname{supp}[P, \check{A}_M] \subset I_{M+1,4}$. As a consequence, we have

$$\frac{Pu}{u}(t, r, \theta) = \sum_{m \geq M} \frac{\mathcal{A}_{m+1,4}(t, r, \theta)}{\mathcal{C}_{m+1,4}(t, r, \theta)} \mathbb{1}_{I_{m+1,4}}(r) + \frac{\mathcal{A}_{m+1,1}(t, r, \theta)}{\mathcal{C}_{m+1,1}(t, r, \theta)} \mathbb{1}_{I_{m+1,1}}(r)$$

where the sums are locally finite: two different terms in the series have disjoint supports. Hence, it now suffices to prove that, for the appropriate value of λ, some $L > 0$ and for all $m \geq M$,

$$\frac{\mathcal{A}_{m+1,4}}{\mathcal{C}_{m+1,4}} \in \mathcal{G}_L^\lambda(\mathbb{R}; C^\ell(I_{m+1,4} \times \mathbb{S}^1)), \quad \frac{\mathcal{A}_{m+1,1}}{\mathcal{C}_{m+1,1}} \in \mathcal{G}_L^\lambda(\mathbb{R}; C^\ell(I_{m+1,1} \times \mathbb{S}^1)),$$

together with the following equality,

$$\begin{aligned} &\left\| \frac{Pu}{u} \right\|_{\mathcal{G}^\lambda(\mathbb{R}; C^\ell([0,R] \times \mathbb{S}^1))} \\ &\qquad = \sup_{m \geq M} \left(\left\| \frac{\mathcal{A}_{m+1,4}}{\mathcal{C}_{m+1,4}} \right\|_{\mathcal{G}_L^\lambda(\mathbb{R}; C^\ell(I_{m+1,4} \times \mathbb{S}^1))}, \left\| \frac{\mathcal{A}_{m+1,1}}{\mathcal{C}_{m+1,1}} \right\|_{\mathcal{G}_L^\lambda(\mathbb{R}; C^\ell(I_{m+1,1} \times \mathbb{S}^1))} \right), \end{aligned} \tag{4.3}$$

where the appropriate definitions of $\mathcal{G}_L^\lambda$ and its related norm can be found in Section B.1. For each of these terms, we will now be in position to apply Corollary B.8 (inverse in Gevrey classes) together with Lemma B.2 (product in Gevrey classes).

Step 4: Regularity of $\frac{\mathcal{A}_{m+1,1}}{\mathcal{C}_{m+1,1}}$ on $\mathbb{R} \times I_{m+1,1} \times \mathbb{S}^1$.
From the definition of E_{m+1} in (4.2) we have

$$\partial_t^j \partial_\theta^{\alpha_2} E_{m+1} = (-i)^j \left(k_{m+1}^2 - k_m^2\right)^j i^{\alpha_2} (\lambda_{m+1} - \lambda_m)^{\alpha_2} E_{m+1}$$

and hence, we can compute for $|\alpha| \leq \ell$,

$$\left|\partial_t^j \partial_r^{\alpha_1} \partial_\theta^{\alpha_2} \mathcal{A}_{m+1,1}(t,r,\theta)\right| = \left|\partial_r^{\alpha_1}\left(\frac{\gamma_{m+1}}{F_m(r)}([P,A_{m+1}]F_{m+1})(r)\right)\left(k_{m+1}^2 - k_m^2\right)^j (\lambda_{m+1}-\lambda_m)^{\alpha_2}\right|$$
$$\leq C_p^j m^{4j/p} m^{2\alpha_2/p}\left|\partial_r^{\alpha_1}\left(\frac{\gamma_{m+1}}{F_m(r)}([P,A_{m+1}]F_{m+1})(r)\right)\right|, \tag{4.4}$$

after having used

$$k_{m+1}^2 - k_m^2 \leq \lambda_{m+1}^2 \leq C m^{4/p}, \quad \text{for all } m \in \mathbb{N}^*, \tag{4.5}$$

which is a consequence of (2.1) and Assumption 2.1. Here, C only depends on p and A.

Using (2.13) and (2.9) we have (using the notation of Lemma B.7 for the pointwise C^ℓ norm)

$$|[P,A_{m+1}]F_{m+1}|_{C^{\alpha_1}(r)} \leq C_\ell m^{2(\alpha_1+2)} |F_{m+1}|_{C^{\alpha_1+1}(r)},$$

and hence the algebra property of $|\cdot|_{C^{\alpha_1}(r)}$ (Lemma B.7) gives

$$\left|\partial_r^{\alpha_1}\left(\frac{1}{F_m(r)}([P,A_{m+1}]F_{m+1})(r)\right)\right| \leq C_\ell \left|\frac{1}{F_m}\right|_{C^{\alpha_1}(r)} m^{2\alpha_1+4} |F_{m+1}|_{C^{\alpha_1+1}(r)}.$$

From Estimate (A.15) in Lemma A.8 together with Lemma A.7, we finally deduce

$$\left|\partial_r^{\alpha_1}\left(\frac{1}{F_m(r)}([P,A_{m+1}]F_{m+1})(r)\right)\right| \leq C_\ell \lambda_m^{\alpha_1^2} \frac{1}{|F_m(r)|} m^{2\alpha_1+4} \lambda_{m+1}^{\alpha_1+1} |F_{m+1}(r)|,$$

for all $1 \leq r \leq 1 + \frac{1}{m+1}$. Coming back to (4.4) , we have now obtained for all $|\alpha| \leq \ell$, for $\beta = \beta(\ell)$ and for all $j \in \mathbb{N}$

$$\left|\partial_t^j \partial_r^{\alpha_1} \partial_\theta^{\alpha_2} \mathcal{A}_{m+1,1}(t,r,\theta)\right| \leq C_\ell^{j+1} m^{4j/p} m^{\beta(\ell)} \left|\frac{\gamma_{m+1}F_{m+1}(r)}{F_m(r)}\right|, \quad \text{for all } 1 \leq r \leq 1 + \frac{1}{m+1}.$$

Thanks to Lemma A.4 and the assumptions $\lambda_m \geq m^{2/p}$ and $p < 2/3$ we obtain finally

$$\left\|\partial_t^j \mathcal{A}_{m+1,1}(t,\cdot)\right\|_{C^\ell(I_{m+1,1}\times\mathbb{S}^1)} \leq C_\ell^{j+1} m^{4j/p} m^{\beta(\ell)} e^{-\mu m^{\frac{2}{p}-3}}. \tag{4.6}$$

The maximization Lemma C.2 and (4.6) give the bound

$$\left\|\partial_t^j \mathcal{A}_{m+1,1}(t,\cdot)\right\|_{C^\ell(I_{m+1,1}\times\mathbb{S}^1)} \leq C C_{p,\ell}^{j+1} L_1^j \Gamma_\lambda(j) \leq C_0 L_0^j \Gamma_\lambda(j), \quad \text{for all } j \in \mathbb{N}, \tag{4.7}$$

for $\lambda = \frac{4}{2-3p}$ and some $L_0 > 0$ and $C_0 > 0$, depending on A, p, ℓ only. This is $\mathcal{A}_{m+1,1} \in \mathcal{G}^\lambda_{L_0}(\mathbb{R}; C^\ell(I_{m+1,1}\times\mathbb{S}^1))$ for a Gevrey radius L_0 depending on A, p, ℓ.

Let now $L > L_0$. Using (4.7) and recalling the definitions of Gevrey norms in Section B.1 we find that

$$|\mathcal{A}'_{m+1,1}|_{\mathcal{G}^\lambda_L(\mathbb{R};C^\ell(I_{m+1,1}\times\mathbb{S}^1))} \leq C_0 L_0 \sup_{j\geq 0}\left(\frac{L_0}{L}\right)^j (j+1)^\lambda,$$

and therefore

$$\|\mathcal{A}_{m+1,1}\|_{\mathcal{G}^\lambda_L(\mathbb{R};C^\ell(I_{m+1,1}\times\mathbb{S}^1))}$$
$$\leq \max\left\{2^6 \|\mathcal{A}_{m+1,1}\|_{L^\infty(\mathbb{R};C^\ell(I_{m+1,1}\times\mathbb{S}^1))}, 2^3 C_0 \frac{L_0}{L} \sup_{j\geq 0}\left(\frac{L_0}{L}\right)^j (j+1)^\lambda\right\}. \tag{4.8}$$

We can now observe that given $\eta > 0$, one can choose $L_1 > L_0$ sufficiently large depending on A, p, ℓ such that $2^3 C_0 \frac{L_0}{L} \sup_{j\geq 0}\left(\frac{L_0}{L}\right)^j (j+1)^\lambda \leq \eta$. Moreover, using (4.6) for $j \leq \ell_0$ we get the bound

$$\|\mathcal{A}_{m+1,1}\|_{C^{\ell_0}(\mathbb{R};C^{\ell_0}(I_{m+1,1}\times\mathbb{S}^1))} \leq C_{\ell_0} m^{\beta(\ell_0)} e^{-\mu m^{\frac{2}{p}-3}}.$$

This implies that since $m \geq M$, up to taking $M \geq M_2$ for some M_2 depending on $p, A, , \ell_0$ we can have

$$\|\mathcal{A}_{m+1,1}\|_{C^{\ell_0}(\mathbb{R};C^{\ell_0}(I_{m+1,1}\times\mathbb{S}^1))} \leq \eta. \tag{4.9}$$

Using (4.8) this implies as well that given $\eta > 0$ there is $M_2 > 0$ and $L_1 > 0$ depending on p, A, ℓ_0 such that if $M \geq M_2$ one has

$$\|\mathcal{A}_{m+1,1}\|_{\mathcal{G}^\lambda_{L_1}(\mathbb{R};C^{\ell_0}(I_{m+1,1}\times\mathbb{S}^1))} \leq \eta. \tag{4.10}$$

We now look at the term $\mathcal{C}_{m+1,1}$ for which we will apply the inversion Corollary B.8. We start by noticing that Lemma A.4 ensures that for all $m \geq M_0$,

$$\begin{aligned}|\mathcal{C}_{m+1,1}| &\geq 1 - \left|\frac{\gamma_{m+1}F_{m+1}(r)}{F_m(r)}A_{m+1}(r)E_{m+1}(t,\theta)\right| \geq 1 - \left|\frac{\gamma_{m+1}F_{m+1}(r)}{F_m(r)}\right| \\ &\geq 1 - Ce^{-\mu\lambda_m m^{-3}} \geq 1 - Ce^{-\mu m^{2/p-3}},\end{aligned}$$

where we have used $\lambda_m \geq m^{\frac{2}{p}}$ and $p < 2/3$. This implies the existence of $M_1 \geq M_0$ depending only on p such that for all $m \geq M \geq M_1$,

$$|\mathcal{C}_{m+1,1}| \geq \frac{1}{2}, \text{ on } \mathbb{R} \times I_{m+1,1} \times \mathbb{S}^1. \tag{4.11}$$

In order to use Corollary B.8 we need to verify that $\mathcal{C}_{m+1,1} \in \mathcal{G}^\lambda(\mathbb{R}; C^\ell(I_{m+1,1} \times \mathbb{S}^1))$ for the appropriate value of λ. We proceed as for $\mathcal{A}_{m+1,1}$ by writing for all $|\alpha| \leq \ell$ and for $j \geq 1$,

$$\begin{aligned}\left|\partial_t^j \partial_r^{\alpha_1}\partial_\theta^{\alpha_2}\mathcal{C}_{m+1,1}(t,r,\theta)\right| &= \left|\partial_r^{\alpha_1}\left(\frac{\gamma_{m+1}F_{m+1}(r)}{F_m(r)}A_{m+1}(r)\right)\left(k_{m+1}^2 - k_m^2\right)^j(\lambda_{m+1}-\lambda_m)^{\alpha_2}\right| \\ &\leq C^j m^{4j/p} m^{2\alpha_2/p}\left|\partial_r^{\alpha_1}\left(\frac{\gamma_{m+1}F_{m+1}(r)}{F_m(r)}A_{m+1}(r)\right)\right| \\ &\leq C^j m^{4j/p} m^{2\alpha_2/p}\left|\frac{\gamma_{m+1}F_{m+1}}{F_m}\right|_{C^{\alpha_1}(r)}|A_{m+1}|_{C^{\alpha_1}(r)} \\ &\leq C_\ell^{j+1} m^{4j/p} m^{\beta(\ell)}\left|\frac{\gamma_{m+1}F_{m+1}}{F_m}\right|_{C^{\alpha_1}(r)} \\ &\leq C_\ell^{j+1} m^{4j/p} m^{\beta'(\ell)} e^{-\mu m^{\frac{2}{p}-3}},\end{aligned} \tag{4.12}$$

where we used (2.9) and then Lemma A.8 and Lemma A.4 combined with $\lambda_m \geq m^{2/p}$ and $p < 2/3$. Recall the constants might depends also on p and A fixed. The maximization Lemma C.2 yields as before, for $j \geq 1$, some C, $L > 0$ depending on p, ℓ and μ, that

$$\left\|\partial_t^j \mathcal{C}_{m+1,1}(t,\cdot)\right\|_{C^\ell(I_{m+1,1}\times\mathbb{S}^1)} \leq C C_{p,\ell}^{j+1} L^j \Gamma_\lambda(j). \tag{4.13}$$

For $j = 0$ we notice that Lemma A.4 implies that for $m \geq M \geq M_1$ one has

$$\mathcal{C}_{m+1,1} \leq 1 + Ce^{-\mu m^{2/p-3}} \leq \frac{3}{2}.$$

We notice as well that (4.12) also holds for $j = 0$ and $|\alpha| \geq 1$. As a consequence (4.13) remains valid for $j = 0$ as well. This gives $\mathcal{C}_{m+1,1} \in \mathcal{G}^\lambda_{L_2}(\mathbb{R}; C^\ell(I_{m+1,1}\times\mathbb{S}^1))$ for $\lambda = \frac{4}{2-3p}$, a Gevrey radius L_2 and Gevrey norm $\|\mathcal{C}_{m+1,1}\|_{L_2}$ depending on p, ℓ only. Thanks to the uniform bound (4.11) we are now in position to apply Corollary B.8 with $\eta_0 = 1/2$. We get that $\frac{1}{\mathcal{C}_{m+1,1}} \in \mathcal{G}^\lambda_{L'}(\mathbb{R}; C^\ell(I_{m+1,1} \times \mathbb{S}^1))$ with

$$L' = L_2 \cdot \max\left(2, C\frac{\|\mathcal{C}_{m+1,1}\|^\ell_{L^\infty(\mathbb{R};C^\ell(I_{m+1,1}\times\mathbb{S}^1))}\|\mathcal{C}_{m+1,1}\|_{L_2}}{\eta_0^{\ell+1}}\right),$$

where $\eta_0 = 1/2$ and $L, C, \|\mathcal{C}_{m+1,1}\|_{L_2}$ depend on λ and ℓ only. Using (4.13) for $j \leq \ell_0$ we find

$$\|\mathcal{C}_{m+1,1}\|_{C^{\ell_0}(\mathbb{R}\times I_{m+1,1}\times\mathbb{S}^1)} \leq C_{\ell_0,p}.$$

Applying (B.6) we have as well

$$\left\|\frac{1}{\mathcal{C}_{m+1,1}}\right\|_{\mathcal{G}^\lambda_{L'}(\mathbb{R};C^\ell(I_{m+1,1}\times\mathbb{S}^1))} \leq D\frac{\|\mathcal{C}_{m+1,1}\|^\ell_{L^\infty(\mathbb{R};C^\ell(I_{m+1,1}\times\mathbb{S}^1))}}{c_0^{\ell+1}} \leq C_{\ell,p}$$

for a constant $C_{\ell,p} > 0$ depending on p, ℓ. In particular, we have shown that

$$\frac{1}{\mathcal{C}_{m+1,1}} \in \mathcal{G}^\lambda(\mathbb{R}; C^\ell(I_{m+1,1} \times \mathbb{S}^1)).$$

Recalling (4.9) (4.10) and that $\mathcal{A}_{m+1,1} \in \mathcal{G}^\lambda(\mathbb{R}; C^\ell(I_{m+1,1}\times\mathbb{S}^1))$ we get that $\frac{\mathcal{A}_{m+1,1}}{\mathcal{C}_{m+1,1}} \in \mathcal{G}^\lambda(\mathbb{R}; C^\ell(I_{m+1,1}\times\mathbb{S}^1))$. The Banach algebra properties of Lemma B.2 and Lemma B.6 imply as well that given $\eta > 0$ there is M_3 and L_3 depending on p, ℓ_0 such that

$$\left\| \frac{\mathcal{A}_{m+1,1}}{\mathcal{C}_{m+1,1}} \right\|_{\mathcal{G}^\lambda_{L_3}(\mathbb{R}; C^{\ell_0}(I_{m+1,1}\times\mathbb{S}^1))} \leq C_{\ell_0,p}\eta, \quad \text{for all } m \geq M \geq M_3. \tag{4.14}$$

and

$$\left\| \frac{\mathcal{A}_{m+1,1}}{\mathcal{C}_{m+1,1}} \right\|_{C^{\ell_0}(\mathbb{R}\times I_{m+1,1}\times\mathbb{S}^1)} \leq C_{\ell_0,p}\eta, \quad \text{for all } m \geq M \geq M_3. \tag{4.15}$$

Step 5: Regularity of $\frac{\mathcal{A}_{m+1,4}}{\mathcal{C}_{m+1,4}}$ on $\mathbb{R}\times I_{m+1,4}\times\mathbb{S}^1$.
This is dealt in a similar fashion as Step 4, with the difference that here one needs to use the estimate of Lemma A.4 in the regime $s \in [3/4, 1]$. This corresponds to working on the interval $r \in I_{m+1,4}$. We find for $\mathcal{A}_{m+1,4}$,

$$\begin{aligned}\left|\partial_t^j\partial_r^{\alpha_1}\partial_\theta^{\alpha_2}\mathcal{A}_{m+1,4}(t,r,\theta)\right| &= \left|\partial_r^{\alpha_1}\left(\frac{1}{\gamma_{m+1}F_{m+1}(r)}([P,A_m]F_m)(r)\right)\left(k_{m+1}^2-k_m^2\right)^j(\lambda_{m+1}-\lambda_m)^{\alpha_2}\right| \\ &\leq C_p^j m^{4j/p}m^{2\alpha_2/p}\left|\partial_r^{\alpha_1}\left(\frac{1}{\gamma_{m+1}F_{m+1}(r)}([P,A_m]F_m)(r)\right)\right|,\end{aligned}$$

thanks to (4.5). Using (2.13) and (2.9) we have

$$|[P,A_m]F_m|_{C^{\alpha_1}(r)} \leq Cm^{2\alpha_1+2}\,|F_m|_{C^{\alpha_1+1}(r)}\,,$$

and hence the algebra property of $|\cdot|_{C^{\alpha_1}(r)}$ gives

$$\left|\partial_r^{\alpha_1}\left(\frac{1}{F_{m+1}(r)}([P,A_m]F_m)(r)\right)\right| \leq C\left|\frac{1}{F_{m+1}}\right|_{C^{\alpha_1}(r)} m^{2\alpha_1+2}\,|F_m|_{C^{\alpha_1+1}(r)}\,.$$

Estimate (A.15) together with Lemma A.7 give

$$\left|\partial_r^{\alpha_1}\left(\frac{1}{F_{m+1}(r)}([P,A_m]F_m)(r)\right)\right| \leq C\lambda_{m+1}^{\alpha_1^2}\frac{1}{|F_{m+1}(r)|}m^{2\alpha_1+2}\lambda_{m+1}^{\alpha_1+1}\,|F_m(r)|\,,$$

for all $1 \leq r \leq 1+\frac{1}{m+1}$. Hence, for all $|\alpha| \leq \ell$, for $\beta = \beta(\ell)$ and for all $j \in \mathbb{N}$

$$\left|\partial_t^j\partial_r^{\alpha_1}\partial_\theta^{\alpha_2}\mathcal{A}_{m+1,4}(t,r,\theta)\right| \leq C_{p,\ell}^{j+1}m^{4j/p}m^{\beta(\ell)}\left|\frac{F_m(r)}{\gamma_{m+1}F_{m+1}(r)}\right|, \quad \text{for all } 1 \leq r \leq 1+\frac{1}{m+1}.$$

Thanks to Lemma A.4 applied this time for $s \in [3/4, 1]$ we get

$$\left\|\partial_t^j\mathcal{A}_{m+1,4}(t,\cdot)\right\|_{C^\ell(I_{m+1,1}\times\mathbb{S}^1)} \leq C_{p,\ell}^{j+1}m^{4j/p}m^{\beta(\ell)}e^{-\mu m^{\frac{2}{p}-3}},$$

and after maximization we find that $\mathcal{A}_{m+1,4} \in \mathcal{G}^\lambda(\mathbb{R}; C^\ell(I_{m+1,4}\times\mathbb{S}^1))$ for $\lambda = \frac{4}{2-3p}$ and a Gevrey radius depending on p, ℓ. In addition, the same argument as in Step 4 yields that given $\eta > 0$ there is $M_2 > 0$ and $L > 0$ depending on p, ℓ such that if $M \geq M_2$ one has

$$\|\mathcal{A}_{m+1,4}\|_{\mathcal{G}^\lambda_L(\mathbb{R}; C^\ell(I_{m+1,4}\times\mathbb{S}^1))} \leq \eta,$$

and

$$\|\mathcal{A}_{m+1,4}\|_{C^{\ell_0}(\mathbb{R}\times I_{m+4,1}\times\mathbb{S}^1)} \leq \eta.$$

Concerning the term $\mathcal{C}_{m+1,1}$ we apply again Lemma A.4 which guarantees that for all $m \geq M_0$,

$$\begin{aligned}|\mathcal{C}_{m+1,4}| &\geq 1 - \left|\frac{F_m(r)}{\gamma_{m+1}F_{m+1}(r)}A_m(r)\tilde{E}_{m+1}(t,\theta)\right| \geq 1 - \left|\frac{F_m(r)}{\gamma_{m+1}F_{m+1}(r)}\right| \\ &\geq 1 - Ce^{-\mu\lambda_m m^{-3}} \geq 1 - Ce^{-\mu m^{2/p-3}},\end{aligned}$$

where we have used $\lambda_m \geq m^{\frac{2}{p}}$ and $p < 2/3$. Therefore, there is $\tilde{M}_1 \geq M_0$ depending only on p such that for all $m \geq M \geq \tilde{M}_1$,

$$|\mathcal{C}_{m+1,4}| \geq \frac{1}{2}, \text{ on } \mathbb{R}\times I_{m+1,4}\times\mathbb{S}^1.$$

Next, we write for all $|\alpha| \leq \ell$ and $j \geq 1$,

$$\begin{aligned}\left|\partial_t^j \partial_r^{\alpha_1} \partial_\theta^{\alpha_2} \mathcal{C}_{m+1,4}(t,r,\theta)\right| &= \left|\partial_r^{\alpha_1}\left(\frac{F_m(r)}{\gamma_{m+1}F_{m+1}(r)}A_m(r)\right)\left(k_{m+1}^2-k_m^2\right)^j\left(\lambda_{m+1}-\lambda_m\right)^{\alpha_2}\right| \\ &\leq C_p^j m^{4j/p} m^{2\alpha_2/p}\left|\partial_r^{\alpha_1}\left(\frac{F_m(r)}{\gamma_{m+1}F_{m+1}(r)}A_m(r)\right)\right| \\ &\leq C_p^j m^{4j/p} m^{2\alpha_2/p}\left|\frac{F_m}{\gamma_{m+1}F_{m+1}}\right|_{C^{\alpha_1}(r)}|A_m|_{C^{\alpha_1}(r)} \\ &\leq C_{p,\ell}^{j+1} m^{4j/p} m^{\beta(\ell)}\left|\frac{F_m}{\gamma_{m+1}F_{m+1}}\right|_{C^{\alpha_1}(r)} \\ &\leq C_{p,\ell}^{j+1} m^{4j/p} m^{\beta'(\ell)} e^{-\mu m^{\frac{2}{p}-3}},\end{aligned}$$

where we used (2.9) and then Lemma A.8 and Lemma A.4 combined with $\lambda_m \geq m^{2/p}$ and $p < 2/3$. From this point on, we can proceed exactly as in Step 4. Corollary B.8 shows that $\frac{1}{\mathcal{C}_{m+1,4}} \in \mathcal{G}^\lambda(\mathbb{R}; C^\ell(I_{m+1,4}\times\mathbb{S}^1))$ and by the algebra property $\frac{\mathcal{A}_{m+1,4}}{\mathcal{C}_{m+1,1}} \in \mathcal{G}^\lambda(\mathbb{R}; C^\ell(I_{m+1,4}\times\mathbb{S}^1))$. We also have the same quantitative bounds on the Gevrey and C^{ℓ_0} norms. Namely, given $\eta > 0$ there is $M_4 > 0$ and $L_4 > 0$ depending on p, ℓ_0 such that

$$\left\|\frac{\mathcal{A}_{m+1,4}}{\mathcal{C}_{m+1,1}}\right\|_{\mathcal{G}^\lambda_{L_4}(\mathbb{R};C^{\ell_0}(I_{m+1,4}\times\mathbb{S}^1))} \leq C_{\ell_0,p}\eta, \quad \text{for all } m \geq M \geq M_4, \tag{4.16}$$

and

$$\left\|\frac{\mathcal{A}_{m+1,4}}{\mathcal{C}_{m+1,4}}\right\|_{C^{\ell_0}(\mathbb{R}\times I_{m+1,4}\times\mathbb{S}^1)} \leq C_{\ell_0,p}\eta, \quad \text{for all } m \geq M \geq M_4. \tag{4.17}$$

Step 6: Conclusion.
The theorem now follows from the definition of V in (2.16) combined with (4.3), (4.14), (4.16) (Gevrey norms) as well as (4.15) (4.17) (C^{ℓ_0} norms) and after renaming the constants. □

Remark 4.2. Note that if instead of the third item of Assumption 2.1 we assume the stronger assumption that there is $c_0 > 0$ such that $\frac{\lambda_{m+1}}{\lambda_m} = 1 + \frac{c_0}{m} + o\left(\frac{1}{m}\right)$, then,

$$k_{m+1}^2 - k_m^2 \sim \lambda_m^2 m^{-1},$$

and the rough bound $k_{m+1}^2 - k_m^2 \leq C m^{4/p}$ we used can be replaced by $k_{m+1}^2 - k_m^2 \leq C m^{\frac{4}{p}-1}$. This leads to maximizing instead

$$(0,+\infty) \ni m \to m^{\frac{4j}{p}-j} e^{-\mu m^{\frac{2}{p}-3}} m^\beta.$$

The calculation of Lemma C.2 gives then a slightly better Gevrey index λ' given by

$$\lambda' = \frac{4-p}{2-3p}.$$

In this case, the Gevrey regularity of the potential V is higher than that of the solution u.

5. End of proof of Theorem 1.1

Before concluding the proof of Theorem 1.1, we establish two lemmata related to preserved quantities for the solution u and boundary conditions.

Lemma 5.1 (Preservation of norms of u)**.** *Assume that $\lambda_m \in \mathbb{N}$ are pairwise distinct. Then for all $M, R > 0$, the function u defined in (2.10) satisfies, for all $\alpha, \beta, \gamma \in \mathbb{N}$ and $w = w(r) \in L^2_{\mathrm{loc}}((0,\infty), r dr)$,*

$$\|w\,\partial_t^\alpha \partial_r^\beta \partial_\theta^\gamma u(t,\cdot)\|_{L^2(\mathbb{D}_R)} = \|w\,\partial_t^\alpha \partial_r^\beta \partial_\theta^\gamma u(s,\cdot)\|_{L^2(\mathbb{D}_R)}, \qquad \text{for all } s,t \in \mathbb{R},$$

where

$$\mathbb{D}_R = \{x \in \mathbb{R}^2, |x| \leq R\}.$$

Note that the fact that the λ_m's are pairwise distinct is part of Assumption 2.1.

Remark 5.2. Note that the proof shows that not only norms are conserved, but also inner products of the form

$$t \mapsto \left(w_1 \, \partial_t^{\alpha_1} \partial_r^{\beta_1} \partial_\theta^{\gamma_1} u(t,\cdot), w_2 \, \partial_t^{\alpha_2} \partial_r^{\beta_2} \partial_\theta^{\gamma_2} u(t,\cdot) \right)_{L^2(\mathbb{D}_R)}.$$

Proof. Set $B_M(r) := \check{A}_M(r) F_M(r)$ together with $B_m(r) := \gamma_{M+1} \cdots \gamma_m A_m(r) F_m(r)$ for $m \geq M+1$. Notice that the B_m's are supported in $[1, +\infty)$ and they are real-valued functions since $A_m, \check{A}_M$ and F_m are. Using the definition of u in (2.10) together with (2.3), we have

$$u(t,r,\theta) = \sum_{m \geq M} B_m(r) e^{i\lambda_m \theta} e^{-ik_m^2 t}.$$

The sum is locally finite (see the proof of Lemma 2.2) and converges in C^∞ as a consequence of Proposition 3.1 (and its proof). We thus have

$$\partial_t^\alpha \partial_r^\beta \partial_\theta^\gamma u(t,r,\theta) = \sum_{m \geq M} (-ik_m^2)^\alpha (i\lambda_m)^\gamma B_m^{(\beta)}(r) e^{i\lambda_m \theta} e^{-ik_m^2 t}.$$

From uniform convergence of the series, we may expand the squared norm, and integrate term by term, thus obtaining

$$\begin{aligned}
&\| w \, \partial_t^\alpha \partial_r^\beta \partial_\theta^\gamma u(t,\cdot) \|^2_{L^2(\mathbb{D}_R)} \\
&= \int_0^R \int_0^{2\pi} \partial_t^\alpha \partial_r^\beta \partial_\theta^\gamma u(t,r,\theta) \overline{\partial_t^\alpha \partial_r^\beta \partial_\theta^\gamma u(t,r,\theta)} |w(r)|^2 r \, dr d\theta \\
&= \sum_{m,j \geq M} k_m^{2\alpha} \lambda_m^\gamma k_j^{2\alpha} \lambda_j^\gamma e^{-i(k_m^2 - k_j^2)t} \int_1^R B_m^{(\beta)}(r) B_j^{(\beta)}(r) |w(r)|^2 r \, dr \int_0^{2\pi} e^{i(\lambda_m - \lambda_j)\theta} \, d\theta.
\end{aligned}$$

Since the λ_m are pairwise distinct integers by assumption, we deduce

$$\int_0^{2\pi} e^{i(\lambda_m - \lambda_j)\theta} \, d\theta = 2\pi \delta_{mj}.$$

Hence

$$\| w \, \partial_t^\alpha \partial_r^\beta \partial_\theta^\gamma u(t,\cdot) \|^2_{L^2(\mathbb{D}_R)} = 2\pi \sum_{m \geq M} k_m^{4\alpha} \lambda_m^{2\gamma} \int_1^R |B_m^{(\beta)}(r)|^2 |w(r)|^2 r \, dr, \tag{5.1}$$

which is indeed independent of t. □

Lemma 5.3 (Dirichlet boundary condition)**.** *There is a sequence $(\mathsf{R}_M)_{M \in \mathbb{N}}$ of real numbers and $M_0 \in \mathbb{N}$, satisfying*

$$\mathsf{R}_M > 1 + \frac{1}{M+1} > 1 \quad \text{for all } M \geq M_0, \quad \text{and} \quad \mathsf{R}_M \to 1^+, \quad \text{as } M \to +\infty, \tag{5.2}$$

such that, for any $M \in \mathbb{N}$, the function u defined in (2.10) satisfies $u(t, \mathsf{R}_M, \theta) = 0$ for all $(t,\theta) \in \mathbb{R} \times \mathbb{S}^1$.

Lemma 5.4 (Neumann boundary condition)**.** *There is a sequence $(\mathsf{R}'_M)_{M \in \mathbb{N}}$ of real numbers and $M_0 \in \mathbb{N}$, satisfying*

$$\mathsf{R}'_M > 1 + \frac{1}{M+1} > 1 \quad \text{for all } M \geq M_0, \quad \text{and} \quad \mathsf{R}'_M \to 1^+, \quad \text{as } M \to +\infty,$$

such that, for any $M \in \mathbb{N}$, the function u defined in (2.10) satisfies $\partial_r u(t, \mathsf{R}'_M, \theta) = 0$ for all $(t,\theta) \in \mathbb{R} \times \mathbb{S}^1$.

The proof of Lemma 5.4 is exactly that of Lemma 5.3, replacing the use of the asymptotics of the first zero $j_{\lambda_M,1}$ of the Bessel function J_{λ_M} in Lemma A.2 by that of the first zero $j'_{\lambda_M,1}$ of its derivative J'_{λ_M}, also given in Lemma A.2.

Proof of Lemma 5.3. We first notice that, recalling the definition of the function u in (2.10) and the cutoff functions A_m in (2.8), we have

$$u(t,r,\theta) = \check{A}_M(r) u_M(t,r,\theta), \quad \text{for all } r > 1 + \frac{1}{M+1}.$$

As a consequence, noticing that $1+\frac{1}{M+1} > 1+\frac{1}{M+2}+\frac{1}{4(M+1)(M+2)}$ for all $M\in\mathbb{N}$, we obtain from (2.7) that $\check{A}_M(r)=1$ for $r>1+\frac{1}{M+1}$ and deduce

$$u(t,r,\theta)=u_M(t,r,\theta)=e^{i\lambda_M\theta-ik_M^2t}F_M(r)=e^{i\lambda_M\theta-ik_M^2t}J_{\lambda_M}(k_Mr),\qquad \text{for } r>1+\frac{1}{M+1}. \tag{5.3}$$

We now fix

$$\mathsf{R}_M:=\frac{j_{\lambda_M,1}}{k_M}=\frac{j_{\lambda_M,1}}{\lambda_M(1-\frac{1}{M})},$$

where $j_{\lambda_M,1}$ is the first zero of the Bessel function J_{λ_M}, defined in Lemma A.2. We deduce from that result that if $\lambda_M>\nu_0$ (that is to say, for $M\geq M_0$), then

$$\frac{1+1.855\,\lambda_M^{-\frac{2}{3}}}{(1-\frac{1}{M})}=\frac{\lambda_M+1.855\,\lambda_M^{\frac{1}{3}}}{\lambda_M(1-\frac{1}{M})}<\mathsf{R}_M=\frac{j_{\lambda_M,1}}{k_M}<\frac{\lambda_M+1.856\,\lambda_M^{\frac{1}{3}}}{\lambda_M(1-\frac{1}{M})}=\frac{1+1.856\,\lambda_M^{-\frac{2}{3}}}{(1-\frac{1}{M})}. \tag{5.4}$$

Noticing now that

$$\left(1+\frac{1}{M+1}\right)\left(1-\frac{1}{M}\right)=1-\frac{2}{M(M+1)}<1,$$

and hence

$$1+\frac{1}{M+1}<\frac{1}{(1-\frac{1}{M})}<\frac{1+1.855\,\lambda_M^{-\frac{2}{3}}}{(1-\frac{1}{M})},$$

we deduce from (5.4) that $1<1+\frac{1}{M+1}<\mathsf{R}_M$. In addition, we notice that $\mathsf{R}_M\to_{M\to+\infty}1^+$ from (5.4) and Assumption 2.1, which concludes the proof of (5.2).

As a consequence, coming back to (5.3), we obtain in particular that

$$u(t,\mathsf{R}_M,\theta)=e^{i\lambda_M\theta-ik_M^2t}J_{\lambda_M}(k_M\mathsf{R}_M)=e^{i\lambda_M\theta-ik_M^2t}J_{\lambda_M}(j_{\lambda_M,1})=0,$$

by definition of $j_{\lambda_M,1}$, which concludes the proof of the lemma. □

We conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. Given $\rho_0\in(0,1),\varepsilon>0,\eta>0,\ell_0\in\mathbb{N}$ as in the statement of Theorem 1.1, we first fix p close to zero so that

$$p\in\left(0,\frac{1}{5}\right)\qquad\text{and}\qquad\frac{4}{2-3p}\leq 2+\varepsilon. \tag{5.5}$$

For these parameters p,η,ℓ_0, the functions u defined in (2.10) and V defined in (2.16) depending on $M\geq M_0$ (to be fixed later on), satisfy the following properties, provided that M_0 is taken sufficiently large (according to Lemma 2.2, Proposition 3.1 and Theorem 4.1):

- $\operatorname{supp} u=\mathbb{R}\times[1,+\infty)\times\mathbb{S}^1$ from Lemma 2.2, and $\operatorname{supp}(V)\subset\mathbb{R}\times\left[1,1+\frac{1}{M+1}\right]\times\mathbb{S}^1$ from (2.15)–(2.16),
- u and V satisfy the Schrödinger equation (2.17),
- $u\in\mathcal{G}^{\frac{4}{2-3p}}\big(\mathbb{R};C^\ell([0,R]\times\mathbb{S}^1;\mathbb{C})\big)$ for all $\ell\in\mathbb{N},R>0$, from Proposition 3.1,
- $V\in\mathcal{G}^{\frac{4}{2-3p}}\big(\mathbb{R};C^\ell([0,R]\times\mathbb{S}^1;\mathbb{C})\big)$ for all $\ell\in\mathbb{N},R>0$, and
$$\|V\|_{C^{\ell_0}(\mathbb{R}\times(0,3)\times\mathbb{S}^1)}\leq\eta,$$
from Theorem 4.1 (where we have fixed $R=3$ in the estimate),
- $t\mapsto\|w\,\partial_t^\alpha\partial_r^\beta\partial_\theta^\gamma u(t,\cdot)\|_{L^2(\mathbb{D}_R)}$ is constant for any $\alpha,\beta,\gamma\in\mathbb{N},R>0$, and $w=w(r)\in L^2((0,R),r\,dr)$, from Lemma 5.1.

We now let R_M be defined by Lemma 5.3, and assume that M_0 satisfies this lemma as well, and notice that $1<\mathsf{R}_M<2$. We set

$$\mathsf{u}(t,r,\theta):=u(\mathsf{R}_M^2t,\mathsf{R}_Mr,\theta),\quad \mathsf{q}(t,r,\theta):=\mathsf{R}_M^2V(\mathsf{R}_M^2t,\mathsf{R}_Mr,\theta),\quad (t,r,\theta)\in\mathbb{R}\times\mathbb{R}_+^*\times\mathbb{S}^1.$$

From the above items, we deduce that

- $\operatorname{supp}\mathsf{u}=\mathbb{R}\times\left[\frac{1}{\mathsf{R}_M},+\infty\right)\times\mathbb{S}^1$ and $\operatorname{supp}(\mathsf{q})\subset\mathbb{R}\times\left[\frac{1}{\mathsf{R}_M},\frac{1}{\mathsf{R}_M}+\frac{1}{\mathsf{R}_M(M+1)}\right]\times\mathbb{S}^1$,
- the functions u,q satisfy
$$(-i\partial_t-\Delta)\mathsf{u}+\mathsf{q}\mathsf{u}=0,\qquad\text{on } (t,r,\theta)\in\mathbb{R}\times\mathbb{R}_+^*\times\mathbb{S}^1,$$

- $\mathsf{u} \in \mathcal{G}^{\frac{4}{2-3p}}\left(\mathbb{R}; C^\ell([0, \frac{R}{\mathsf{R}_M}] \times \mathbb{S}^1; \mathbb{C})\right)$ for all $\ell \in \mathbb{N}, R > 0$,
- $\mathsf{q} \in \mathcal{G}^{\frac{4}{2-3p}}\left(\mathbb{R}; C^\ell([0, \frac{R}{\mathsf{R}_M}] \times \mathbb{S}^1; \mathbb{C})\right)$ for all $\ell \in \mathbb{N}, R > 0$, and $\|\mathsf{q}\|_{C^{\ell_0}\left(\mathbb{R}\times(0,\frac{3}{\mathsf{R}_M})\times\mathbb{S}^1\right)} \leq C_0\eta$, where C_0 depends only on ℓ_0 (since $1 < \mathsf{R}_M < 2$),
- $t \mapsto \|w\, \partial_t^\alpha \partial_r^\beta \partial_\theta^\gamma \mathsf{u}(t,\cdot)\|_{L^2(\mathbb{D}_R)}$ is constant for any $\alpha, \beta, \gamma \in \mathbb{N}, R > 0$, and $w = w(r) \in L^2((0,R), r\,dr)$,
- from Lemma 5.3, we in addition obtain that $\mathsf{u}(t,1,\theta) := u(\mathsf{R}_M^2 t, \mathsf{R}_M, \theta) = 0$.

Now, recalling that $\rho_0 \in (0,1)$ and $\mathsf{R}_M \to 1^+$ (from Lemma 5.3), we fix $M \geq M_0$ such that $\min(3, \frac{1}{\rho_0}) > \mathsf{R}_M$. We then set

$$\rho_1 := \frac{1}{\mathsf{R}_M}, \quad \rho_2 := \frac{1}{\mathsf{R}_M} + \frac{1}{\mathsf{R}_M(M+1)},$$

and we have $\rho_0 < \rho_1 < \rho_2 < 1$, where the last inequality is a consequence of (5.2) in Lemma 5.3. With this notation, we have obtained that $\operatorname{supp} \mathsf{u} = \mathbb{R}\times[\rho_1, +\infty)\times\mathbb{S}^1$ and $\operatorname{supp}(\mathsf{q}) \subset \mathbb{R}\times[\rho_1,\rho_2]\times\mathbb{S}^1$.

Finally, all statements of the theorem are satisfied if we replace u by $\frac{\mathsf{u}|_{\mathbb{D}}}{\|\mathsf{u}(0,\cdot)\|_{L^2(\mathbb{D})}}$ (from the support properties of u together with conservation of the L^2–norm, we have $\|\mathsf{u}(0,\cdot)\|_{L^2(\mathbb{D})} > 0$), and undo polar coordinates (which is a fixed real-analytic diffeomorphism away from $r = 0$ where u and q are supported). □

6. The wave equation

This section is devoted to the proof of Theorem 1.3. The method of proof follows exactly the same lines as the one of Theorem 1.1. The only difference is that in the wave case the oscillation in the time variable t is chosen equal to k_m. As a consequence, a j-th order derivative in time contributes a power of order $m^{2j/p}$ instead of $m^{4j/p}$, which in view of the optimization step of Lemma C.2 yields a Gevrey regularity of $\frac{2}{2-3p}$ instead of $\frac{4}{2-3p}$. We write here the exact differences with respect to the Schrödinger case.

We use the notations and definitions of Section 2.1. Let

$$u_m^W(t,r,\theta) = F_m(r) e^{i(\lambda_m\theta + k_m t)},$$

so that

$$\Box u_m^W = (\partial_t^2 - \Delta) u_m^W = 0, \qquad \text{for} \qquad (t,r,\theta) \in \mathbb{R}\times\mathbb{R}_+^*\times\mathbb{S}^1, m \in \mathbb{N}.$$

We define then

$$u^W(t,r,\theta) := \check{A}_M(r) u_M^W(t,r,\theta) + \sum_{m=M+1}^{\infty} \left(\prod_{n=M+1}^{m} \gamma_n\right) A_m(r) u_m^W(t,r,\theta),$$

and

$$V^W(t,r,\theta) = \begin{cases} 0, & r \in K, \\ -\frac{Pu^W}{u^W}, & r \in [0,\infty)\backslash K, \end{cases}$$

where the set K is defined in Section 2.1. One can immediately verify the following points.

- The function u^W is well defined and satisfies $\operatorname{supp}(u^W) = \mathbb{R}\times[1,+\infty)\times\mathbb{S}^1$. The proof is exactly the same as the one of Lemma 2.2.
- Following the steps of Proposition 3.1 we find that there exists $M_1 \in \mathbb{N}$ such that if $M \geq M_1$ in the definition of u^W, we have $u^W \in \mathcal{G}^{\frac{2}{2-3p}}\left(\mathbb{R}; C^\ell([0,R]\times\mathbb{S}^1;\mathbb{C})\right)$ for all $\ell \in \mathbb{N}$ and for all $R > 0$. Indeed, in the wave equation case the bound (3.1) is replaced by

$$\left|\partial_t^j u^W(t)\right|_{C^\ell((r,\theta))} \leq \sum_{m\in\{m_0,m_0-1\}} \gamma_{M+1}\cdot\ldots\cdot\gamma_m k_m^j \left|A_m\right|_{C^\ell(r)} \left|F_m(r)\right|_{C^\ell(r)} \left|e^{i\lambda_m\theta}\right|_{C^\ell(\theta)},$$

  which in turn gives

$$\left\|\partial_t^j u^W(t)\right\|_{C^\ell([0,R]\times\mathbb{S}^1;\mathbb{C})} \leq \sup_{m\geq M+1} C m^{\frac{2j}{p}+\beta(\ell,p)} e^{-cm^{\frac{2}{p}-3}}, \quad \forall j \in \mathbb{N},$$

  and hence the stated regularity by means of Lemma C.2.

- Following the proof of Theorem 4.1 we get an analogous statement for
$$V^W \in \mathcal{G}^{\frac{2}{2-3p}}\big(\mathbb{R}; C^\ell([0,R]\times\mathbb{S}^1;\mathbb{C})\big).$$
Indeed, we have
$$\frac{Pu^W}{u^W} = \frac{\mathcal{A}^W_{m+1,1}}{\mathcal{C}^W_{m+1,1}}, \quad \text{on} \quad \mathbb{R}\times I_{m+1,1}\times\mathbb{S}^1, \quad m\geq M,$$
with
$$\mathcal{A}^W_{m+1,1}(t,r,\theta) = \frac{\gamma_{m+1}}{F_m(r)}([P,A_{m+1}]F_{m+1})(r)E^W_{m+1}(t,\theta), \quad \text{defined for } (t,r,\theta)\in\mathbb{R}\times\mathbb{R}^*_+\times\mathbb{S}^1,$$
$$\mathcal{C}^W_{m+1,1}(t,r,\theta) = 1+\frac{\gamma_{m+1}F_{m+1}(r)}{F_m(r)}A_{m+1}(r)E^W_{m+1}(t,\theta), \quad \text{defined for } (t,r,\theta)\in\mathbb{R}\times\mathbb{R}^*_+\times\mathbb{S}^1,$$
and
$$E^W_{m+1}(t,\theta) = e^{i(\lambda_{m+1}-\lambda_m)\theta+i(k_{m+1}-k_m)t}.$$
The bound (4.5) is replaced by
$$k_{m+1}-k_m\leq\lambda_{m+1}\leq C_p m^{2/p}, \quad \text{for all } m\in\mathbb{N},$$
and this leads to
$$\Big\|\partial_t^j\mathcal{A}^W_{m+1,1}(t,\cdot)\Big\|_{C^\ell(I_{m+1,1}\times\mathbb{S}^1)}\leq C^{j+1}_{p,\ell}m^{2j/p}m^{\beta(\ell)}e^{-\mu m^{\frac{2}{p}-3}}.$$
This shows that $\mathcal{A}^W_{m+1,1}\in\mathcal{G}^{\frac{2}{2-3p}}(\mathbb{R};C^\ell(I_{m+1,1}\times\mathbb{S}^1))$. Similarly, the bound (4.12) is replaced by
$$\Big|\partial_t^j\partial_r^{\alpha_1}\partial_\theta^{\alpha_2}\mathcal{C}^W_{m+1,1}(t,r,\theta)\Big|\leq C^{j+1}_{p,\ell}m^{2j/p}m^{\beta'(\ell)}e^{-\mu m^{\frac{2}{p}-3}},$$
which implies that $\mathcal{C}^W_{m+1,1}\in\mathcal{G}^{\frac{2}{2-3p}}(\mathbb{R};C^\ell(I_{m+1,1}\times\mathbb{S}^1))$. The same modifications are applied in the intervals $I_{m+1,4}$. The rest of the proof remains as in Theorem 4.1.

Equipped with the functions u^W and V^W we can conclude the proof of Theorem 1.3 as in Section 5. Given $\varepsilon>0$ as in the statement of Theorem 1.3, we fix p close to zero so that
$$p\in\left(0,\frac{1}{5}\right) \quad \text{and} \quad \frac{2}{2-3p}\leq 2+\varepsilon.$$
We take the R_M given by Lemma 5.3 and observe that
$$u^W(t,R_M,\theta)=0.$$
The proof is exactly as in Lemma 5.3 with k_m^2 replaced by k_m. We define
$$\mathsf{u}^W(t,r,\theta):=u^W(\mathsf{R}_Mt,\mathsf{R}_Mr,\theta), \quad \mathsf{q}^W(t,r,\theta):=\mathsf{R}_M^2V^W(\mathsf{R}_Mt,\mathsf{R}_Mr,\theta), \quad (t,r,\theta)\in\mathbb{R}\times\mathbb{R}^*_+\times\mathbb{S}^1,$$
and observe that u^W and q^W satisfy all the properties of Theorem 1.3 with the exception of the energy preservation property. In order to conclude, it suffices to observe that Lemma 5.1 remains valid for u^W as well. The proof uses exactly the same calculation with the sole difference that identity (5.1) is replaced by
$$\|w\,\partial_t^\alpha\partial_r^\beta\partial_\theta^\gamma u^W(t,\cdot)\|^2_{L^2(\mathbb{D}_R)} = 2\pi\sum_{m\geq M}k_m^{2\alpha}\lambda_m^{2\gamma}\int_1^R|B_m^{(\beta)}(r)|^2|w(r)|^2rdr.$$
The preservation of the above norms together with the fact that $w(r)=\frac{1}{r}\in L^2_{\mathrm{loc}}((0,+\infty),rdr)$ show that the energy
$$\|\partial_t\mathsf{u}^W(t,\cdot)\|^2_{L^2(\mathbb{D})}+\|\nabla\mathsf{u}^W(t,\cdot)\|^2_{L^2(\mathbb{D})}, \quad t\in\mathbb{R},$$
is preserved. Remark finally that $\|\partial_t\mathsf{u}^W(0,\cdot)\|^2_{L^2(\mathbb{D})}+\|\nabla\mathsf{u}^W(0,\cdot)\|^2_{L^2(\mathbb{D})}>0$ (since the support of u^W implies in particular that it is not constant). We can therefore replace u^W by
$$\frac{\mathsf{u}^W|_{\mathbb{D}}}{(\|\partial_t\mathsf{u}^W(0,\cdot)\|^2_{L^2(\mathbb{D})}+\|\nabla\mathsf{u}^W(0,\cdot)\|^2_{L^2(\mathbb{D})})^{1/2}}.$$
This finishes the proof of Theorem 1.3.

Appendix A. Bessel functions and the functions F_m

This first appendix is devoted first to recall classical properties of the Bessel functions J_ν, and second to collect properties of the functions F_m, all used in the main part of the article.

A.1. **Bessel functions J_ν.** The Bessel functions J_ν for $\nu \in \mathbb{R}_+$, see e.g. [OLBC10, Equation 10.2.2 p217] are defined by

$$J_\nu(r) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma(\nu+p+1)} \left(\frac{r}{2}\right)^{2p+\nu}, \quad r > 0, \tag{A.1}$$

where Γ is the Γ function (satisfying $\Gamma(\nu+p+1) = (\nu+p)!$ if $\nu \in \mathbb{N}$). For $\nu \in \mathbb{N}$, it can be rewritten as $J_\nu(r) = \frac{1}{\pi}\int_0^\pi \cos(\nu\tau - r\sin(\tau))d\tau$. For all $\nu \in \mathbb{R}_+$ the function J_ν solves the Bessel differential equation [OLBC10, Equation 10.2.1 p217]

$$r^2 \frac{d^2 J_\nu}{dr^2}(r) + r\frac{dJ_\nu}{dr}(r) + (r^2 - \nu^2)J_\nu(r) = 0, \quad r > 0. \tag{A.2}$$

We will need the following asymptotic, which is taken from [Tak21, Lemma 2.1] for $\delta = 1/4$ (which yields the range $p \in (0, 1/5)$ in the lemma).

Lemma A.1. *Let $p \in (0, 1/5)$ and J_λ be a Bessel function of order λ. We then have the asymptotic formula uniformly for $a \in (0, 1-\lambda^{-p}]$,*

$$J_\lambda(\lambda a) = \frac{1}{\sqrt{2\pi\lambda\tanh\alpha}} \exp\left(\lambda(\tanh\alpha - \alpha)\right)\left(1 + O(\lambda^{-1/4})\right),$$

as $\lambda \to \infty$ where $\alpha > 0$ is defined by

$$\cosh\alpha = \frac{1}{a} > 1.$$

Note that a related estimate is proved and used in [LL21, Lemma 3.12], where the function $a \mapsto -(\tanh\alpha(a) - \alpha(a))$ with $\alpha(a) = \cosh^{-1}\left(\frac{1}{a}\right)$ is interpreted as an Agmon distance.

The following properties of the first zero of the functions J_ν, J'_ν follow from [AS64, Section 9.5].

Lemma A.2. *For $\nu \in \mathbb{R}_+^*$, the functions J_ν, J'_ν have an infinite number of real zeros on $\mathbb{R}_+^*$, all of which are simple and denoted $(j_{\nu,k})_{k\in\mathbb{N}^*}, (j'_{\nu,k})_{k\in\mathbb{N}^*}$ ordered increasingly, and both tend to $+\infty$. Moreover, concerning the first zeros, there is $\nu_0 > 0$ such that for all $\nu \geq \nu_0$,*

$$\nu + 1.855\,\nu^{\frac{1}{3}} < j_{\nu,1} < \nu + 1.856\,\nu^{\frac{1}{3}},$$
$$\nu + 1.808\,\nu^{\frac{1}{3}} < j'_{\nu,1} < \nu + 1.809\,\nu^{\frac{1}{3}}.$$

A.2. **The functions F_m.** We now deduce properties of the functions F_m. We start with the following results taken from [Kg63, Tak21], and consequences of Lemma A.1. Note that Assumption 2.1 on the sequence λ_m is slightly stronger than the assumptions made in [Tak21, Equations (2.1) and (2.2)], so that results taken from [Tak21] apply.

Lemma A.3. *With r_m defined in (2.5), we have uniformly for $s \in [0,1]$,*

$$F_m(r_m(s)) = (1+o(1))\frac{\sqrt{m}e^{-(1+o(1))\frac{2\sqrt{2}}{3}(1+s)^{\frac{3}{2}}\lambda_m m^{-3}}}{(2\pi^2(1+s))^{\frac{1}{4}}\sqrt{\lambda_m}}, \quad \text{as } m \to \infty.$$

A consequence of Lemma A.3 is the following result.

Lemma A.4. *There exist $C, M_0 > 0$ and $\mu > 0$ such for all $m > M_0$ one has, with γ_{m+1} defined in (2.6),*

$$0 < \gamma_{m+1} \leq e^{-\lambda_m m^{-3}}, \tag{A.3}$$
$$0 < \gamma_{m+1}F_{m+1}(r_{m+1}(s)) \leq Ce^{-\mu\lambda_m m^{-3}}F_m(r_{m+1}(s)), \quad \text{for } s \in [0, 1/4], \tag{A.4}$$
$$0 < F_m(r_{m+1}(s)) \leq Ce^{-\mu\lambda_m m^{-3}}\gamma_{m+1}F_{m+1}(r_{m+1}(s)), \quad \text{for } s \in [3/4, 1]. \tag{A.5}$$

We refer to [Tak21, Lemma 2.3] for a proof of Lemma A.3 and Lemma A.4. Note that the fact that $F_{m+1}(r_{m+1}(s)) > 0$ in (A.4) is a straightforward consequence of Lemma A.3. The

fact that $F_m(r_{m+1}(s)) > 0$ in (A.5) (and $\gamma_{m+1} > 0$ in (A.3)) follows from Lemma A.1. Indeed $F_m(r_{m+1}(s)) = J_{\lambda_m}\left(\lambda_m(1-\frac{1}{m})r_{m+1}(s)\right)$ where

$$(1-\frac{1}{m})r_{m+1}(s) \leq (1-\frac{1}{m})(1+\frac{1}{m+1}) \leq 1-\frac{1}{m^2} \leq 1-\lambda_m^{-p}, \quad \text{for all } s \in [3/4,1],$$

since, from Assumption 2.1, we have $m^{\frac{2}{p}} \leq \lambda_m$. Lemma A.1 thus applies and implies $F_m(r_{m+1}(s)) > 0$ in (A.5) and $\gamma_{m+1} > 0$ (together with the sought upper bounds).

Remark A.5. The last two estimates above can be written as:

$$\begin{aligned} 0 <& \gamma_{m+1}F_{m+1}(r) \leq Ce^{-\mu\lambda_m m^{-3}}F_m(r), \quad \text{for } r \in I_{m+1,1},\\ 0 <& F_m(r) \leq Ce^{-\mu\lambda_m m^{-3}}\gamma_{m+1}F_{m+1}(r), \quad \text{for } r \in I_{m+1,4}. \end{aligned}$$

We also need to obtain a bound on the derivatives of F_m. We start with the following lemma.

Lemma A.6. *There are $\tilde{M}_1 > 0$ and $C > 0$ such that for $m \geq \tilde{M}_1$ and $1 \leq r \leq 1+\frac{1}{m}$ one has*

$$|F_m'(r)| \leq C\lambda_m|F_m(r)|.$$

Proof of Lemma A.6. Let us recall that $F_m(r) = J_{\lambda_m}(k_m r)$, with $\frac{k_m}{\lambda_m} = 1-\frac{1}{m}$. Remark in particular that since $k_m r < \lambda_m$ Lemma A.2 implies that $F_m(r) \neq 0$ for $1 \leq r \leq 1+\frac{1}{m}$. We notice the recurrence formulae for Bessel functions, namely (see [OLBC10, Equation 10.6.2 p. 222])

$$J_\nu'(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x),$$

and apply it to $F_m'(r) = k_m J_{\lambda_m}'(k_m r)$ to find

$$\begin{aligned} F_m'(r) &= k_m\left(\frac{\lambda_m}{k_m r}J_{\lambda_m}(k_m r) - J_{\lambda_m+1}(k_m r)\right) \qquad \text{(A.6)}\\ \Longrightarrow \frac{F_m'(r)}{F_m(r)} &= k_m\left(\frac{\lambda_m}{k_m r} - \frac{J_{\lambda_m+1}(k_m r)}{J_{\lambda_m}(k_m r)}\right). \end{aligned}$$

We need therefore to study the ratio $\frac{J_{\lambda_m+1}(k_m r)}{J_{\lambda_m}(k_m r)}$. Notice that

$$J_{\lambda_m}(k_m r) = J_{\lambda_m}(\lambda_m a), \quad a = (1-1/m)r.$$

Remark that the assumption $1 \leq r \leq 1+1/m$ implies that $0 < a \leq 1-\frac{1}{m^2} \leq 1-\lambda_m^{-p}$ since $\lambda_m^p \geq m^2$, hence we are in the regime of Lemma A.1. This implies that we have, uniformly as $m \to \infty$,

$$J_{\lambda_m}(k_m r) = J_{\lambda_m}(\lambda_m a) = \frac{1}{\sqrt{2\pi\lambda_m\sqrt{1-a^2}}}\exp\left(\lambda_m\Phi(a)\right)\left(1+O(\lambda_m^{-1/4})\right), \qquad \text{(A.7)}$$

with

$$\Phi(a) := \sqrt{1-a^2} - \log\left(\frac{1+\sqrt{1-a^2}}{a}\right).$$

Here we used the relations

$$\tanh\alpha = \frac{\sinh\alpha}{\cosh\alpha} = \sqrt{1-a^2}, \quad \alpha = \operatorname{arcosh}\frac{1}{a} = \log\left(\frac{1+\sqrt{1-a^2}}{a}\right).$$

We now look at the term $J_{\lambda_m+1}(k_m r)$ and write it as

$$J_{\lambda_m+1}(k_m r) = J_{\lambda_m+1}((\lambda_m+1)b),$$

with

$$b = \frac{\lambda_m}{\lambda_m+1}a. \qquad \text{(A.8)}$$

Again, we notice that

$$0 < b < a \leq 1-\lambda_m^{-p} \leq 1-(\lambda_m+1)^{-p},$$

so we are in the regime of Lemma A.1 which yields

$$J_{\lambda_m+1}(k_m r) = J_{\lambda_m+1}((\lambda_m+1)b) \tag{A.9}$$
$$= \frac{1}{\sqrt{2\pi(\lambda_m+1)\sqrt{1-b^2}}} \exp\left((\lambda_m+1)\Phi(b)\right)\left(1+O((\lambda_m+1)^{-1/4})\right).$$

Observe that in particular a, b lie in the compact interval $a, b \in [\frac{1}{2}, 1]$ in which Φ is smooth. Recalling that $a - b = \frac{a}{\lambda_m+1}$ we can then bound

$$|(\lambda_m+1)\Phi(b) - \lambda_m\Phi(a)| \leq \lambda_m|\Phi(b)-\Phi(a)| + \Phi(b) \leq C\lambda_m|b-a| + C' \tag{A.10}$$
$$\leq C\left(\frac{\lambda_m}{\lambda_m+1} + C\right) \leq C'.$$

Notice as well that using the definition of b in (A.8) we can write

$$\frac{1-b^2}{1-a^2} = 1 + \frac{a^2}{1-a^2}\frac{2\lambda_m+1}{(\lambda_m+1)^2}.$$

Therefore, for large m

$$\frac{1-b^2}{1-a^2} \leq 2.$$

This yields

$$\frac{1/\sqrt{2\pi\lambda_m\sqrt{1-a^2}}}{1/\sqrt{2\pi(\lambda_m+1)\sqrt{1-b^2}}} = \sqrt{\frac{\lambda_m+1}{\lambda_m}}\sqrt{\frac{\sqrt{1-b^2}}{\sqrt{1-a^2}}} \leq 2, \tag{A.11}$$

for large m. We can then use the expressions (A.7) and (A.9) combined with the bounds (A.11) and (A.10) to find that there is $\tilde{M}_1$ large enough such that for $m \geq \tilde{M}_1$ one has the uniform bound

$$\frac{J_{\lambda_m+1}(k_m r)}{J_{\lambda_m}(k_m r)} \leq C, \tag{A.12}$$

for some $C > 0$. Going back to (A.6) we use (A.12) to get that for $m \geq \tilde{M}_1$

$$\left|\frac{F_m'(r)}{F_m(r)}\right| \leq k_m\left(2 + \left|\frac{J_{\lambda_m+1}(k_m r)}{J_{\lambda_m}(k_m r)}\right|\right) \leq C\lambda_m,$$

since $\frac{k_m}{\lambda_m} = 1 - \frac{1}{m}$. This proves the lemma. □

We can use this to prove an estimate for the derivatives of any order for F_m.

Lemma A.7. *With $\tilde{M}_1 > 0$ given by Lemma A.6, for $\ell \in \mathbb{N}$ there is $C_\ell > 0$ such that one has for $m \geq \tilde{M}_1$ and $1 \leq r \leq 1 + \frac{1}{m}$*

$$|F_m^{(\ell)}(r)| \leq C_\ell \lambda_m^\ell |F_m(r)|.$$

In particular, using the seminorm given in Lemma B.7, we have

$$|F_m|_{C^\ell(r)} \leq C_\ell \lambda_m^\ell |F_m(r)| \tag{A.13}$$

Proof. The key point is to make repetitive use of the differential equation (2.2) satisfied by F_m, that is

$$F_m''(r) = -\frac{1}{r}F_m'(r) - \left(k_m^2 - \frac{\lambda_m^2}{r^2}\right)F_m(r). \tag{A.14}$$

Note that on the interval $r \in \left[1, 1+\frac{1}{m}\right]$, all derivatives of $1/r$ and $1/r^2$ are uniformly bounded. We proceed by induction on ℓ. For $\ell = 0$, there is nothing to prove, and for $\ell = 1$ this is Lemma A.6.

Assume now that there is $\ell \geq 0$ such that for all $0 \leq j \leq \ell + 1$,

$$|F_m^{(j)}(r)| \leq C\lambda_m^j |F_m(r)|.$$

We differentiate the equation (A.14) ℓ times

$$F_m^{(\ell+2)} = -\partial_r^\ell\left(\frac{1}{r}F_m'\right) - \partial_r^\ell\left(\left(k_m^2 - \frac{\lambda_m^2}{r^2}\right)F_m\right).$$

By Leibniz' rule and uniform boundedness of derivatives of the coefficients,

$$|F_m^{(\ell+2)}| \leq C\sum_{j=0}^{\ell+1}|F_m^{(j)}| + C\lambda_m^2\sum_{j=0}^{\ell}|F_m^{(j)}|.$$

Using the induction hypothesis,

$$|F_m^{(\ell+2)}| \leq C|F_m|\left(\sum_{j=0}^{\ell+1}\lambda_m^j + \lambda_m^2\sum_{j=0}^{\ell}\lambda_m^j\right).$$

Since $\lambda_m^j \leq \lambda_m^{\ell+2}$ for $0 \leq j \leq \ell+2$ this gives

$$|F_m^{(\ell+2)}(r)| \leq C\lambda_m^{\ell+2}|F_m(r)|,$$

for a constant C depending on ℓ and $\tilde{M}_1$ only. This completes the induction and the proof. □

Lemma A.8. *With $\tilde{M}_1 > 0$ given by Lemma A.6, for all $\ell \in \mathbb{N}$, there is $C_\ell > 0$ such that for all $m \geq \tilde{M}_1$,*

$$\left|\frac{1}{F_m}\right|_{C^\ell(r)} \leq C_\ell\lambda_m^{\ell^2}\frac{1}{|F_m(r)|}, \quad \text{for all } 1 \leq r \leq 1+\frac{1}{m}, \tag{A.15}$$

$$\left|\left(\frac{F_{m+1}}{F_m}\right)^{(\ell)}(r)\right| \leq C_\ell\lambda_m^{\ell^2}\left|\frac{F_{m+1}}{F_m}(r)\right|, \quad \text{for all } 1 \leq r \leq 1+\frac{1}{m+1}.$$

Here, the norm $|\cdot|_{C^\ell(r)}$ is the seminorm given in Lemma B.7.

Proof. With the chosen seminorm, Lemma B.7 provides the following algebra property.

$$\left|\left(\frac{F_{m+1}}{F_m}\right)^{(\ell)}(r)\right| \leq C\left|\frac{F_{m+1}}{F_m}\right|_{C^\ell(r)} \leq C\,|F_{m+1}|_{C^\ell(r)}\left|\frac{1}{F_m}\right|_{C^\ell(r)}.$$

Using now the inversion result of Lemma B.7, we deduce

$$\left|\frac{1}{F_m}\right|_{C^\ell(r)} \leq C_\ell\frac{|F_m|^\ell_{C^\ell(r)}}{|F_m(r)|^{\ell+1}}, \quad \text{for all } 1 \leq r \leq 1+\frac{1}{m}.$$

Estimate (A.13) of Lemma A.7 implies $|F_m|_{C^\ell(r)} \leq C_\ell\lambda_m^\ell|F_m(r)|$ and hence (A.15).

Using again Lemma A.7 we have $|F_{m+1}|_{C^\ell(r)} \leq C_\ell\lambda_{m+1}^\ell|F_{m+1}(r)| \leq C_\ell C^\ell\lambda_m^\ell|F_{m+1}(r)|$ for all $m \geq \tilde{M}_1$ and $1 \leq r \leq 1+\frac{1}{m+1}$, after having used that there is $C > 1$ such that $\lambda_{m+1} \leq C\lambda_m$ for all $m \in \mathbb{N}$, from Assumption 2.1.

Combining the last four inequalities, we deduce (changing the value of the constant C_ℓ)

$$\left|\left(\frac{F_{m+1}}{F_m}\right)^{(\ell)}(r)\right| \leq C_\ell\lambda_m^\ell|F_{m+1}(r)|\frac{\left(C_\ell\lambda_m^\ell|F_m(r)|\right)^\ell}{|F_m(r)|^{\ell+1}} \leq \widetilde{C_\ell}\lambda_m^{\ell^2}\left|\frac{F_{m+1}}{F_m}(r)\right|,$$

for all $m \geq \tilde{M}_1$ and $1 \leq r \leq 1+\frac{1}{m+1}$, which concludes the proof of the lemma. □

Appendix B. Gevrey type functional spaces

B.1. **Abstract Banach valued Gevrey spaces.** This appendix is devoted to technical results concerning Gevrey regularity of $\frac{1}{f}$ for a Gevrey function f, and we follow closely [Yam89, Yam92], see also [LR20, LRR25]. In all this section $\mathcal{A}$ is a Banach algebra with unit. In Appendix B.2 below (and in the main part of the article), we specify to the case where $\mathcal{A}$ is the set of C^ℓ functions, with a well chosen norm making it an algebra, see Lemma B.6.

We will define Banach spaces of Gevrey λ functions for $\lambda > 1$ with values in $\mathcal{A}$. All the spaces in this section depend on the algebra $\mathcal{A}$ and on the Gevrey coefficient $\lambda > 1$, but it will be fixed, so we will sometimes omit this dependence. We introduce the function

$$\Gamma_\lambda(k) = 2^{-5}(k!)^\lambda(1+k)^{-2}.$$

For any $L > 0$, we consider the space of functions $u \in C^\infty(\mathcal{J}, \mathcal{A})$, where $\mathcal{J} \subset \mathbb{R}$ is an open interval, such that

$$|u|_L := \sup \left\{ \frac{\left\|u^{(k)}(t)\right\|_{\mathcal{A}}}{L^k \Gamma_\lambda(k)}, t \in I, k \in \mathbb{N} \right\} < \infty.$$

The following space is defined in Yamanaka [Yam92].

Definition B.1. Let $\lambda > 1, L > 0$ and $\mathcal{J} \subset \mathbb{R}$ be an open interval. We denote by $\mathcal{G}_L^\lambda = \mathcal{G}_L^\lambda(\mathcal{J}; \mathcal{A})$ the (Banach) space of functions $u \in C^\infty(\mathcal{J}, \mathcal{A})$ such that the norm

$$\|u\|_{\mathcal{G}_L^\lambda(\mathcal{J};\mathcal{A})} = \|u\|_L := \max \left\{ 2^6 \|u\|_{L^\infty(\mathcal{J},\mathcal{A})}, 2^3 L^{-1} |u'|_L \right\}$$

is finite.

Note that, with this definition, we have in particular $\bigcup_{L>0} \mathcal{G}_L^\lambda(\mathcal{J}; \mathcal{A}) = \mathcal{G}^\lambda(\mathcal{J}; \mathcal{A})$, where the latter is defined in Definition 1.2. The following result is a specific case of [Yam92, Theorem 5.4] and will be used several times in the sequel.

Lemma B.2. *(Algebra property) For any $u, v \in \mathcal{G}_L^\lambda(\mathcal{J}, \mathcal{A})$,we have $uv \in \mathcal{G}_L^\lambda(\mathcal{J}, \mathcal{A})$ together with the estimate*

$$\|uv\|_L \leq \|u\|_L \|v\|_L \quad \forall u, v \in \mathcal{G}_L^\lambda(\mathcal{J}, \mathcal{A}).$$

Note that since $\mathcal{A}$ is an algebra, the product uv is the function defined by $t \mapsto (uv)(t) := u(t)v(t)$, where the product in the right hand side is that of $\mathcal{A}$.

Proof. The result is essentially [Yam92, Theorem 5.4], which actually considered linear applications. We only translate this result to our context. Setting $\mathcal{J} = (t_1, t_2)$, and using the notation of [Yam92, Theorem 5.4] for $X = Y = Z, \mathbb{G}(\mathcal{J}, \mathcal{A})$, we set $X = Y = Z = \mathcal{A}$. In that notation, we have $\mathbb{G}(\mathcal{J}, \mathcal{A}) = \mathcal{G}_L^\lambda(\mathcal{J}, \mathcal{A})$ with equality of the norms. For any $v \in \mathcal{G}_L^\lambda(\mathcal{J}, \mathcal{A})$, $t \in \mathcal{J}$ define $(L_v)(t) \in \mathcal{L}(\mathcal{A})$, the application of multiplication by $v(t)$, that is $(L_v)(t)(u_0) = v(t)u_0$ for any $u_0 \in \mathcal{A}$. If $v \in \mathcal{G}_L^\lambda(\mathcal{J}, \mathcal{A})$, we see that $\frac{d^k}{dt^k} L_v(t) = L_{v^{(k)}(t)}$ together with $\left\| \frac{d^k}{dt^k} L_v(t) \right\|_{\mathcal{L}(\mathcal{A})} \leq \left\| v^{(k)}(t) \right\|_{\mathcal{A}}$ by the algebra property of $\mathcal{A}$. In particular, for $v \in \mathcal{G}_L^\lambda(\mathcal{J}, \mathcal{A})$, we have $L_v \in \mathbb{G}(\mathcal{J}, \mathcal{L}(\mathcal{A}))$ with

$$\|L_v\|_{\mathbb{G}(\mathcal{J},\mathcal{L}(\mathcal{A}))} \leq \|v\|_{\mathcal{G}_L^\lambda(\mathcal{J},\mathcal{A})}. \tag{B.2}$$

Then, [Yam92, Theorem 5.4] implies

$$\|L_v u\|_{\mathbb{G}(\mathcal{J},\mathcal{A})} \leq \|L_v\|_{\mathbb{G}(\mathcal{J},\mathcal{L}(\mathcal{A}))} \|u\|_{\mathbb{G}(\mathcal{J},\mathcal{A})}.$$

But we have $\|L_v u\|_{\mathbb{G}(\mathcal{J},\mathcal{A})} = \|vu\|_{\mathcal{G}_L^\lambda(\mathcal{J},\mathcal{A})}$ and $\|u\|_{\mathbb{G}(\mathcal{J},\mathcal{A})} = \|u\|_{\mathcal{G}_L^\lambda(\mathcal{J},\mathcal{A})}$. This gives the result using (B.2). □

The next lemma is straightforward, but we write it in order to stress the fact that making L large enough allows to make the norm as small as desired if the L^∞ norm is small.

Lemma B.3. *For any $0 < L < L'$, $\mathcal{G}_L^\lambda(\mathcal{J}, \mathcal{A}) \subset \mathcal{G}_{L'}^\lambda(\mathcal{J}, \mathcal{A})$, together with the following estimates: for all $u \in \mathcal{G}_L^\lambda(\mathcal{J}, \mathcal{A})$*

$$|u|_{L'} \leq |u|_L, \quad \|u\|_{L'} \leq \|u\|_L$$
$$\|u\|_{L'} \leq \max \left\{ 2^6 \|u\|_{L^\infty}, 2^3 L'^{-1} |u'|_L \right\} \leq \max \left\{ 2^6 \|u\|_{L^\infty}, 2^3 \frac{L}{L'} \|u\|_L \right\}$$

Proof. The first inequalities are straightforward, and the last one is a consequence of $|u'|_L \leq L\|u\|_L$. □

The following proposition shows that the inverse of a Gevrey function is a Gevrey function. The proof is a consequence of [Yam89, Lemma 6.4], which we recall in Lemma B.5 below.

Corollary B.4. *Let $\lambda > 1, L > 0$ and $\mathcal{J} \subset \mathbb{R}$ be an open interval. Then, there exist constants B_λ and $E_\lambda > 0$, only depending on λ, so that the following statement holds.*

Assume $f \in \mathcal{G}_L^\lambda(\mathcal{J},\mathcal{A})$ is such that $f(t)$ is invertible in $\mathcal{A}$ for any $t \in \mathcal{J}$, denote by f^{-1} the function $t \mapsto (f(t))^{-1}$ where $(f(t))^{-1}$ is the inverse of $f(t)$ in the Banach algebra $\mathcal{A}$, and assume in addition that $f^{-1} \in L^\infty(\mathcal{J},\mathcal{A})$. Then, for all

$$\widetilde{L} \geq L \max\left(2, B_\lambda \left\|f^{-1}\right\|_{L^\infty(\mathcal{J},\mathcal{A})} \|f\|_L\right),$$

we have $f^{-1} \in \mathcal{G}_{\tilde{L}}^\lambda(\mathcal{J},\mathcal{A})$ together with

$$\left\|f^{-1}\right\|_{\tilde{L}} \leq E_\lambda \left\|f^{-1}\right\|_{L^\infty(\mathcal{J},\mathcal{A})}. \tag{B.3}$$

Proof. We consider $E = F = \mathcal{A}$ and let $\Phi : U \mapsto \mathrm{Iso}(E,F)$ be the multiplication by f defined by $\Phi(t)u := f(t)u$ for $t \in \mathcal{J}, u \in \mathcal{A}$. The inverse $\Phi(t)^{-1}$ is the multiplication by $f(t)^{-1}$. By assumption, we have $\Phi \in C^\infty(\mathcal{J};\mathcal{L}(\mathcal{A}))$ and for all $t \in \mathcal{J}$

$$\begin{aligned}
\left\|\Phi^{-1}(t)\right\|_{\mathcal{L}(\mathcal{A})} &\leq \left\|f^{-1}\right\|_{L^\infty(\mathcal{J},\mathcal{A})} < \infty,\\
\|\Phi(t)\|_{\mathcal{L}(\mathcal{A})} &\leq \|f\|_{L^\infty(\mathcal{J},\mathcal{A})} \leq 2^{-6}\|f\|_L\\
\left\|\Phi^{(k)}(t)\right\|_{\mathcal{L}(\mathcal{A})} &\leq \left\|f^{(k)}(t)\right\|_{\mathcal{A}} \leq L^{k-1}\Gamma_\lambda(k-1)|f'|_L, \quad \forall k \geq 1\\
&\leq C_\lambda L^k \Gamma_\lambda(k)\|f\|_L,
\end{aligned}$$

where $C_\lambda = 2^{-3}\sup_{k\geq 1}\frac{\Gamma_\lambda(k-1)}{\Gamma_\lambda(k)} \in (0,+\infty)$ and we have used the definition of $\|\cdot\|_L$ in (B.1). In particular, for any $L' > L$, since $k \geq 1$, we have $L^k = \left(\frac{L}{L'}\right)^k (L')^k \leq \frac{L}{L'}(L')^k$. This implies

$$\left\|\Phi^{(k)}(t)\right\|_{\mathcal{L}(\mathcal{A})} \leq B(L')^k\Gamma_\lambda(k), \quad \forall t \in \mathcal{J}, \forall k \geq 1,$$

with $B = \frac{LC_\lambda\|f\|_L}{L'}$. Taking $L' \geq \max\left(L, 2\left\|f^{-1}\right\|_{L^\infty(\mathcal{J},\mathcal{A})} LC_\lambda \|f\|_L\right)$ and denoting $A = \left\|f^{-1}\right\|_{L^\infty(\mathcal{J},\mathcal{A})}$, we obtain $AB \leq 1/2$. Applying Lemma B.5 for any $n \in \mathbb{N}$, we get for $\Psi(t) = \Phi(t)^{-1}$, and for all $t \in \mathcal{J}$ and $k \geq 1$

$$\left\|\Psi^{(k)}(t)\right\|_{\mathcal{L}(\mathcal{A})} \leq A(L')^k\Gamma_\lambda(k).$$

Since $\Psi(t)$ is the multiplication by $v(t) = f(t)^{-1}$, we have $\left\|\Psi^{(k)}(t)\right\|_{\mathcal{L}(\mathcal{A})} = \left\|v^{(k)}(t)\right\|_{\mathcal{A}}$ and we obtain

$$\left\|v^{(k)}(t)\right\|_{\mathcal{A}} \leq A(L')^k\Gamma_\lambda(k), \quad \forall t \in \mathcal{J}, \forall k \geq 1.$$

In particular, this yields

$$\left\|(v')^{(i)}(t)\right\|_{\mathcal{A}} \leq A(L')^{i+1}\Gamma_\lambda(i+1) \leq D_\lambda A(2L')^{i+1}\Gamma_\lambda(i), \quad \forall t \in \mathcal{J}, \forall i \geq 0. \tag{B.4}$$

where we have denoted some constant $D_\lambda = \sup_{i\geq 0} 2^{-(i+1)}\frac{\Gamma_\lambda(i+1)}{\Gamma_\lambda(i)}$ which is finite since $(i+1)! = (i+1)i!$. Now, Estimate (B.4) can be written as

$$(2L')^{-1}|v'|_{2L'} \leq D_\lambda A.$$

Recalling the definition (B.1) of the norm together with $\|v\|_{L^\infty(\mathcal{J},\mathcal{A})} = \left\|f^{-1}\right\|_{L^\infty(\mathcal{J},\mathcal{A})} = A$, we have obtained

$$\|v\|_{2L'} \leq A\max(2^6, 2^3 D_\lambda).$$

This is the sought estimate (B.3), after having changed the name of the constants: $\widetilde{L} = 2L'$ and $B_\lambda = 4C_\lambda$, and $E_\lambda = \max(2^6, 2^3 D_\lambda)$. □

The following result is a particular case of [Yam89, Lemma 6.4].

Lemma B.5. *Let E, F be Banach spaces and $\mathcal{J}$ an open interval of $\mathbb{R}$. Let $A > 1$, $L > 0$, $A > 0$, $B > 0$ be real constants and n be a positive integer. Assume that $AB < 1$. Let $\Phi : \mathcal{J} \longrightarrow \mathrm{Iso}(E,F)$ be a C^n-map such that*

$$\left\|\Phi(t)^{-1}\right\|_{\mathcal{L}(F;E)} \leq A, \quad t \in \mathcal{J}$$

and

$$\left\|\Phi^{(k)}(t)\right\|_{\mathcal{L}(E;F)} \leq BL^k\Gamma_\lambda(k), \quad t \in \mathcal{J}, \quad 1 \leq k \leq n.$$

Setting $\Psi(t) = \Phi(t)^{-1}$ *for* $t \in \mathcal{J}$*, the map* $\Psi : \mathcal{J} \longrightarrow \mathrm{Iso}(F, E)$ *is a* C^n*-map and satisfies*

$$\left\|\Psi^{(k)}(x)\right\|_{\mathcal{L}(F;E)} \leq \frac{A^2 B}{1 - AB} L^k \Gamma_\lambda(k), \quad t \in \mathcal{J}, \quad 1 \leq k \leq n.$$

Note that this lemma is written in [Yam89, Lemma 6.4] in a more general context where $\mathcal{J}$ is replaced by a Banach space, so that $\Phi^{(k)}(t)$ is a multilinear operator with value in $\mathcal{L}(E, F)$, for which the norm is more intricate to define. We can check that if $\mathcal{J}$ is an interval, $\Phi^{(k)}(t)$ can be identified with an element of $\mathcal{L}(E, F)$.

B.2. **Gevrey spaces with values in C^ℓ.** In this section, we give the applications of the results of Section B.1 to Gevrey functions with values the Banach algebra of functions of class C^ℓ. Before this, we start with preliminaries concerning the C^ℓ norms we consider.

Lemma B.6. *Let* $\Omega \subset \mathbb{R}^d$ *be open and let* $\ell \in \mathbb{N}$*. There exists* $D_{\ell,d} > 0$ *such that the following statement holds. Let* $\mathcal{A}$ *be either*

- $\mathcal{A} = C^\ell_b(\Omega)$*,*
- *(in case* $\overline{\Omega}$ *compact)* $\mathcal{A} = C^\ell(\overline{\Omega})$*, the set of restrictions to* $\overline{\Omega}$ *of functions that are* C^ℓ *in a neighborhood of* Ω*,*
- *(in case* $\overline{\Omega}$ *compact)* $\mathcal{A} = C^\ell(\overline{\Omega} \times \mathbb{S}^1)$*, the set of restrictions to* $\overline{\Omega} \times \mathbb{S}^1$ *of functions that are* C^ℓ *in a neighborhood of* $\Omega \times \mathbb{S}^1$ *in* $\mathbb{R}^d \times \mathbb{S}^1$ *.*

Then, $\mathcal{A}$ *endowed with the norm*

$$\|f\|_{C^\ell} := \sum_{|\alpha| \leq \ell} \frac{\|\partial^\alpha f\|_{L^\infty}}{\alpha!},$$

is a Banach algebra. More precisely, we have $\|fg\|_{C^\ell} \leq \|f\|_{C^\ell} \|g\|_{C^\ell}$*, for every* $f, g \in \mathcal{A}$*. Moreover, assume that* $f \in \mathcal{A}$ *satisfies* $|f(x)| \geq \eta_0 > 0$ *for* $x \in \Omega$ *(resp. for* $x \in \overline{\Omega} \times \mathbb{S}^1$ *in the last case). Then* $\frac{1}{f} \in \mathcal{A}$ *and we have*

$$\left\|\frac{1}{f}\right\|_{C^\ell} \leq D_{\ell,d} \frac{\|f\|^\ell_{C^\ell}}{\eta_0^{\ell+1}}.$$

Lemma B.6 directly follows from the pointwise version in Lemma B.7 below.

Lemma B.7. *Let* U *be either an open set* $U \subset \mathbb{R}^d$ *or* $U = V \times \mathbb{S}^1$ *where* $V \subset \mathbb{R}^{d-1}$ *is an open set, and let* $\ell \in \mathbb{N}$*. There exists* $D_\ell > 0$ *such that the following statement holds. For all* $x \in U$ *and all* $f, g \in C^\ell$ *in a neighborhood of* x*, defining the pointwise quantity*

$$|f|_{C^\ell(x)} := \sum_{|\alpha| \leq \ell} \frac{|\partial^\alpha f(x)|}{\alpha!},$$

we have

$$|fg|_{C^\ell(x)} \leq |f|_{C^\ell(x)} \, |g|_{C^\ell(x)} \,. \tag{B.5}$$

If in addition $f(x) \neq 0$*, then* $\frac{1}{f}$ *is* C^ℓ *in a neighborhood of* x*, together with the estimate*

$$\left|\frac{1}{f}\right|_{C^\ell(x)} \leq D_\ell \frac{|f|^\ell_{C^\ell(x)}}{|f(x)|^{\ell+1}}.$$

Proof. Let us first prove (B.5). For every multi-index α with $|\alpha| \leq \ell$, Leibniz' rule gives

$$\partial^\alpha (fg)(x) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f(x) \, \partial^{\alpha-\beta} g(x).$$

Hence

$$\frac{|\partial^\alpha (fg)(x)|}{\alpha!} \leq \sum_{\beta \leq \alpha} \frac{\binom{\alpha}{\beta}}{\alpha!} |\partial^\beta f(x)| |\partial^{\alpha-\beta} g(x)| = \sum_{\beta \leq \alpha} \frac{|\partial^\beta f(x)|}{\beta!} \frac{|\partial^{\alpha-\beta} g(x)|}{(\alpha-\beta)!}.$$

Summing over all multi-indices α such that $|\alpha| \leq \ell$, we obtain

$$|fg|_{C^\ell(x)} \leq \sum_{|\alpha|\leq\ell} \sum_{\beta\leq\alpha} \frac{|\partial^\beta f|}{\beta!} \frac{|\partial^{\alpha-\beta} g(x)|}{(\alpha-\beta)!}$$
$$\leq \sum_{|\beta|\leq\ell} \sum_{|\gamma|\leq\ell} \frac{|\partial^\beta f(x)|}{\beta!} \frac{|\partial^\gamma g(x)|}{\gamma!} = \left(\sum_{|\beta|\leq\ell} \frac{|\partial^\beta f(x)|}{\beta!} \right) \left(\sum_{|\gamma|\leq\ell} \frac{|\partial^\gamma g(x)|}{\gamma!} \right) = |f|_{C^\ell(x)} \, |g|_{C^\ell(x)} \,,$$

where we used the change of index $\gamma = \alpha - \beta$, so that the summation set $\{(\beta,\gamma) : |\beta| + |\gamma| \leq \ell\}$ is contained in $\{(\beta,\gamma) : |\beta| \leq \ell,\ |\gamma| \leq \ell\}$. This proves (B.5).

To bound the inverse let us set $g = 1/f$. Since $|f| \neq 0$, g is well defined and in C^ℓ in a neighborhood of x.

For a multi-index $\alpha \neq 0$, the Faà di Bruno formula (see for instance [Ler19, Lemma B8]) applied to $g = \Phi(f)$ with $\Phi(s) = 1/s$ satisfying $\Phi^{(m)}(s) = \frac{(-1)^m m!}{s^{m+1}}$, yields

$$\partial^\alpha g(x) = \sum_{1\leq m\leq|\alpha|} \frac{C_{\alpha,\beta_1,\dots,\beta_m}}{f(x)^{m+1}} \prod_{j=1}^m \partial^{\beta_j} f(x),$$

where β_j is a non-zero multi-index for $1 \leq j \leq m$ with

$$\beta_1 + \cdots + \beta_m = \alpha,$$

and $C_{\alpha,\beta_1,\dots,\beta_m}$ is a combinatorial constant. Using $|\partial^\beta f(x)| \leq C_\ell |f|_{C^\ell(x)}$ for $|\beta| \leq \ell$, we obtain for $|\alpha| \leq \ell$

$$|\partial^\alpha g(x)| \leq \sum_{m=1}^{|\alpha|} \sum_{\beta_1+\cdots+\beta_m=\alpha} C_{\alpha,\beta_1,\dots,\beta_m} \frac{\prod_{j=1}^m |\partial^{\beta_j} f(x)|}{|f(x)|^{m+1}} \leq C_\ell \sum_{m=1}^{\ell} \frac{|f|^m_{C^\ell(x)}}{|f(x)|^{m+1}}.$$

Finally noticing that $0 < |f(x)| \leq |f|_{C^\ell(x)}$, whence $1 \leq \frac{|f|_{C^\ell(x)}}{|f(x)|}$, we deduce $\frac{|f|^m_{C^\ell(x)}}{|f(x)|^{m+1}} \leq \frac{|f|^\ell_{C^\ell(x)}}{|f(x)|^{\ell+1}}$ for all $m \leq \ell$. Recalling that $g = 1/f$, the result follows up to changing the constant. □

We finally formulate a corollary of Corollary B.4, in the setting of C^ℓ functions, using Lemma B.6.

Corollary B.8. *Let $\lambda > 1, L > 0, \ell \in \mathbb{N}$, $\mathcal{J} \subset \mathbb{R}$ be an open interval and $I = I_{m+1,4}$ or $I_{m+1,1}$ a compact interval of $\mathbb{R}$. There are $C, D > 0$ depending only on λ and ℓ such that the following statement holds. We let $\mathcal{A} := C^\ell(I \times \mathbb{S}^1)$ be endowed with the norm in Lemma B.6. Assume $f \in \mathcal{G}^\lambda_L(\mathcal{J};\mathcal{A})$ satisfies $|f(t,r,\theta)| \geq \eta_0 > 0$ for $(t,r,\theta) \in \mathcal{J} \times I \times \mathbb{S}^1$. Then, with $L' = L \max\left(2, C\frac{\|f\|^\ell_{L^\infty(\mathcal{J},\mathcal{A})}\|f\|_L}{\eta_0^{\ell+1}}\right) > 0$ we have $\frac{1}{f} \in \mathcal{G}^\lambda_{L'}(\mathcal{J};\mathcal{A})$ with*

$$\left\| \frac{1}{f} \right\|_{\mathcal{G}^\lambda_{L'}(\mathcal{J};\mathcal{A})} \leq D \frac{\|f\|^\ell_{L^\infty(\mathcal{J},\mathcal{A})}}{\eta_0^{\ell+1}}. \tag{B.6}$$

Proof. The product in the algebra $\mathcal{A}$ is the product of functions, so the inverse of f for this product is $\frac{1}{f}$. From the assumption $|f(t,r,\theta)| \geq \eta_0 > 0$ for $(t,r,\theta) \in \mathcal{J} \times I \times \mathbb{S}^1$, together with Lemma B.6, we have

$$\left\| \frac{1}{f} \right\|_{L^\infty(\mathcal{J},\mathcal{A})} \leq D \frac{\|f\|^\ell_{L^\infty(\mathcal{J},\mathcal{A})}}{\eta_0^{\ell+1}} \leq D \frac{\|f\|^\ell_{\mathcal{G}^\lambda_L(\mathcal{J},\mathcal{A})}}{\eta_0^{\ell+1}}.$$

Corollary B.4 writes for $L' \geq L \max\left(2, B_\lambda \left\|f^{-1}\right\|_{L^\infty(\mathcal{J},\mathcal{A})} \|f\|_L\right)$. Moreover, we have

$$\left\|f^{-1}\right\|_{L^\infty(\mathcal{J},\mathcal{A})} \|f\|_L \leq D \frac{\|f\|^\ell_{L^\infty(\mathcal{J},\mathcal{A})}}{\eta_0^{\ell+1}} \|f\|_L \,.$$

So, picking $C = B_\lambda D$ provides the expected result. □

Appendix C. The maximization lemma

We write here the maximization lemma which is used in order to calculate the Gevrey index. Its proof is straightforward.

Lemma C.1. *Let $\alpha > 0$. The maximum of the function $(0, +\infty) \ni x \mapsto x^\alpha e^{-x}$ is attained for $x = \alpha$ and is equal to $\alpha^\alpha e^{-\alpha}$.*

The following Lemma will be used with $\epsilon = 4$ for the Schrödinger equation and $\epsilon = 2$ for the wave equation.

Lemma C.2. *Let $0 < p < 2/3$ and $\epsilon, \mu, \beta > 0$. Then, there exists $C, L > 0$ so that one has*

$$m^{\frac{\epsilon j}{p}+\beta} e^{-\mu m^{\frac{2}{p}-3}} \leq C L^j \Gamma_\lambda(j), \quad \text{for all } j \in \mathbb{N}, m > 0,$$

where $\lambda = \frac{\epsilon}{2-3p} > 0$.

Proof. We denote $Q_m^j := m^{\frac{\epsilon j}{p}+\beta} e^{-\mu m^{\frac{2}{p}-3}}$ and notice that $Q_m^j = \frac{1}{\mu^\alpha} x^\alpha e^{-x}$ with $\alpha = \frac{\epsilon j + \beta p}{2-3p} > 0$, $x = \mu m^{\frac{2}{p}-3} > 0$. Lemma C.1 then yields for any $j \in \mathbb{N}$, $m > 0$

$$Q_m^j \leq \left(\frac{1}{e\mu}\right)^{\frac{\epsilon j+\beta p}{2-3p}} \left(\frac{\epsilon j + \beta p}{2-3p}\right)^{\frac{4j+\beta p}{2-3p}},$$

which has the advantage not to depend on the parameter m anymore. We set $b = \frac{\beta p}{2-3p} > 0$ and $\lambda = \frac{\epsilon}{2-3p} > 0$. Then the previous estimate reads

$$Q_m^j \leq \left(\frac{1}{e\mu}\right)^{\lambda j + b} (\lambda j + b)^{\lambda j + b}.$$

For $j \geq 1$, we have

$$\begin{aligned}(\lambda j + b)^{\lambda j + b} &= (\lambda j + b)^{\lambda j} (\lambda j + b)^b = (\lambda j)^{\lambda j} \left(1 + \frac{b}{\lambda j}\right)^{\lambda j} (\lambda j + b)^b \\ &\leq e^b \lambda^{\lambda j} j^{\lambda j} (\lambda + b)^b j^b,\end{aligned}$$

where we used $\left(1 + \frac{b}{\lambda j}\right)^{\lambda j} \leq e^b$. Consequently,

$$Q_m^j \leq C_0 L_0^j \left(\frac{j}{e}\right)^{\lambda j} j^b,$$

for some C_0, L_0 depending on p, ϵ, μ, β. Using the classical inequality (see e.g [Rod93, p10-11]) $\left(\frac{j}{e}\right)^j \leq j!$ we find that

$$Q_m^j \leq C_0 L_0^j (j!)^\lambda j^b.$$

Finally, there is $C > 0$ such that for $j \geq 1$ it holds $j^b \leq C\, 2^j (1+j)^{-2}$ and therefore, after recalling that $\Gamma_\lambda(j) = 2^{-5}(j!)^\lambda (1+j)^{-2}$, we obtain

$$Q_m^j \leq C (2L_0)^j \Gamma_\lambda(j), \quad j \geq 1.$$

This proves the lemma for $L \geq 2L_0$ and $j \geq 1$. It only remains to include the case $j = 0$ after renaming the constant since $m^\beta e^{-\mu m^{\frac{2}{p}-3}} \leq C$ uniformly in $m > 0$ for some other constant depending on p, ϵ, μ, β. □

Appendix D. Perturbation of observability

This section is devoted to proving the following result, stating that observability (which is a linear quantitative version of unique continuation) is stable by small perturbations of the evolution.

Proposition D.1 (Observability for small bounded perturbations)**.** *Let $\mathcal{X}, \mathcal{Y}$ be two Banach spaces. Assume that an operator $(A, D(A))$ generates a strongly continuous semigroup $(e^{tA})_{t \in \mathbb{R}_+}$ on $\mathcal{X}$, and that $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a bounded observation operator. Let $T > 0$ and assume that there is $C_{obs} > 0$ such that*

$$\|u_0\|_{\mathcal{X}}^2 \leq C_{obs} \int_0^T \|C e^{tA} u_0\|_{\mathcal{Y}}^2 dt, \quad \text{for all } u_0 \in \mathcal{X}. \tag{D.1}$$

Then, there is $K_0 > 0$ such that for all $B \in L^1(0,T;\mathcal{L}(\mathcal{X}))$, we have

$$\big(1 - K_0\|B\|^2_{L^1(0,T;\mathcal{L}(\mathcal{X}))}\big)\|v_0\|^2_{\mathcal{X}} \leq 2C_{obs}\int_0^T \|Cv(t)\|^2_{\mathcal{Y}}dt, \quad \text{for all } v_0 \in \mathcal{X}, \tag{D.2}$$

where v denotes the unique solution to

$$\partial_t v - (A + B(t))v = 0, \quad t \in (0,T), \quad v(0) = v_0. \tag{D.3}$$

The output of Proposition D.1 is that observability continues to hold for the system perturbed by B if $\|B\|_{L^1(0,T;\mathcal{L}(\mathcal{X}))}$ is sufficiently small.

Proof. A fixed point argument in the Duhamel formula

$$v(t) = e^{tA}v_0 + \int_0^t e^{(t-s)A}B(s)v(s)ds,$$

implies well-posedness in $C^0([0,T];\mathcal{X})$ for (D.3), together the estimate (for some $C_0 = C_0(A,B,T)$)

$$\|v(t)\|_{\mathcal{X}} \leq C_0\|v_0\|_{\mathcal{X}}, \quad t \in [0,T].$$

The Duhamel formula also implies the estimate (for some $C_1 = C_1(A,B,T)$)

$$\|v(t) - e^{tA}v_0\|_{\mathcal{X}} \leq C_1\|B\|_{L^1(0,T;\mathcal{L}(\mathcal{X}))}\|v\|_{L^\infty(0,T;\mathcal{X})}, \qquad \text{for all } t \in (0,T).$$

Combining the last two lines implies

$$\|v(t) - e^{tA}v_0\|_{\mathcal{X}} \leq C_1C_0\|B\|_{L^1(0,T;\mathcal{L}(\mathcal{X}))}\|v_0\|_{\mathcal{X}}, \qquad \text{for all } t \in (0,T).$$

Applying now (D.2) to v_0 and using the last inequality, we obtain

$$\begin{aligned}\|v_0\|^2_{\mathcal{X}} &\leq C_{obs}\int_0^T \|Ce^{tA}v_0\|^2_{\mathcal{Y}}dt \\ &\leq 2C_{obs}\int_0^T \|Cv(t)\|^2_{\mathcal{Y}}dt + 2C_{obs}C_1^2C_0^2T\|C\|^2_{\mathcal{L}(\mathcal{X};\mathcal{Y})}\|B\|^2_{L^1(0,T;\mathcal{L}(\mathcal{X}))}\|v_0\|^2_{\mathcal{X}},\end{aligned}$$

which concludes the proof with $K_0 = 2C_{obs}C_1^2C_0^2T\|C\|^2_{\mathcal{L}(\mathcal{X};\mathcal{Y})}$. □

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